Aztec castles and other instances of brane tilings in combinatorics

Gregg Musiker (U. Minnesota)

International Centre for Mathematical Sciences

Workshop on Gauge theories: quivers, tilings and Calabi-Yaus

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Work done during 2011-2013 Research Experiences for Undergraduates in Combinatorics at University of Minnesota, Twin Cities

Outline

1. Introduction to Cluster Algebras
2. Markoff Quiver
3. Graphs from Cluster Algebras from Surfaces
4. Brane Tilings associated to the Del Pezzo 3 Quiver
5. Aztec Castles

http//math.umn.edu/∼musiker/ICMS.pdf

Introduction to Cluster Algebras

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Cluster algebras are a certain class of commutative rings which have a distinguished set of generators that are grouped into overlapping subsets, called clusters, each having the same cardinality.

What is a Cluster Algebra?

Definition (Sergey Fomin and Andrei Zelevinsky 2001) A cluster algebra $\mathcal{A}$ (of geometric type) is a subalgebra of $k(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m})$ constructed cluster by cluster by certain exchange relations.

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The set of all such generators are known as Cluster Variables, and the initial pattern of exchange relations (described as a *valued* quiver, i.e. a directed graph) determines the Seed.

Relations:

Induced by the Binomial Exchange Relations.

Example: Rank 2 Cluster Algebras

Let $B = \begin{bmatrix} 0 & b \\ -c & 0 \end{bmatrix}$, $b, c \in \mathbb{Z}_{>0}$. $(\{x_1, x_2\}, B)$ is a seed for a cluster algebra $\mathcal{A}(b, c)$ of rank 2.

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$$\mu_1(B) = \mu_2(B) = -B \text{ and } x_1 x_1' = x_2^c + 1, \quad x_2 x_2' = 1 + x_1^b.$$

Thus the cluster variables in this case are

$$\{x_n : n \in \mathbb{Z}\} \text{ satisfying } x_n x_{n-2} = \begin{cases} x_{n-1}^b + 1 \text{ if } n \text{ is odd} \\ x_{n-1}^c + 1 \text{ if } n \text{ is even} \end{cases}.$$

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Example ($b = c = 1$): (Finite Type, of Type A_2)

$$x_3 = \frac{x_2 + 1}{x_1}. \quad x_4 = \frac{x_3 + 1}{x_2} = \frac{\frac{x_2+1}{x_1} + 1}{x_2} = \frac{x_1 + x_2 + 1}{x_1 x_2}.$$

$$x_5 = \frac{x_4 + 1}{x_3} = \frac{\frac{x_1+x_2+1}{x_1 x_2} + 1}{(x_2 + 1)/x_1} = \frac{x_1(x_1 + x_2 + 1 + x_1 x_2)}{x_1 x_2 (x_2 + 1)} = \frac{x_1 + 1}{x_2}. \quad x_6 = x_1.$$

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The next number in the sequence is $x_7 = \frac{34^2+1}{13} = \frac{1157}{13} = 89$, an integer!

Example: The Markoff Quiver and Cluster Algebra

Consider the cyclic quiver

$$\begin{tikzcd} & 1 \arrow[dl, shift right] \arrow[dl, shift left] & \\ 2 \arrow[rr, shift left] \arrow[rr, shift right] & & 3 \arrow[ul, shift left] \arrow[ul, shift right] \end{tikzcd}$$

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We will return to this example later.

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We will return to this example later. Called the Markoff quiver since starting with the seed $\{1, 1, 1\}$, every seed mutation-equivalent is a Markoff triple (i.e. satisfying $3xyz = x^2 + y^2 + z^2$).

$$\{1, 2, 5\}, \quad \{1, 5, 13\}, \quad \{2, 5, 29\}, \quad \{2, 29, 169\}, \quad \{13, 34, 1325\}, \ldots$$

Example: Cluster Algebras from Surfaces

Theorem (Fomin-Shapiro-Thurston 2006), (Based on earlier work of Fock-Goncharov and Gekhtman-Shapiro-Vainshtein):

Given a Riemann surface with marked points (S, M), one can define a corresponding cluster algebra $\mathcal{A}(S, M)$.

$$\text{Seed} \leftrightarrow \text{Triangulation } T = \{\tau_1, \tau_2, \ldots, \tau_n\}$$

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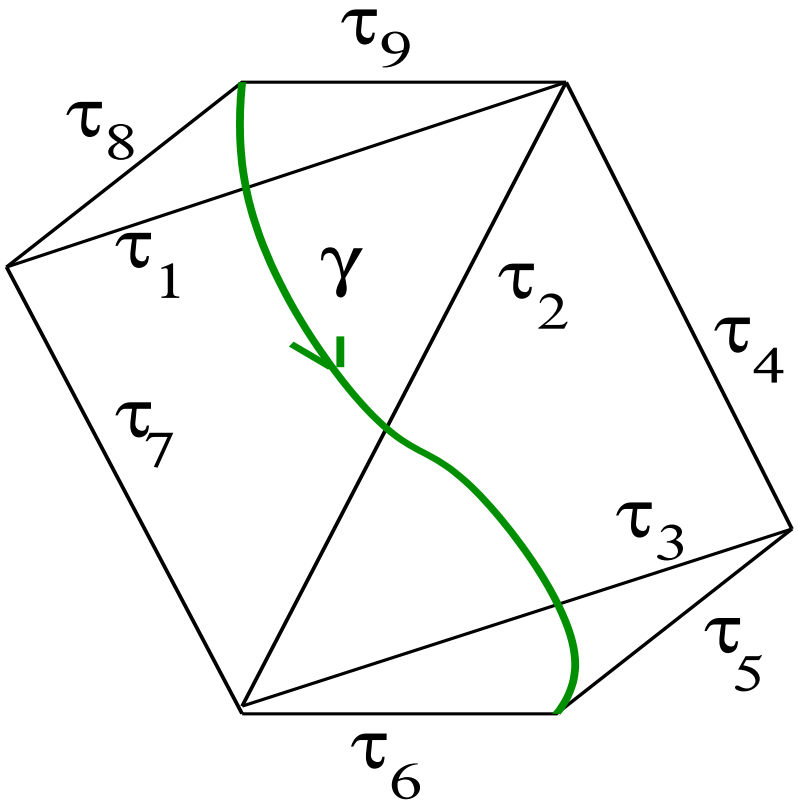
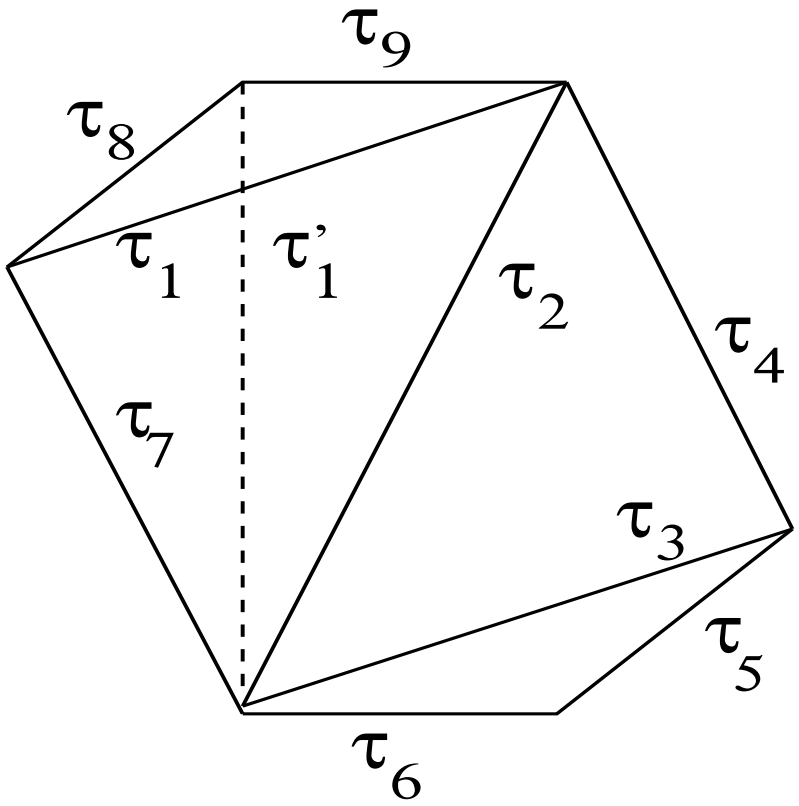
Theorem. (M-Schiffler-Williams 2009) Let $\mathcal{A}(S, M)$ be any cluster algebra arising from a surface (with or without punctures), where the coefficient system is of geometric type, and let Σ be any initial seed.

Then the Laurent expansion of every cluster variable with respect to the seed Σ has non-negative coefficients. Proof via explicit combinatorial formulas in terms of graphs.

Example of Hexagon (Type A_3 or $\mathbb{C}[Gr_{2,6}]$)

Consider the triangulated hexagon (S, M) by the triangulation T.





