Outline.

1. Introduction

2. Snake Graphs for Surfaces without Punctures

3. Graphs for the Classical Types (Bipartite Seeds)

4. Other Examples of Graph Theoretic Interpretations
Definition (Sergey Fomin and Andrei Zelevinsky 2001) A cluster algebra $\mathcal{A}$ is a certain subalgebra of $k(x_1, \ldots, x_m)$, the field of rational functions over $\{x_1, \ldots, x_m\}$. Generators constructed by a series of exchange relations, which in turn induce all relations satisfied by the generators.
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Theorem. (The Laurent Phenomenon FZ 2001) For any cluster algebra defined by initial seed $(\{x_1, x_2, \ldots, x_m\}, B)$, all cluster variables of $\mathcal{A}(B)$ are Laurent polynomials in $\{x_1, x_2, \ldots, x_m\}$

(with no coefficient $x_{n+1}, \ldots, x_m$ in the denominator).

Thus, any cluster variable $x_\alpha = \frac{P_\alpha(x_1, \ldots, x_m)}{x_1^{\alpha_1} \cdots x_n^{\alpha_n}}$ where $P_\alpha \in \mathbb{Z}[x_1, \ldots, x_n]$.

(We use the notation $x_\alpha$ since we only consider cases in this talk where denominator defines cluster variable.)
Cluster Expansions

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(We use the notation $x_\alpha$ since we only consider cases in this talk where denominator defines cluster variable.)

**Conjecture.** (Positivity Conjecture FZ 2001) For any cluster variable $x_\alpha$ the polynomial $P_\alpha(x_1, \ldots, x_n)$ has nonnegative integer coefficients.

[FZ 2002] proved positivity for finite type with bipartite seed.
Some Prior Work on Positivity Conjecture


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Some Prior Work on Positivity Conjecture


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Positivity also proven for those cluster variables for an acyclic seed [Caldero-Reineke 2006],

as well as for Cluster algebras arising from unpunctured surfaces [Schiffler-Thomas 2007, Schiffler 2008], generalizing Trails model of Carroll-Price.
We follow (Fomin-Shapiro-Thurston) and have a surface \((S, M)\). We assume marked points \(M \subset \partial S\) (no punctures).

Recall an arc \(\gamma\) satisfies (we care about arcs up to isotopy)
Cluster Algebras of Triangulated Surfaces

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1. The endpoints of \(\gamma\) are in \(M\).
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3. Relative interior of \(\gamma\) is disjoint from \(M\) and the boundary of \(S\).
We follow (Fomin-Shapiro-Thurston) and have a surface $(S, M)$. We assume marked points $M \subset \partial S$ (no punctures).

Recall an arc $\gamma$ satisfies (we care about arcs up to isotopy)

1. The endpoints of $\gamma$ are in $M$.
2. $\gamma$ does not cross itself.
3. relative interior of $\gamma$ is disjoint from $M$ and the boundary of $S$.
4. $\gamma$ does not cut out a monogon or digon.

Seed $\leftrightarrow$ Triangulation $T = \{\tau_1, \tau_2, \ldots, \tau_n\}$

Cluster Variable $\leftrightarrow$ Arc $\gamma$

$x_i \leftrightarrow \tau_i \in T$.

For $\gamma \not\in T$ let $e_i(T : \gamma)$ be the minimal intersection number of $\tau_i$ and $\gamma$. 

A Graph Theoretic Approach

Recall from Ralf Schiffler’s Talk:

**Theorem.** (M-Schiffler 2008) For every triangulation $T$ (in a surface without punctures) and arc $\gamma$, we construct a snake graph $G_{\gamma,T}$ such that

$$x_\gamma = \frac{\sum_{\text{perfect matching } M \text{ of } G_{\gamma,T}} x(M)y(M)}{x_1^{e_1(T,\gamma)}x_2^{e_2(T,\gamma)} \cdots x_n^{e_n(T,\gamma)}}$$

where $e_i(T,\gamma)$ is the crossing number of $\tau_i$ and $\gamma$, and $x(M)$, $y(M)$ are each monomials. ($x_\gamma$ is cluster variable with principal coefficients.)
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where $e_i(T, \gamma)$ is the crossing number of $\tau_i$ and $\gamma$, and $x(M), y(M)$ are each monomials. ($x_{\gamma}$ is cluster variable with principal coefficients.)