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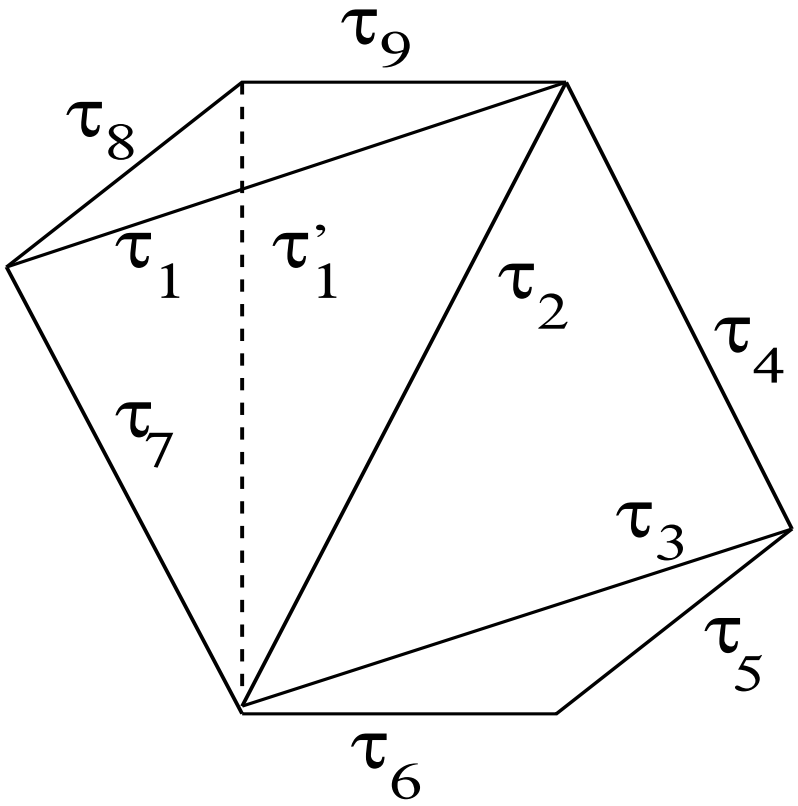

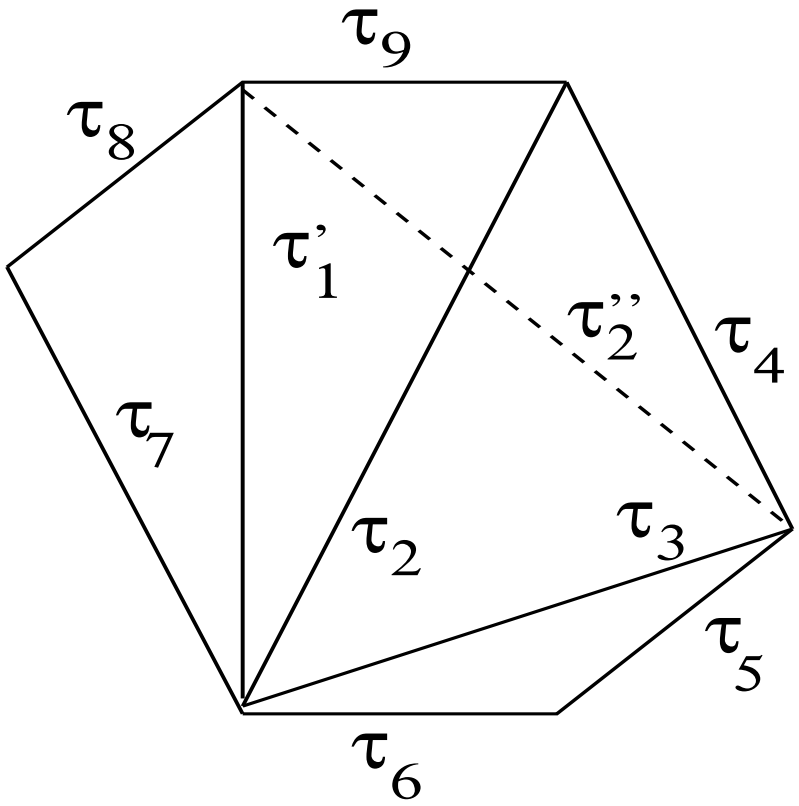

$$
\begin{aligned}
x_1 x_1' &= (x_7 x_9) + x_2(x_8) \\
x_2 x_2'' &= x_3(x_9) + x_1'(x_4)
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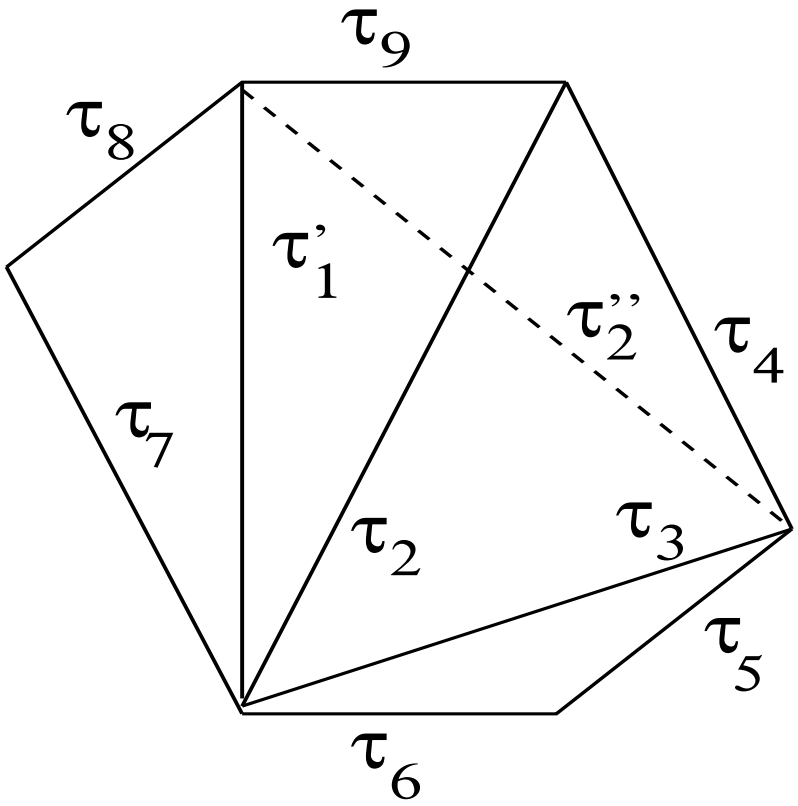
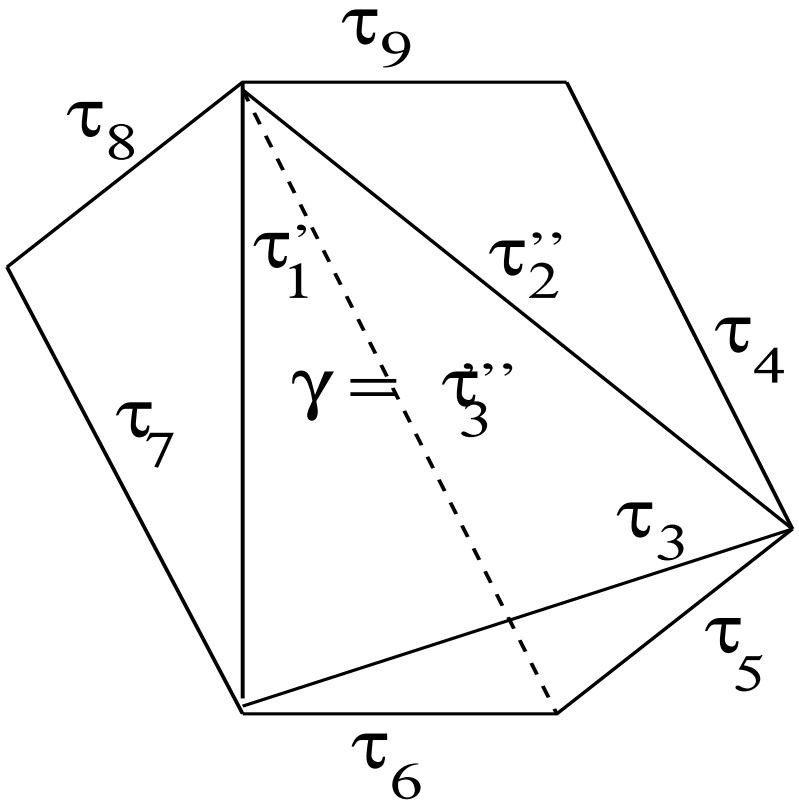
Consider the triangulated hexagon (S, M) by the triangulation T.





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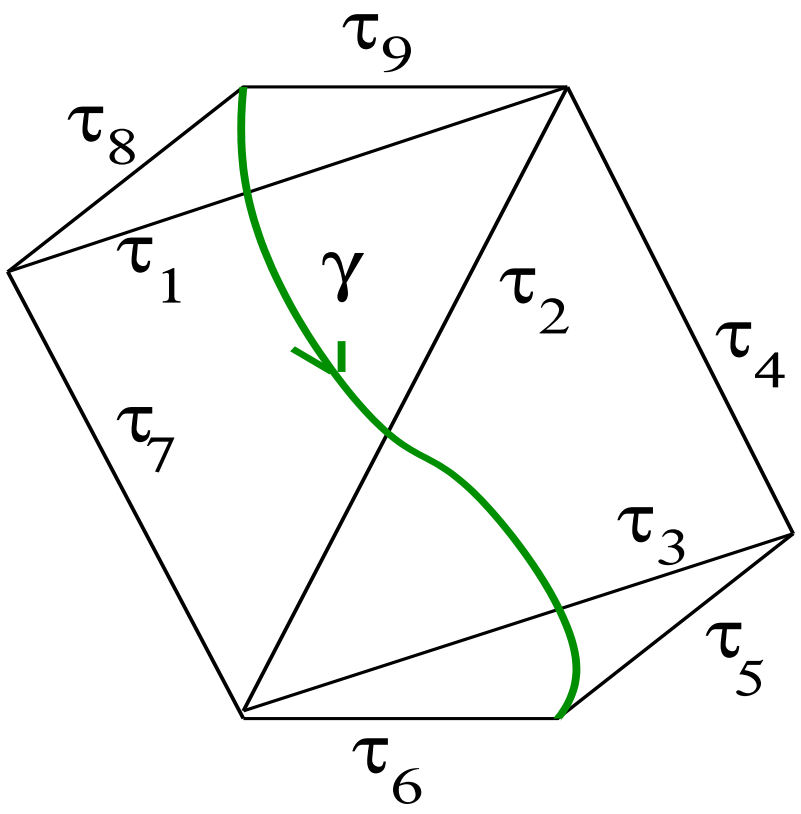
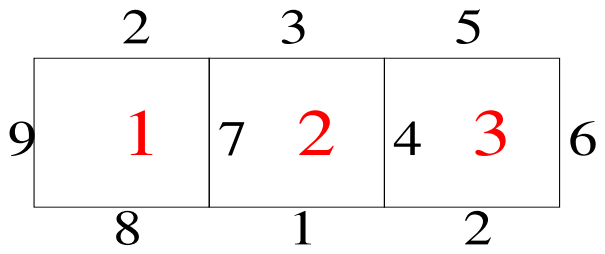


By using the Ptolemy exchange relations on τ_1, τ_2, then τ_3, we obtain

$$\begin{aligned} x_3''' &= x_\gamma = \frac{1}{x_1 x_2 x_3}\Big(x_2^2(x_5 x_8) + x_2(x_5 x_7 x_9) + x_2(x_4 x_6 x_8) \\ &+ (x_4 x_6 x_7 x_9) + x_1 x_3 (x_6 x_9) \Big). \end{aligned}$$

Example of Hexagon (Type A_3 or $\mathbb{C}[Gr_{2,6}]$)

Consider the graph $G_{T_H,\gamma} =$ 

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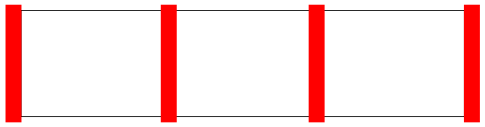
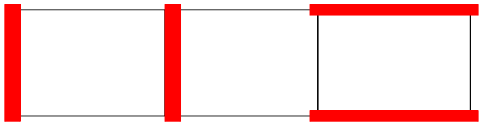
$G_{T_H,\gamma}$ has five perfect matchings $(x_4, x_5, \ldots, x_9 = 1)$:


$(x_9)x_1x_3(x_6)$,


$(x_9x_7x_4x_6)$,


$x_2(x_8)(x_4x_6)$,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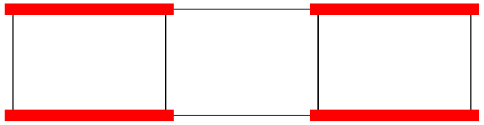
$x_2(x_8)x_2(x_5)$.

A perfect matching $M \subseteq E$ is a set of distinguished edges so that every vertex of V is covered exactly once. The weight of a matching M is the product of the weights of the constituent edges, i.e. $x(M) = \prod_{e \in M} x(e)$.

Principal coefficients: for $i \in Q_0$, add $i' \in Q_0$, $i' \to i \in Q_1$

$$\frac{x_1x_3(x_6x_9)y_1y_2y_3 + (x_4x_6x_7x_9)y_1y_3 + x_2(x_4x_6x_8)y_3 + x_2(x_5x_7x_9)y_1 + x_2^2(x_5x_8)}{x_1x_2x_3}$$

$G_{T_H,\gamma}$ has five perfect matchings $(x_4, x_5, \ldots, x_9 = 1)$:

$(x_9)x_1x_3(x_6)$,

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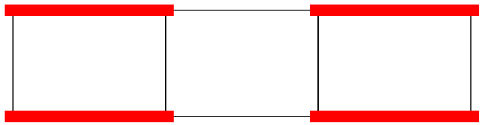

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These five monomials exactly match those appearing in the numerator of the expansion of x_γ. The denominator of $x_1x_2x_3$ corresponds to the labels of the three tiles. The y_i's correspond to principal coefficients (heights).

More on Height Functions

Letting $x_4, x_5, \ldots, x_9 = 1$:

$$x_\gamma = \frac{x_1 x_3 y_1 y_2 y_3 + y_1 y_3 + x_2 y_3 + x_2 y_1 + x_2^2}{x_1 x_2 x_3}$$

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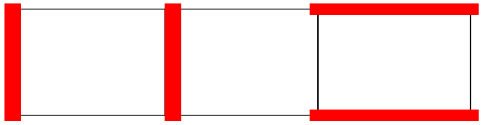
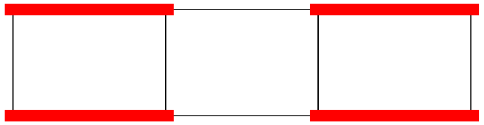


y_1,



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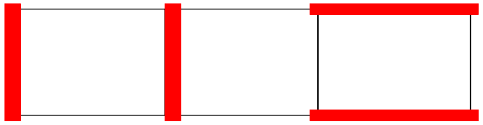

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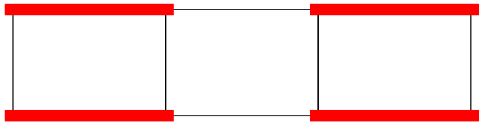

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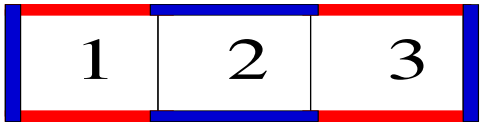

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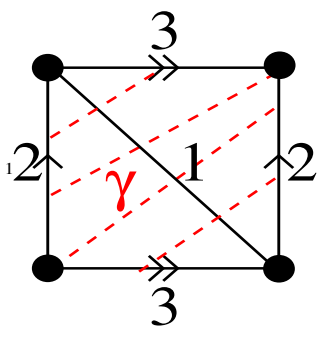
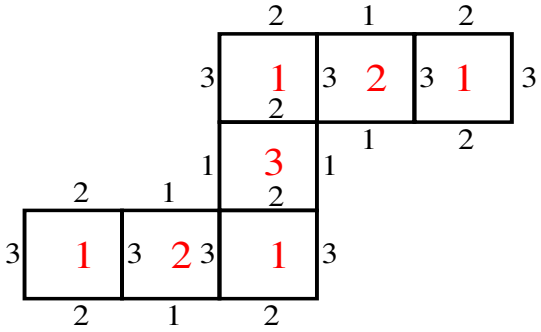
, and

Another Height Function Example (Markoff Quiver)

For $T, \gamma =$ 

, the snake graph is $G_{T,\gamma} =$ 

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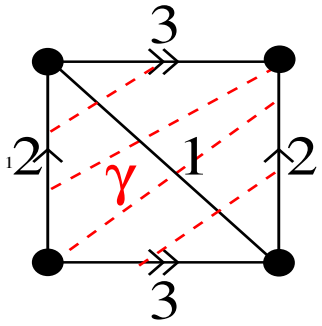
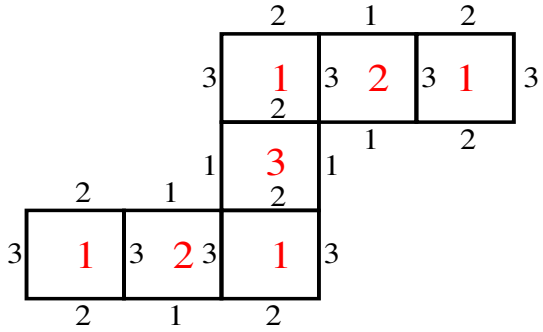
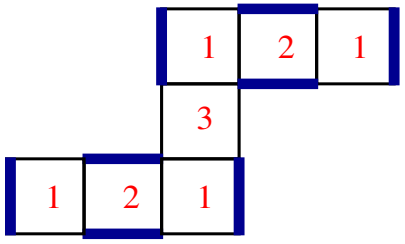
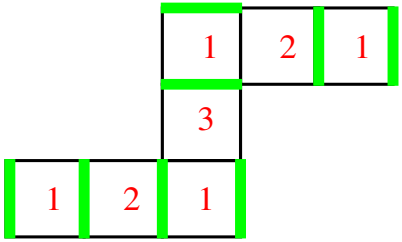
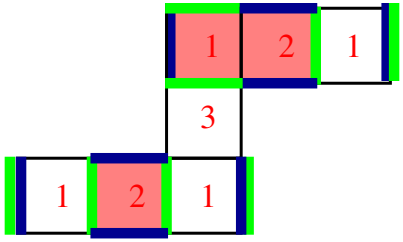
For $T, \gamma =$ 

, the snake graph is $G_{T,\gamma} =$ 

.

For this graph, M_- is 

, and one matching, M, is 

, so $M \ominus M_- =$ 

which has height $y_1 y_2^2$. So one of the 29 terms in the cluster expansion of x_γ is $\frac{x_2^2 x_3^6}{x_1^4 x_2^2 x_3}(y_1 y_2^2)$.

Graph Theoretic Summary

For every triangulation T (in a surface with or without punctures) and an ordinary arc γ through ordinary triangles, we construct a snake graph $G_{T,\gamma}$ such that

$$x_\gamma = \frac{\sum_{\text{perfect matching } M \text{ of } G_{T,\gamma}} x(M)y(M)}{x_1^{e_1(T,\gamma)} x_2^{e_2(T,\gamma)} \cdots x_n^{e_n(T,\gamma)}}.$$

x_γ is cluster variable (corresp. to γ w.r.t. seed given by T) with principal coefficients.

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$e_i(T,\gamma)$ is the crossing number of τ_i and γ (min. int. number),

$x(M)$ is the weight of M,

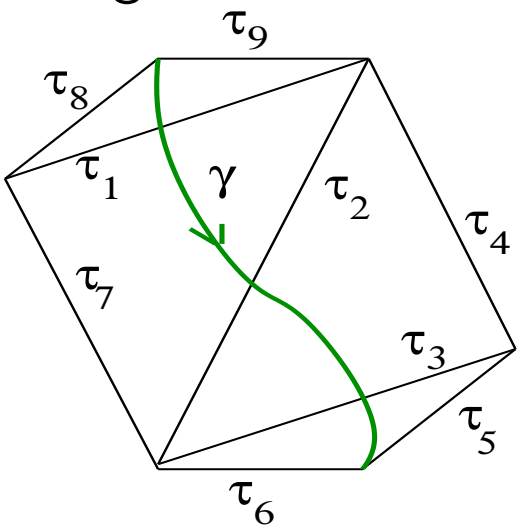
$y(M)$ is the height of M,

Similar formula holds for non-ordinary arcs (or with self-folded triangles).

Height functions are due to William Thurston, and Conway-Lagarias.