**Definition.** Given a simple undirected graph $G = (V, E)$, a perfect matching $M \subseteq E$ is a set of distinguished edges so that every vertex of $V$ is covered exactly once. (Each edge has weight $x(e)$ where $x(e)$ is allowed to be 1 (unweighted) or some variable $x_i$.)
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**Theorem.** (M-Schiffler 2008) For every triangulation $T$ (in a surface without punctures) and arc $\gamma$, we construct a snake graph $G_{\gamma,T}$ such that

$$x\gamma = \sum_{\text{perfect matching } M \text{ of } G_{\gamma,T}} x(M)y(M)$$

where $e_i(T, \gamma)$ is the crossing number of $\tau_i$ and $\gamma$, and $x(M), y(M)$ are each monomials. ($x\gamma$ is cluster variable with *principal coefficients*.)

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The weight of a matching $M$ is the product of the weights of the constituent edges, i.e. $x(M) = \prod_{e \in M} x(e)$. 

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December 19, 2008
Example of Octagon
Recall that there are 5 completed \((T, \gamma)\)-paths of this octagon, with weights

\[
\frac{x_3^2 + x_3x_4 + x_2x_3 + x_2x_4 + x_1x_5}{x_1x_3x_5}.
\]
Consider the graph $G_{TO, \gamma} = 
\begin{array}{cccc}
7 & 1 & 2 & 3 \\
8 & 1 & 4 & 5 \\
\end{array}
$

$G_{TO, \gamma}$ has five perfect matchings ($x_7, x_8, \ldots, x_{13} = 1$):

- $x_3(x_8)x_3(x_{13}),$
- $x_3(x_8)x_4(x_{11}),$
- $(x_7)x_1x_5(x_{11}).$
- $x_2(x_7)x_3(x_{13}),$
- $(x_7)x_2x_4(x_{11}).$
Consider the graph $G_{T_0, \gamma} =$

\[
\begin{array}{cccc}
3 & 5 & 12 \\
7 & 2 & 3 & 4 & 5 & 11 \\
8 & 1 & 3
\end{array}
\]

$G_{T_0, \gamma}$ has five perfect matchings $(x_7, x_8, \ldots, x_{13} = 1)$:

- $x_3(x_8)x_3(x_{13})$,
- $x_3(x_8)x_4(x_{11})$,
- $(x_7)x_1x_5(x_{11})$,
- $x_2(x_7)x_3(x_{13})$,
- $(x_7)x_2x_4(x_{11})$.

Dividing each monomial by $x_1x_3x_5$, we obtain weights of $(T, \gamma)$-paths.
How to construct $G_{\mathcal{T}, \gamma}$'s (unpunctured surfaces)

**Definition.** For $1 \leq i \leq n$ (i.e. all $\tau_i \in \mathcal{T}$), define Tile $\overline{S_i}$ to be (weighted) triangulated quadrilateral homeomorphic to the quadrilateral bounding arc $\tau_i$ in surface $S$. (Diagonal $NW - SE$ and opposite sides still opposite)
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1. Now given arc $\gamma$: Pick orientation of $\gamma : s \rightarrow t$.
2. Label $p_0 = s, p_1, \ldots, p_d, p_{d+1} = t$, the intersection points of $\gamma$ with $T$ ($p_j \in \tau_i$).
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1. Now given arc $\gamma$: Pick orientation of $\gamma : s \to t$.
2. Label $p_0 = s$, $p_1, \ldots, p_d, p_{d+1} = t$, the intersection points of $\gamma$ with $T$ ($p_j \in \tau_{i_j}$).
3. Let $\Delta_i$ (for $1 \leq j \leq d - 1$) denote the triangles bounded by arcs $\tau_{i_j}$ and $\tau_{i_{j+1}}$. ($\Delta_0$ and $\Delta_d$ denote the first and last triangles that $\gamma$ traverses.)
4. Let $[\gamma_j]$ denote the third side of $\Delta_j$ for $1 \leq j \leq d - 1$. 
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5. By convention Let $G_{\gamma,1} := \overline{S_i}$.
**How to construct $G_{T, \gamma}$'s (unpunctured surfaces)**

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5. By convention Let $\overline{G_{\gamma,1}} := \overline{S_{i_1}}$.
6. Inductively attach tile $\overline{S_{i_{j+1}}}$ to graph $\overline{G_{\gamma,j}}$ to obtain $\overline{G_{\gamma,j+1}}$. 