How to build $G_{T,\gamma}$

Using the above construction for



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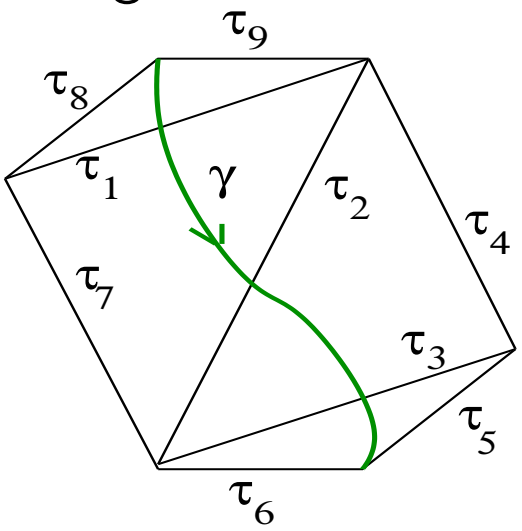

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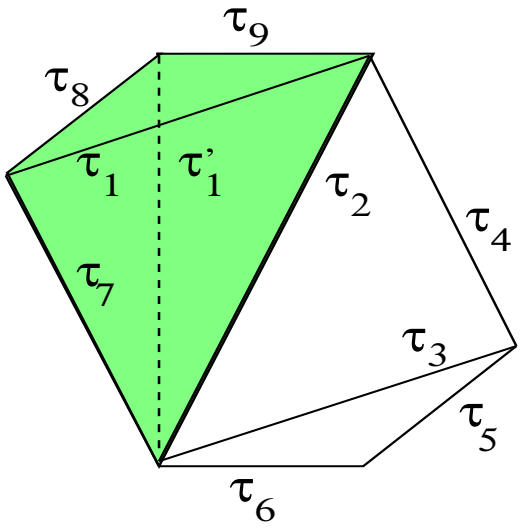

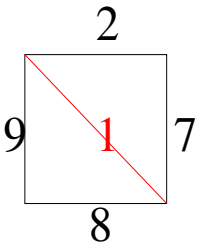

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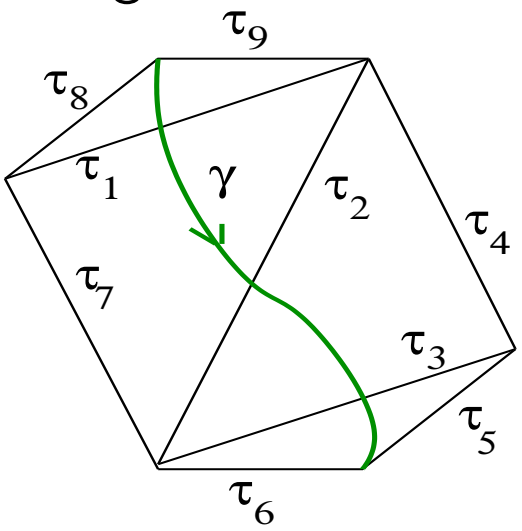

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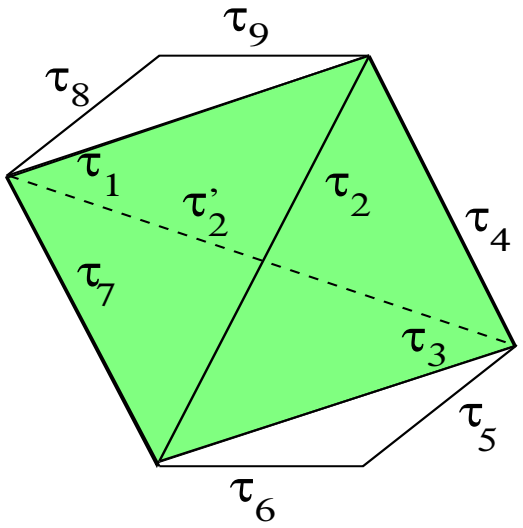

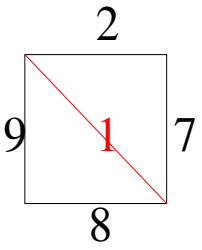

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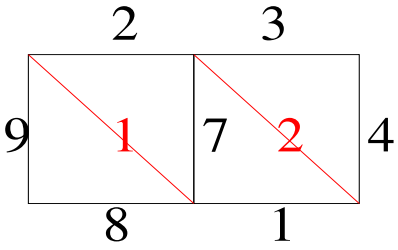

How to build $G_{T,\gamma}$

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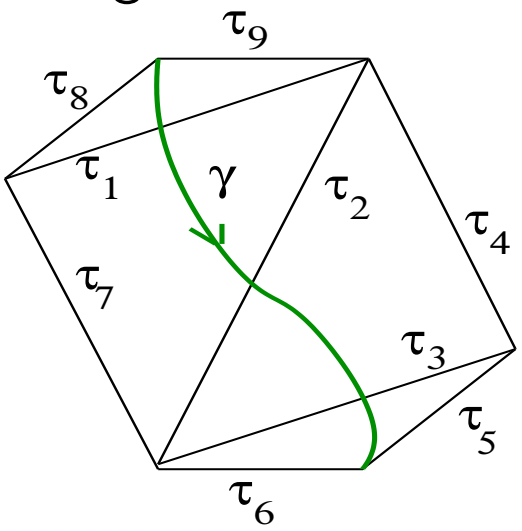

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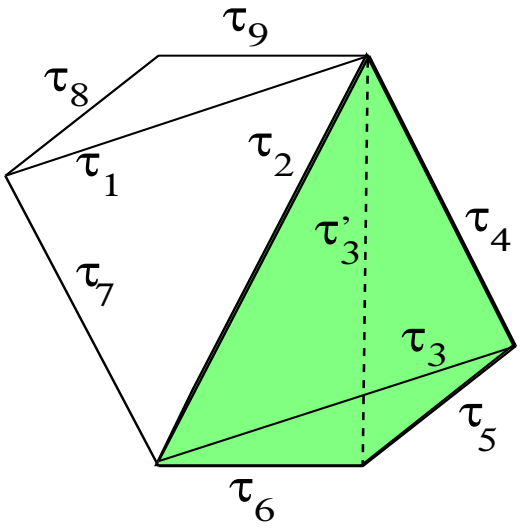

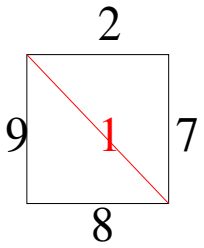

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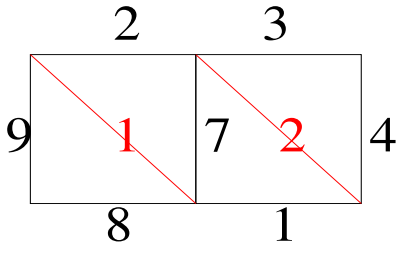

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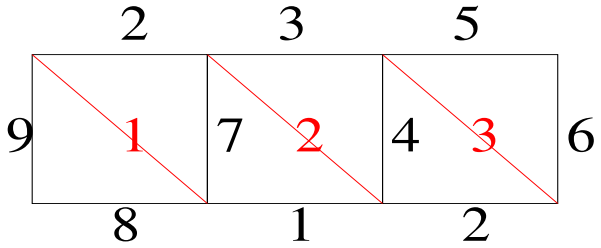
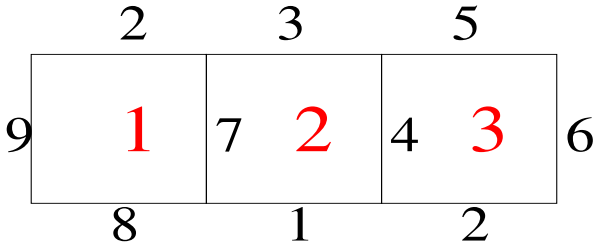


. Thus

$G_{T_H,\gamma} =$



. $x_\gamma|_{y_1=y_2=y_3=1} = \dfrac{x_1x_3 + 1 + 2x_2 + x_2^2}{x_1x_2x_3}$

Brane Tilings and Cluster Algebras

Most simply stated, a Brane Tiling is a **Bipartite graph on a torus**. We view such a tiling as a doubly-periodic tiling of its universal cover.

A **Brane Tiling** can be associated to a pair (Q, W), where Q is a quiver and W is a potential (a.k.a. superpotential).

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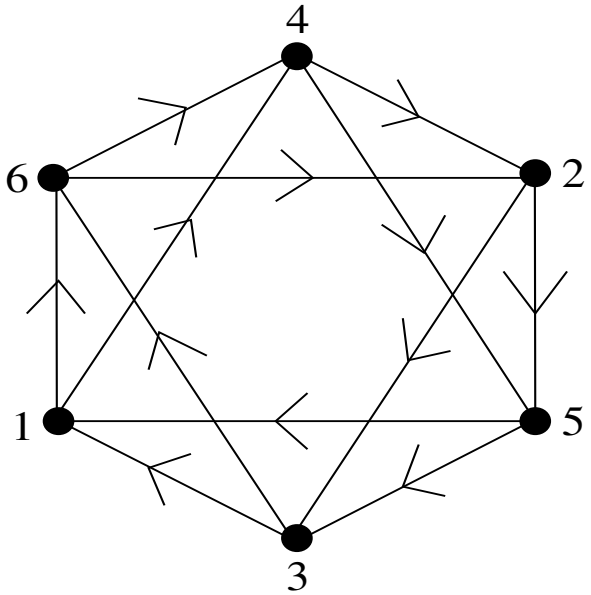
For combinatorial purposes, we assume other conditions on (Q, W):

• Each arrow of Q appears in one term of W with a positive sign, and one term with a negative sign.

• The number of terms of W with a positive sign equals the number with a negative sign. All coefficients in W are ± 1.

Example (The dP_3 Quiver): $Q_{dP_3} = Q =$



,

$$\begin{aligned} W &= A_{16}A_{64}A_{42}A_{25}A_{53}A_{31} + A_{14}A_{45}A_{51} + A_{23}A_{36}A_{62} \\ &- A_{16}A_{62}A_{25}A_{51} - A_{36}A_{64}A_{45}A_{53} - A_{14}A_{42}A_{23}A_{31}. \end{aligned}$$

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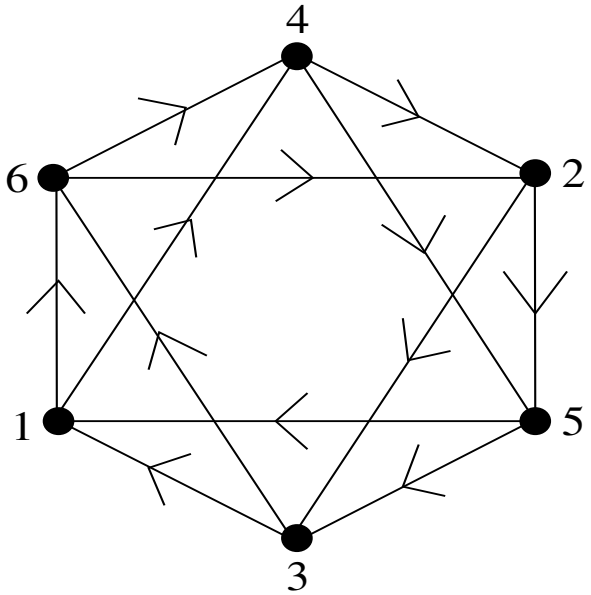

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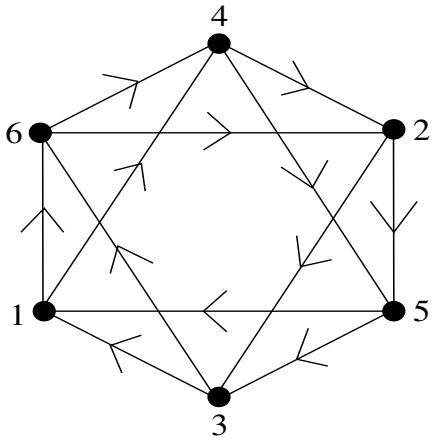
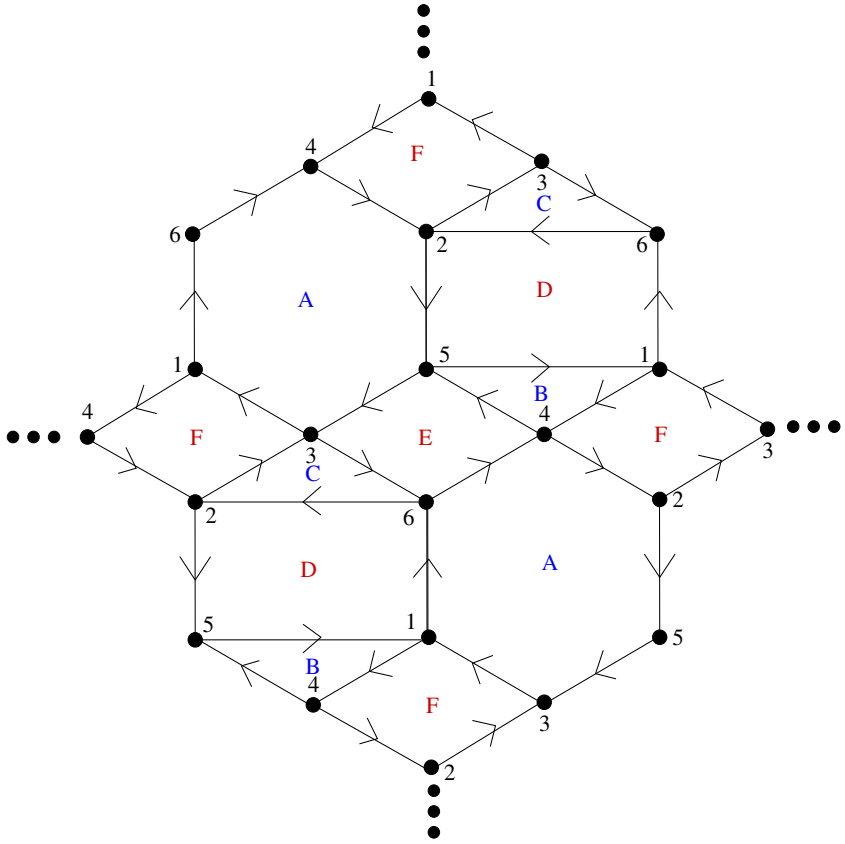
We now unfold Q onto the plane, letting the three positive (resp. negative) terms in W depict clockwise (resp. counter-clockwise) cycles on $\widetilde{Q}$.

Brane Tilings and Cluster Algebras

Example (continued):

$Q =$ 

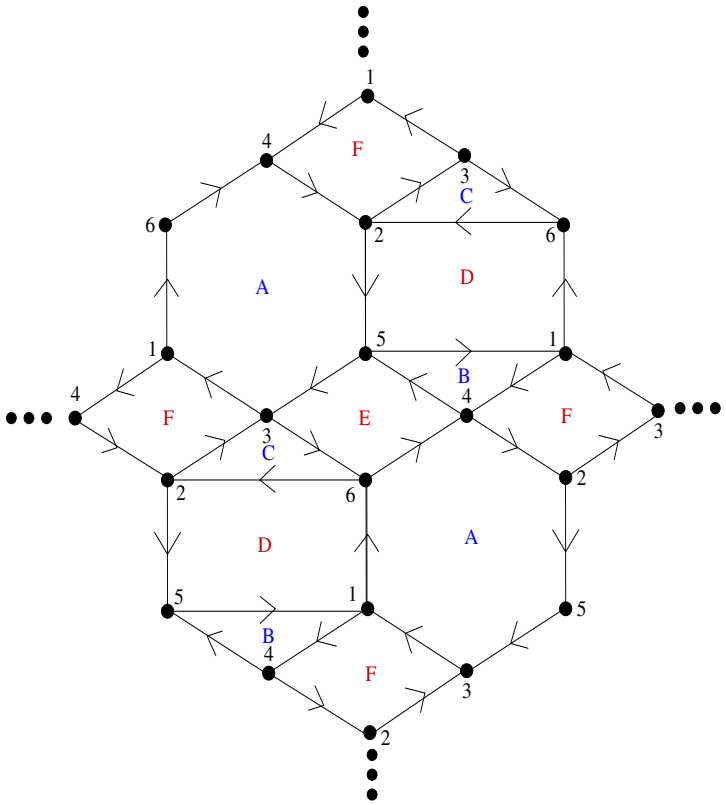
unfolds to $\widetilde{Q} =$ 

$$W = A_{16}A_{64}A_{42}A_{25}A_{53}A_{31}(A) + A_{14}A_{45}A_{51}\ (B) + A_{23}A_{36}A_{62}\ (C)$$
$$- A_{16}A_{62}A_{25}A_{51}\ (D) - A_{36}A_{64}A_{45}A_{53}\ (E) - A_{14}A_{42}A_{23}A_{31}(F).$$

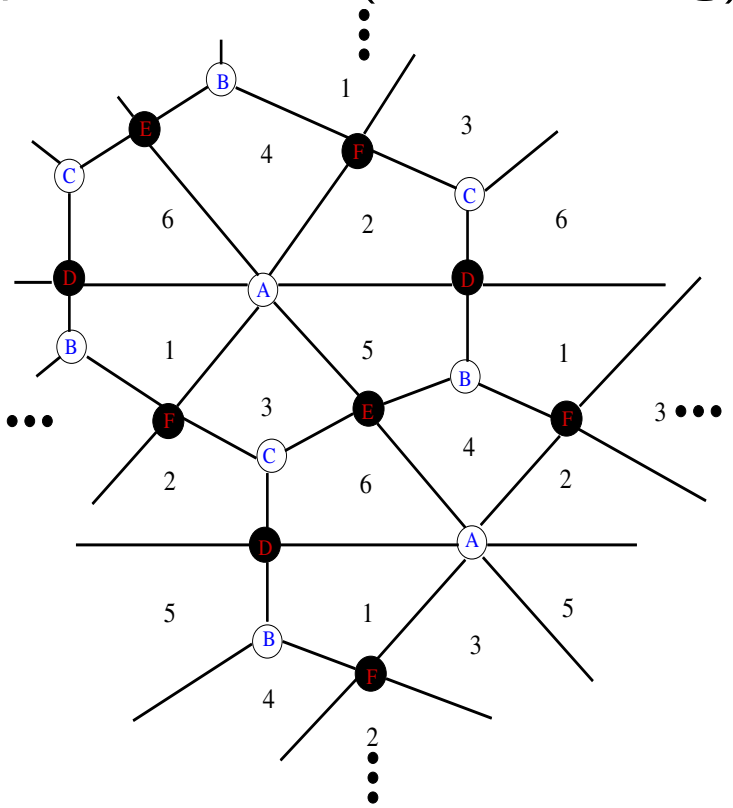
Locally, the configurations around vertices of Q and $\widetilde{Q}$ are identical.