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December 19, 2008  
9 / 39
How to construct $G_{T, \gamma}$'s (unpunctured surfaces)

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6. Inductively attach tile $\overline{S_{i_{j+1}}}$ to graph $G_{\gamma, j}$ to obtain $G_{\gamma, j+1}$. ($N$ or $E$ edge of $G_{\gamma, j}$ agrees with tile $\overline{S_{i_{j+1}}}: N \leftrightarrow E \text{ and } S \leftrightarrow W$)
How to construct $G_{T,\gamma}$'s (unpunctured surfaces)

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   ($N$ or $E$ edge of $G_{\gamma, j}$ agrees with tile $\overline{S_{i_{j+1}}}$: $N \leftrightarrow E$ and $S \leftrightarrow W$)
7. We define $G_{T,\gamma}$ to be $G_{\gamma,d}$.
How to construct $G_{T,\gamma}$'s (unpunctured surfaces)

**Definition.** For $1 \leq i \leq n$ (i.e. all $\tau_i \in T$), define Tile $\overline{S_i}$ to be (weighted) triangulated quadrilateral homeomorphic to the quadrilateral bounding arc $\tau_i$ in surface $S$. (Diagonal $NW - SE$ and opposite sides still opposite)

1. Now given arc $\gamma$: Pick orientation of $\gamma : s \to t$.
2. Label $p_0 = s, p_1, \ldots, p_d, p_{d+1} = t$, the intersection points of $\gamma$ with $T$ ($p_j \in \tau_{i_j}$).
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6. Inductively attach tile $\overline{S_{i_{j+1}}}$ to graph $\overline{G_{\gamma,j}}$ to obtain $\overline{G_{\gamma,j+1}}$. ($N$ or $E$ edge of $\overline{G_{\gamma,j}}$ agrees with tile $\overline{S_{i_{j+1}}}$: $N \leftrightarrow E$ and $S \leftrightarrow W$)
7. We define $\overline{G_{T,\gamma}}$ to be $\overline{G_{\gamma,d}}$. (Erase diagonals to obtain $G_{T,\gamma}$.)
Examples of $G_{T,\gamma}$

Example 1. Use above construction for $\gamma$.
Examples of $G_{T,\gamma}$

Example 1. Use above construction for

```
  3
  2
  1
  4
  5
  6
  7
  8
```

```plaintext
Example 1. Use above construction for
```
Examples of $G_{T,\gamma}$

**Example 1.** Use above construction for

$$
\begin{array}{c}
3 \\
7 & 2 \\
8 & 7 \\
\end{array}, \quad
\begin{array}{c}
3 & 5 \\
7 & 4 \\
1 & 8 & 2 \\
\end{array}
$$
Example 1. Use above construction for

\[
G_{T_0, \gamma} = \begin{array}{c}
\begin{array}{ccc}
7 & 1 & 2 \\
8 & 1 & 3 \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{ccc}
3 & 5 \\
12 & 31 \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{ccc}
8 & 1 & 3 \\
1 & 5 \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{ccc}
7 & 1 & 2 \\
8 & 1 & 3 \\
\end{array}
\end{array}
\end{array}
\]
Example 2. We now construct graph $G_{T_A, \gamma}$. 
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We now wish to give formula for $y(M)$’s, i.e. the terms in the $F$-polynomials.
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**Definition.** [W. Thurston-Conway] (Following description of [Elkies-Larsen-Kuperberg-Propp])

Given a snake graph $G$, up to orientation, there is a choice of **minimal matching** $(M_-)$ which consists of every-other edge on the boundary.
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Given a snake graph $G$, up to orientation, there is a choice of minimal matching $(M_-)$ which consists of every-other edge on the boundary.

Given any other matching $M$, let $M \ominus M_-$ denote the symmetric difference.