Brane Tilings from a Quiver Q with Potential W

Taking the planar dual yields a bipartite graph on a torus (**Brane Tiling**):



$\widetilde{Q} \longrightarrow \mathcal{T}_Q =$



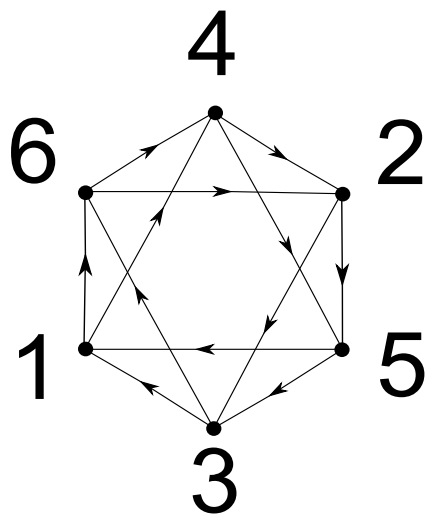
Negative Term in $W \longleftrightarrow$ Counter-Clockwise cycle in $\widetilde{Q} \longleftrightarrow \bullet$ in $\mathcal{T}_Q$

Positive Term in $W \longleftrightarrow$ Clockwise cycle in $\widetilde{Q} \longleftrightarrow \circ$ in $\mathcal{T}_Q$

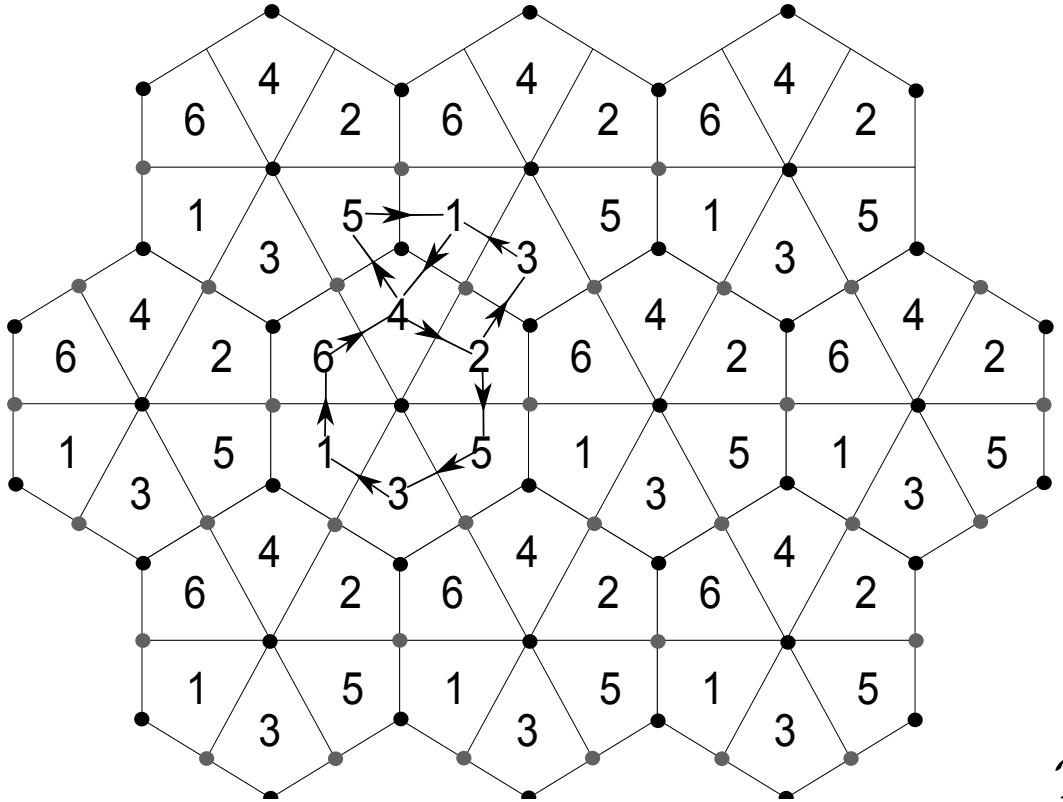
(To obtain $\widetilde{Q}$ from $\mathcal{T}_Q$, we dualize edges so that white is on the right.)

Brane Tilings from a Quiver Q with Potential W

Summarizing the dP_3 Example:



Q



$\mathcal{T}_Q$

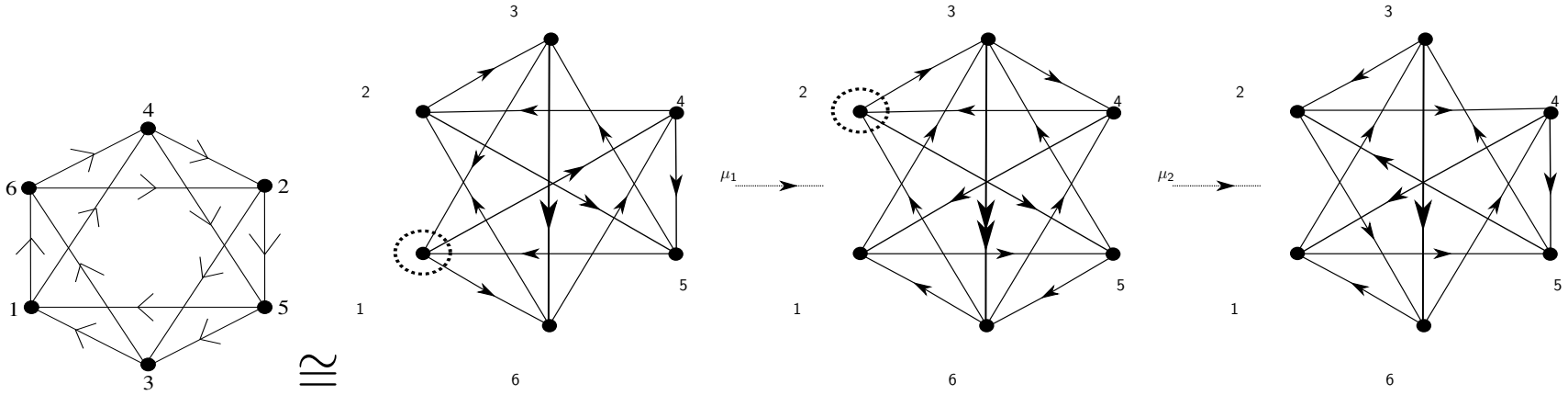
Negative Term in $W \longleftrightarrow$ Counter-Clockwise cycle in $\widetilde{Q} \longleftrightarrow \bullet$ in $\mathcal{T}_Q$

Positive Term in $W \longleftrightarrow$ Clockwise cycle in $\widetilde{Q} \longleftrightarrow \circ$ in $\mathcal{T}_Q$

(To obtain $\widetilde{Q}$ from $\mathcal{T}_Q$, we dualize edges so that white is on the right.)

Enter Combinatorics

If we mutate Q_{dP_3} by $1, 2, 3, 4, 5, 6, 1, 2, \ldots$, after the first two mutations, we obtain same quiver back up to cyclically permuting the vertex labels.



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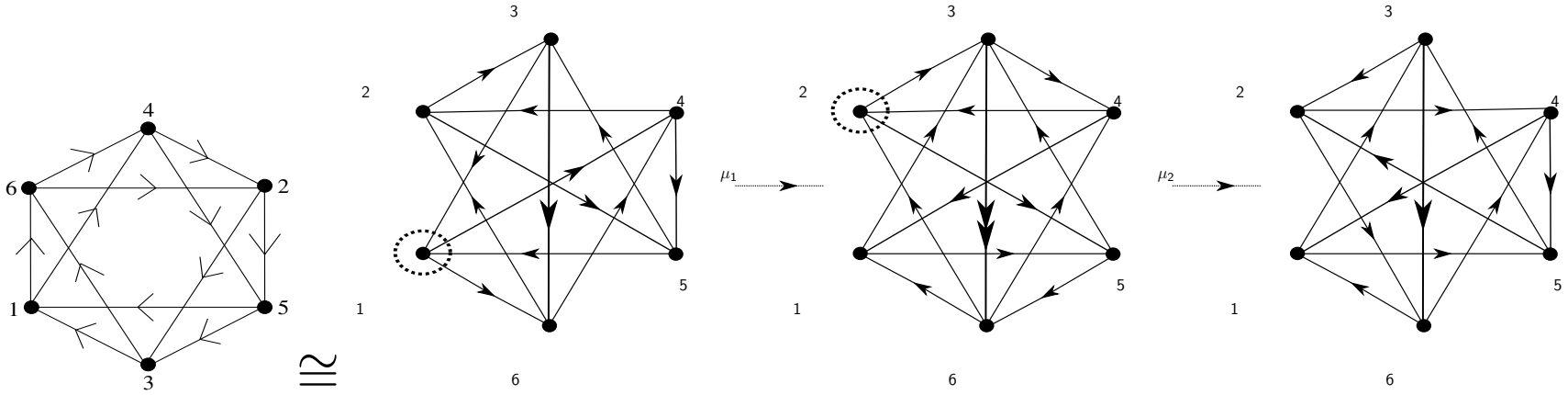

Point: Mutating once in the $Q_N^{(r,s)}$ case, or twice in the Q_{dP_3} case, yields a quiver with potential that is equivalent up to cyclic rotation.

Such quivers are called periodic in the Fordy-Marsh sense.

Cluster Variable Mutation

In addition to the mutation of quivers, there is also a complementary **cluster mutation** that can be defined.

Cluster mutation yields a sequence of Laurent polynomials in $\mathbb{Q}(x_1, x_2, \ldots, x_n)$ known as **cluster variables**.

Given a quiver Q (the potential is irrelevant here) and an initial cluster $\{x_1, \ldots, x_N\}$, then mutating at vertex 1 yields a new cluster variable x_{N+1}

$$\text{defined by} \qquad x_{N+1} = \left(\prod_{1 \to i \in Q} x_i + \prod_{i \to 1 \in Q} x_i \right) \Big/ x_1 .$$

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Example (dP_3): $x_{2n+7}x_{2n+1} = x_{2n+3}x_{2n+5} + x_{2n+4}x_{2n+6}$ and

$$x_{2n+8}x_{2n+2} = x_{2n+3}x_{2n+5} + x_{2n+4}x_{2n+6}.$$

Example (Gale-Robinson): $x_{n+N}x_n = x_{n+r}x_{n+N-r} + x_{n+s}x_{n+N-s}$.

Main Theorem (Jeong-M-Zhang)

For certain periodic quivers Q, which include the Gale-Robison quiver family, the dP_3 quiver, and some other 2-periodic quivers, we can use the Brane Tiling $\mathcal{T}_Q$ to obtain combinatorial formulas for an infinite sequence of cluster variables in $\mathcal{A}_Q$.

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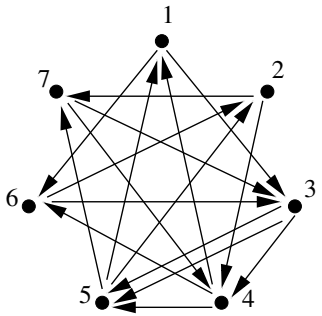
For certain periodic quivers Q, which include the Gale-Robison quiver family, the dP_3 quiver, and some other 2-periodic quivers, we can use the Brane Tiling $\mathcal{T}_Q$ to obtain combinatorial formulas for an infinite sequence of cluster variables in $\mathcal{A}_Q$.

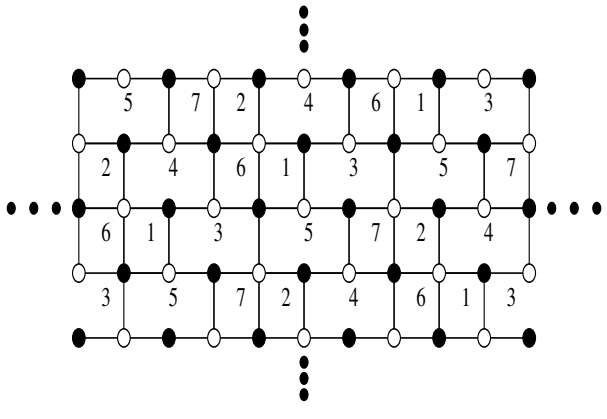
$$\text{For } n > N, \quad x_n = cm(G_n) \sum_{M = \text{ perfect matching of } G_n} x(M)y(M), \text{ where}$$

$\{G_n : n > N\}$'s are a collection of subgraphs of $\mathcal{T}_Q$, $x(M) = \prod_{\text{edge } e \in M} \frac{1}{x_i x_j}$ (for edge e straddling faces i and j), $y(M) =$ height of M (recording what faces need to be twisted to obtain matching M starting from the minimal matching, and $cm(G_n) =$ the covering monomial of the graph G_n (which records what face labels are contained in G_n and along its boundary).

Remark: This weighting scheme is a reformulation of schemes appearing in works of Speyer ("Octahedron Recurrence") and Goncharov-Kenyon.

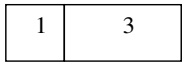
Gale-Robinson Example ($Q_7^{(2,3)}$, Mutating $1, 2, \ldots, 7, \ldots$)

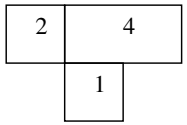
$Q =$ 

$\mathcal{T}_Q =$ 

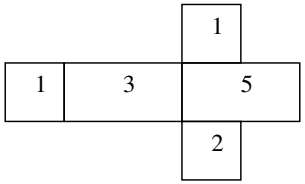
$x_8 \leftrightarrow$ 

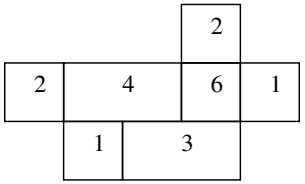
, $x_9 \leftrightarrow$ 

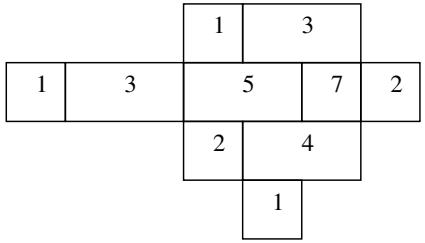
, $x_{10} \leftrightarrow$ 

, $x_{11} \leftrightarrow$ 

,

$x_{12} \leftrightarrow$ 

, $x_{13} \leftrightarrow$ 

, $x_{14} \leftrightarrow$ 

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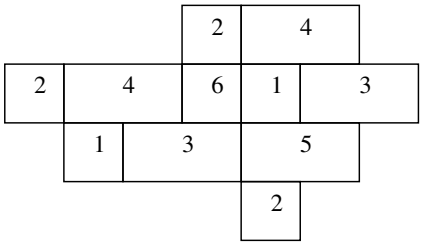
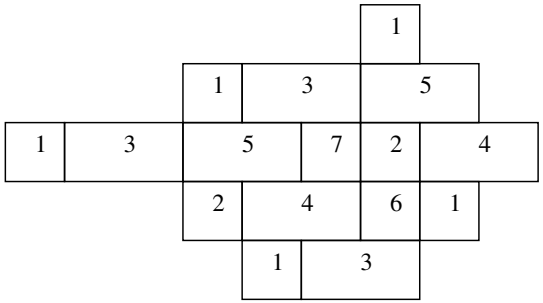
$x_{15} \leftrightarrow$ 

, $x_{16} \leftrightarrow$ 

, $\ldots$

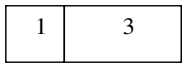
Gale-Robinson Example ($Q_7^{(2,3)}$, Mutating $1, 2, \ldots, 7, \ldots$)

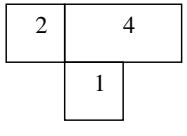
Obtain **pinecone graphs** from Bousquet-Mélou, Propp, and West in terms of Brane Tilings Terminology.

Furthermore, to get cluster variable formulas with coefficients, need only use weights (Goncharov-Kenyon, Speyer) and heights (Kenyon-Propp-...)

$x_8 \leftrightarrow$ 

, $x_9 \leftrightarrow$ 

, $x_{10} \leftrightarrow$ 

, $x_{11} \leftrightarrow$ 

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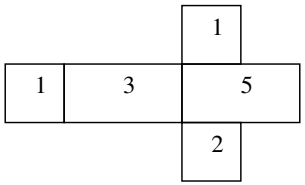
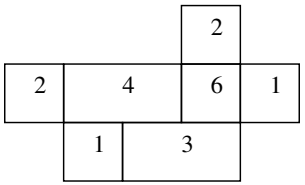
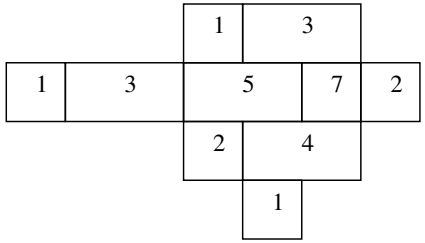
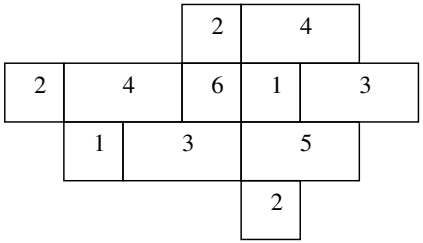
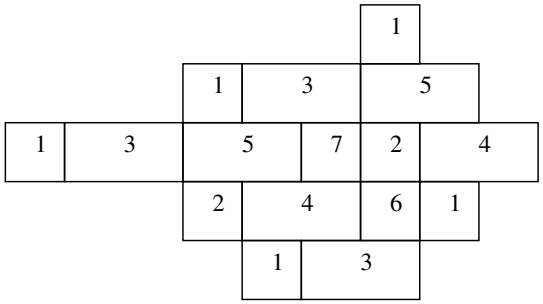
$x_{12} \leftrightarrow$ 

, $x_{13} \leftrightarrow$ 

, $x_{14} \leftrightarrow$ 

,

$x_{15} \leftrightarrow$ 

, $x_{16} \leftrightarrow$ 

, ...

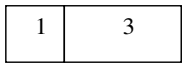
Gale-Robinson Example ($Q_7^{(2,3)}$, Mutating $1, 2, \ldots, 7, \ldots$)

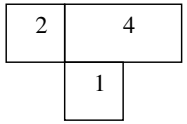
Similar connections (without principal coefficients) also observed in "Brane tilings and non-commutative geometry" by Richard Eager.

Eager uses physics terminology where he looks at $Y^{p,q}$ and $L^{a,b,c}$ quiver gauge theories, and their periodic Seiberg duality (i.e. quiver mutations).

$x_8 \leftrightarrow$ 

, $x_9 \leftrightarrow$ 

, $x_{10} \leftrightarrow$ 

, $x_{11} \leftrightarrow$ 

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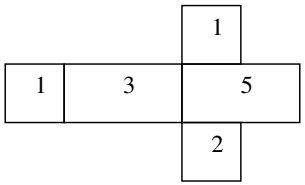
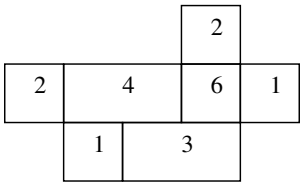
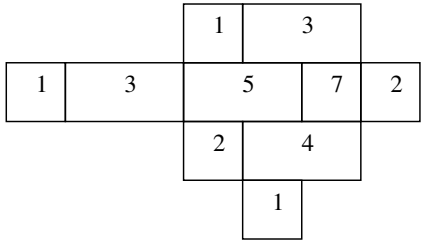
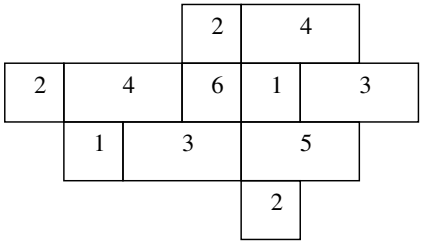
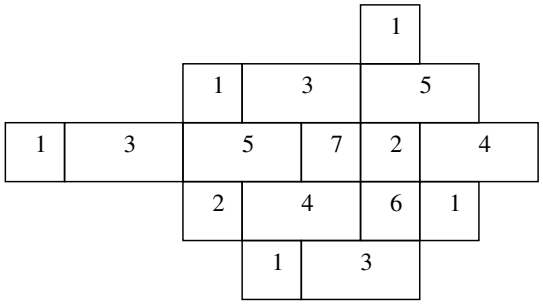
$x_{12} \leftrightarrow$ 