The **height** $h_M : \text{Faces}(G) \rightarrow \mathbb{Z}_{\geq 0}$ of matching $M$ is a function recording which faces are enclosed by $M \ominus M_-$. 
Height Functions (of Perfect Matchings of *Snake* Graphs)

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Given a snake graph $G$, up to orientation, there is a choice of **minimal matching** ($M_\ominus$) which consists of every-other edge on the boundary.

Given any other matching $M$, let $M \ominus M_\ominus$ denote the **symmetric difference**.

The **height** $h_M : \text{Faces}(G) \to \mathbb{Z}_{\geq 0}$ of matching $M$ is a function recording which faces are enclosed by $M \ominus M_\ominus$.

For snake graphs, $h_M(F) \in \{0, 1\}$ and we obtain the formula

$$y(M) := \prod_i y_i \sum_{\text{Faces Labeled } i} h_M(F).$$
Recall that $G_{T_0,\gamma}$ has three faces, labeled 1, 3 and 5. $G_{T_0,\gamma}$ has five perfect matchings ($x_7, x_8, \ldots, x_{13} = 1$):

$x_3^2 y_1 y_3 y_5,$
$x_3 x_4 y_1 y_3,$
$x_1 x_5 (1).$

(← This matching is $M_-$.)
Recall that $G_{T_0, \gamma}$ has three faces, labeled 1, 3 and 5. $G_{T_0, \gamma}$ has five perfect matchings ($x_7, x_8, \ldots, x_{13} = 1$):

$x_3^2 y_1 y_3 y_5$, $x_2 x_3 y_3 y_5$, $x_3 x_4 y_1 y_3$, $x_2 x_4 y_3$, $x_1 x_5 (1)$. (← This matching is $M_-$.)

For example, we get heights $y_1 y_3$, $y_3$, and $y_3 y_5$ because of superpositions:
For $G_{T_{A,\gamma}}$, $M_-$ is
Height Function Examples (continued)

For $G_{T_A, \gamma}$, $\gamma = 1 2 3 4 1 2 4 7 1 3 4 6 2 3 8 2 5 3$, $M_-$ is $1 2 3$. One of the 17 matchings, $M$, is $1 2 3$, $1 2$, $2 3$, $4$.
For $G_{TA, \gamma}$, $M_-$ is one of the 17 matchings, $M$, is, so $M \ominus M_-$.
Height Function Examples (continued)

For $G_{T_A, \gamma}$, $M_-$ is one of the 17 matchings. One of the 17 terms in the cluster expansion of $x_\gamma$ is $\frac{x_4^2 x_2}{x_1^2 x_2 x_3 x_4} (y_1 y_2^2)$. Which has height $y_1 y_2^2$. So one of the 17 terms in the cluster expansion of $x_\gamma$ is $\frac{x_4^2 x_2}{x_1^2 x_2 x_3 x_4} (y_1 y_2^2)$. 

Gregg Musiker (MIT)  
Graph Theoretical Cluster Expansions  
December 19, 2008  
14 / 39
Theorem. (M-Schiffler 2008) For every triangulation $T$ of unpunctured surface and arc $\gamma$, we construct a snake graph $G_{\gamma,T}$ such that

$$x_\gamma = \frac{\sum_{\text{perfect matching } M \text{ of } G_{\gamma,T}} x(M)y(M)}{x_1^{e_1(T,\gamma)}x_2^{e_2(T,\gamma)} \cdots x_n^{e_n(T,\gamma)}}$$

where $e_i(T,\gamma)$ is the crossing number of $\tau_i$ and $\gamma$, $x(M)$ is the edge-weight of perfect matching $M$, and $y(M)$ is the height of perfect matching $M$. ($x_\gamma$ is cluster variable with principal coefficients.)
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$$x_\gamma = \sum_{\text{perfect matching } M \text{ of } G_{\gamma,T}} \frac{x(M)y(M)}{x_1^{e_1(T,\gamma)}x_2^{e_2(T,\gamma)}\cdots x_n^{e_n(T,\gamma)}}$$

where $e_i(T,\gamma)$ is the crossing number of $\tau_i$ and $\gamma$, $x(M)$ is the edge-weight of perfect matching $M$, and $y(M)$ is the height of perfect matching $M$. ($x_\gamma$ is cluster variable with principal coefficients.)

Corollary. The $F$-polynomial equals $\sum_M y(M)$, is positive, and has constant term 1.

The $g$-vector satisfies $x^g = x(M_-)$.

Corollary. The Laurent expansion of cluster variable $x_\gamma$ is positive for any cluster algebra (of geometric type) arising from a triangulated surface without punctures.
**Theorem.** (M 2007) For every classical root system, let $B_\Phi$ denote the corresponding bipartite seed (without coefficients). Then there exists a family of graphs $G_\Phi = \{ G_\alpha \}_{\alpha \in \Phi_+}$ such that $x_\alpha$, the cluster variable of $A(B_\Phi)$ corresponding to $\alpha \in \Phi_+$, can be expressed as

$$x_\alpha = \frac{P_{G_\alpha}(x_1, \ldots, x_n)}{x_1^{\alpha_1} \cdots x_n^{\alpha_n}}.$$ 

Further, we will construct the graphs in a very simple manner using the tiles $T_k$. 
Tiles for the four classical types

$A_5$

$C_5$

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Graphs for $A_n$ and $C_n$

$A_5$

\[
\begin{array}{ccc}
1 & 3 & 5 \\
1 & 2 & 3 & 3 & 4 & 5 & 3 & 4 \\
2 & 3 & 1 & 2 & 3 & 4 & 5 & 3 & 4 \\
4 & 5 & 2 & 3 & 4 & & & 1 & 2 \\
4 & & & & & & & & \\
\end{array}
\]
$C_3$ folds onto $A_5$ (Take right-half including middle)
Tiles for the four classical types (cont.)

\( B_5 \)

\( D_5 \)
The $B_n$ and $D_n$ cases

$B_4$ After mutating with respect to $x_1$ and $x_3$ ($x_2$ and $x_4$), we obtain
The $B_n$ and $D_n$ cases (cont.)
The $B_n$ and $D_n$ cases (cont.)
The $B_n$ and $D_n$ cases (cont.)

$D_5$

$\begin{array}{c}
\begin{array}{c}
1 \\
\end{array}
\end{array}$

$\begin{array}{c}
\begin{array}{c}
1 \\
\end{array}
\end{array}$

$\begin{array}{c}
\begin{array}{c}
3 \\
\end{array}
\end{array}$

$\begin{array}{c}
\begin{array}{c}
2 \\
\end{array}
\end{array}$

$\begin{array}{c}
\begin{array}{c}
1 \\
\end{array}
\end{array}$

$\begin{array}{c}
\begin{array}{c}
4 \\
3 \\
\end{array}
\end{array}$
The $B_n$ and $D_n$ cases (cont.)

**$D_5$ (cont.)**

![Diagram of $D_5$ case](image-url)
The $B_n$ and $D_n$ cases (cont.)

$D_5$ (cont.)

\[ \begin{array}{c}
1 & 2 \\
4 & 3 \\
\end{array} \quad \begin{array}{c}
1 & 2 \\
4 & 3 \\
\end{array} \]
The $B_n$ and $D_n$ cases (cont.)

$D_5$ (cont.)

Diagram of $D_5$ with labeled nodes.
Seed matrix is $B = \begin{bmatrix} 0 & 1 \\ -3 & 0 \end{bmatrix}$

Hexagon has $x_1$ on NW, NE, and S sides, Trapezoid has $x_2$ on N side.
Joint work with Jim Propp.

Let \( B = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} \) or \( \begin{bmatrix} 0 & -4 \\ 1 & 0 \end{bmatrix} \).

Here we also exploit invariance of matrices \( B \) under mutation.

So we are considering \((b, c)\)-sequence

\[
x_n x_{n-2} = \begin{cases} 
  x^b_{n-1} + 1 & \text{if } n \text{ odd} \\
  x^c_{n-1} + 1 & \text{if } n \text{ even}
\end{cases}
\]

for \((b, c) = (2, 2)\) or \((1, 4)\).
Since cluster algebra structure, \((b, c)\) sequence consists of Laurent polynomials.

Work of Sherman and Zelevinsky verifies positive coefficients for \((1, 4)\) and \((2, 2)\) using Newton polytope, and Caldero-Zelevinsky give another proof of positivity for \((2, 2)\) case via Quiver Grassmannians.

This cluster algebra also comes from an annulus with one marked point on each boundary (no punctures).

Equivalently, this is a cluster algebra of affine type \(\tilde{A}_{1,1}\).

We give proof of positivity via graph theoretical interpretation similar to above.
Affine Rank 2 (cont.)

(2, 2): all cluster variables have denominators $x_1^d x_2^{d+1}$ (resp. $x_1^{d+1} x_2^d$)
We string together corresponding number of squares

in an intertwining fashion.
Affine Rank 2 (cont.)

(2, 2): all cluster variables have denominators $x_1^d x_2^{d+1}$ (resp. $x_1^{d+1} x_2^d$)

We string together corresponding number of squares in an intertwining fashion.

Examples:

$$
\frac{x_2^4 + 2 x_2^2 + 1 + x_1^2}{x_1^2 x_2} \leftrightarrow \begin{array}{c}
1 \\
2 \\
1
\end{array}
$$

$$
\frac{x_1^6 + 3 x_1^4 + 3 x_1^2 + 2 x_2^2 x_1^2 + x_2^4 + 1 + 2 x_2^2}{x_2^3 x_1^2} \leftrightarrow \begin{array}{c}
2 \\
1 \\
2 \\
1 \\
2
\end{array}
$$
(1, 4): Tiles are a square and an octagon:
$x_4$ 17 terms

$x_5$ 9 terms

$x_6$ 386 terms

$x_7$ 43 terms
Sequence Continues (cont.)

\(x_8\) 8857 terms

\(x_9\) 206 terms

\(x_{10}\) 203321 terms

\(x_{11}\) 987 terms
Running the \((1, 4)\) sequence backwards

\[
\begin{align*}
    x_{-1} & \quad 3 \text{ terms} \\
    x_{-2} & \quad 41 \text{ terms} \\
    x_{-3} & \quad 14 \text{ terms} \\
    x_{-4} & \quad 937 \text{ terms}
\end{align*}
\]
Running the $(1, 4)$ sequence backwards (cont.)

- $x_{-5}$: 67 terms
- $x_{-6}$: 21506 terms
- $x_{-7}$: 321 terms
- $x_{-8}$: 493697 terms
Markoff polynomials

Joint work by Carroll, Itsara, Le, M, Price, Thurston, and Viana under Propp in REACH program.

\[
B = \begin{bmatrix}
0 & 2 & -2 \\
-2 & 0 & 2 \\
2 & -2 & 0
\end{bmatrix}, \quad \text{Exchange graph is free ternary tree.}
\]

\(B\) invariant under mutation. All exchanges have form \((x, y, z) \mapsto (x', y, z)\) where \(xx' = y^2 + z^2\).

(Cluster algebra corresponds to once punctured torus.)
Markoff polynomials

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\(B\) invariant under mutation. All exchanges have form \((x, y, z) \mapsto (x', y, z)\) where \(xx' = y^2 + z^2\).

(Cluster algebra corresponds to once punctured torus.)

These also have graph theoretic interpretation: Snake Graphs, e.g.

\[
\text{Polynomial}(x, y, z) \quad \xymatrix{ & x & y & x \\ x & y & x & z } \quad \xymatrix{ & y & x & y \\ z & x & z & y } \quad \xymatrix{ & X & Z & Z \\ z & x & z & y }
\]

with tiles

\[
\xymatrix{ & y \\ y & x & Z }
\]

\[x^4 y^2 z^1\]
Theorem. Formulas for $F$-polynomials and $g$-vectors for types $A$, $B$, $C$, $D$ with respect to any seed (not nec. acyclic).

In Progress. Snake Graph Interpretations for Triangulated Surfaces (even in presence of punctures).
Cluster Expansion Formulas and Perfect Matchings (with Ralf Schiffler), arXiv:math.CO/0810.3638


The Combinatorics of Frieze Patterns and Markoff Numbers (by Jim Propp), arXiv:math.CO/0511633

Slides Available at http://math.mit.edu/~musiker/GraphTalk.pdf