, $x_{13} \leftrightarrow$ 

, $x_{14} \leftrightarrow$ 

,

$x_{15} \leftrightarrow$ 

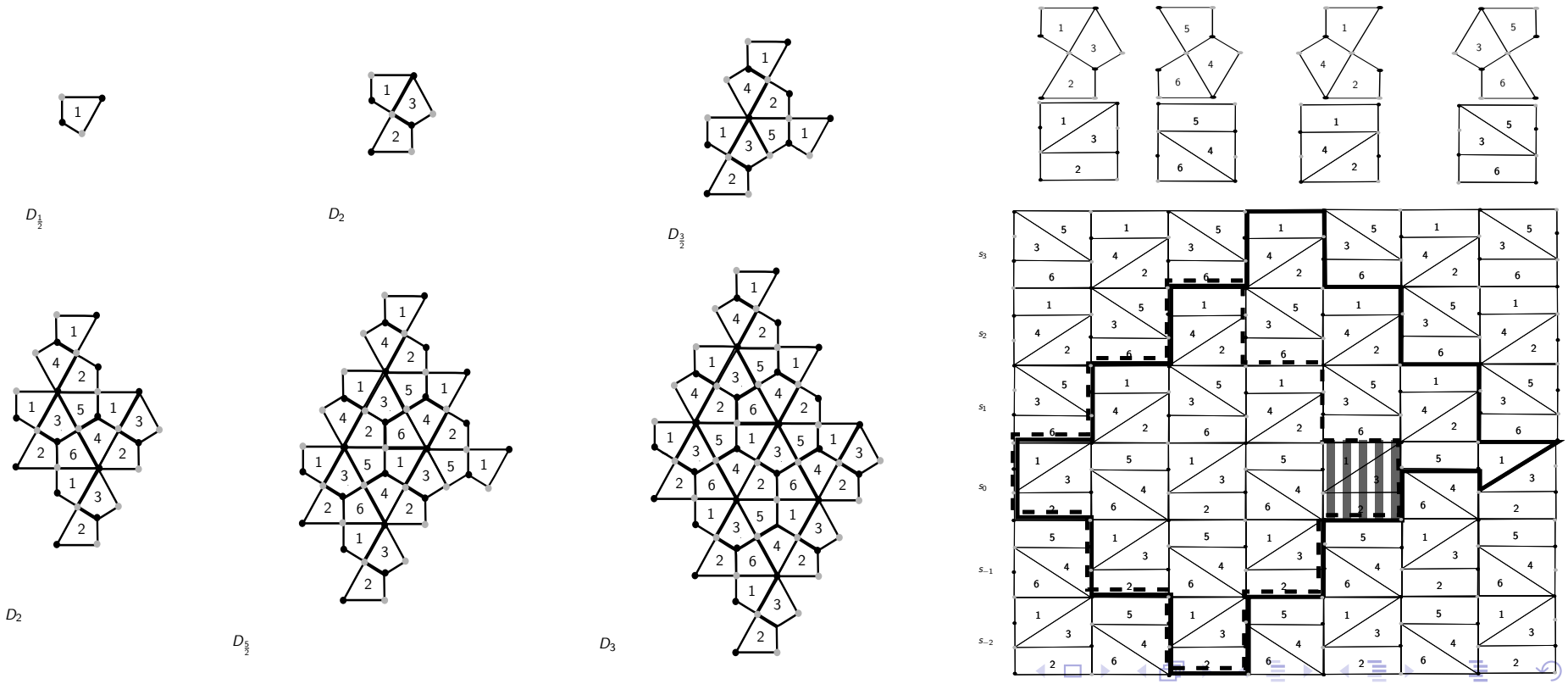
, $x_{16} \leftrightarrow$ 

, ...

dP_3 Example (Mutating $1, 2, 3, 4, 5, 6, 1, 2, \ldots$)

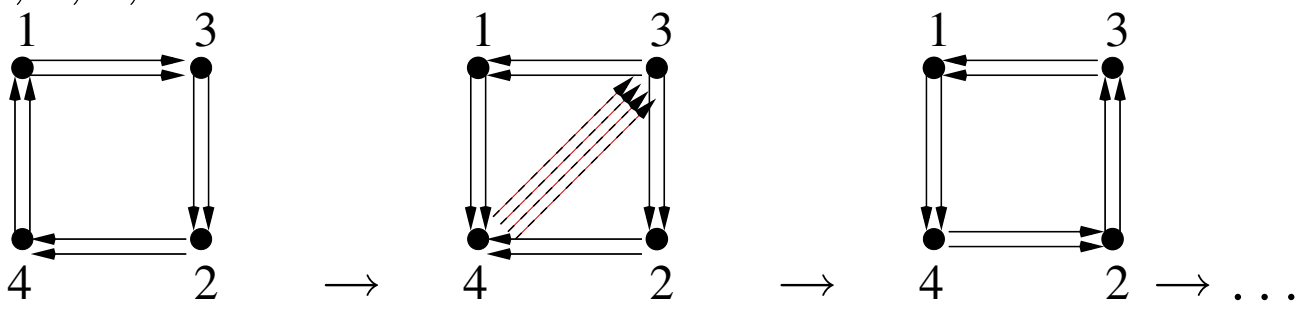
These subgraphs appear in work by Cottrell-Young and a subsequence of them appear in M. Ciucu's work "Perfect matchings and perfect powers", where they are called **Aztec Dragons**.

More on Aztec Dragons and the dP_3 lattice shortly.



Hirzebruch Quiver F_0: **Aztec Diamonds** and Fortresses

The quiver Q below is 2-periodic, as illustrated by mutating in order $1, 2, 3, 4, 1, 2, \ldots$

$Q =$ 

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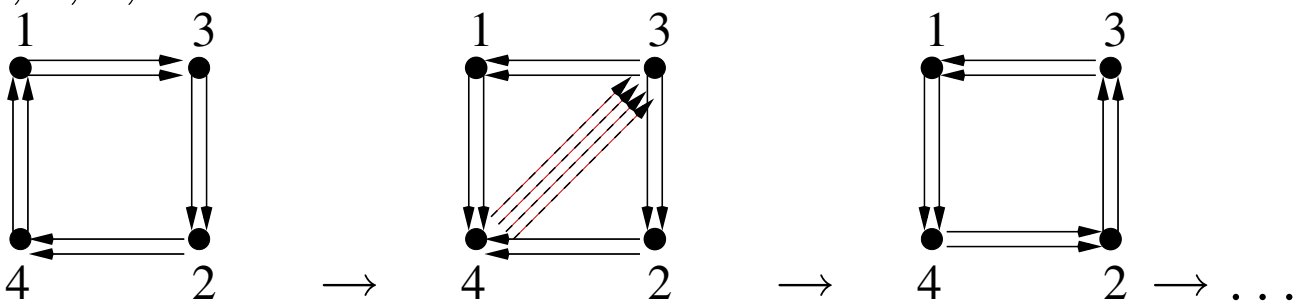
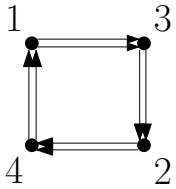
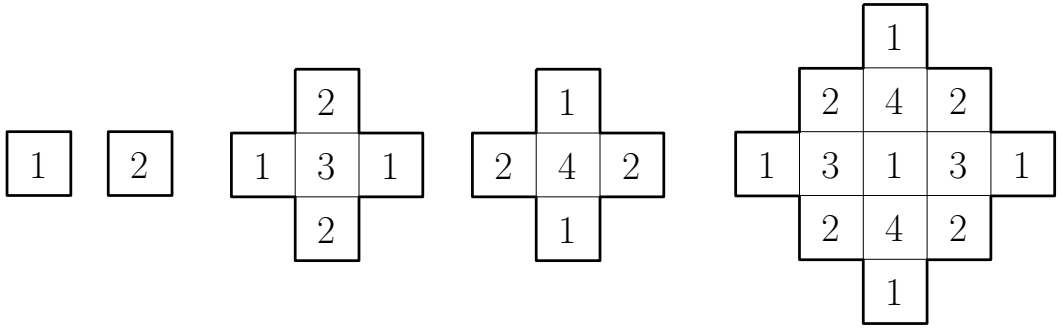
$Q =$ 

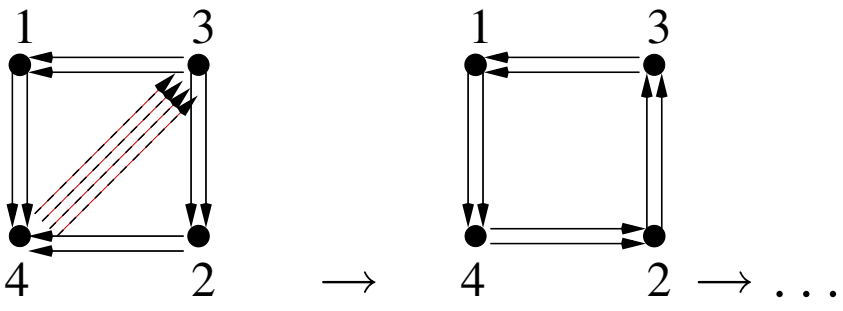
2	4	2	4	2	4	2
3	1	3	1	3	1	3
2	4	2	4	2	4	2
3	1	3	1	3	1	3
2	4	2	4	2	4	2
3	1	3	1	3	1	3
2	4	2	4	2	4	2





Hirzebruch Quiver F_0: Aztec Diamonds and **Fortresses**

The quiver Q' below is 2-periodic, as illustrated by mutating in order $1, 2, 3, 4, 1, 2, \ldots$

$Q' =$ 

$\ldots$

Hirzebruch Quiver F_0: Aztec Diamonds and **Fortresses**

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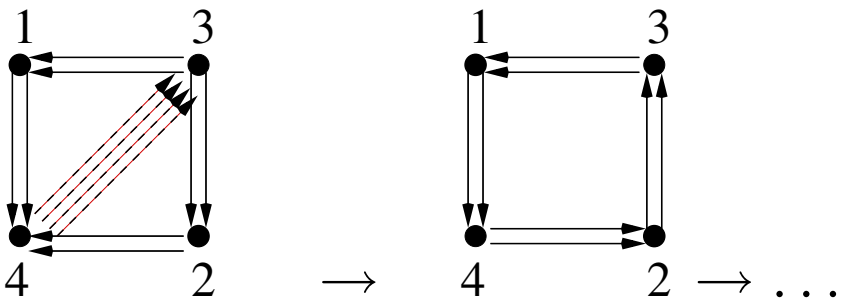
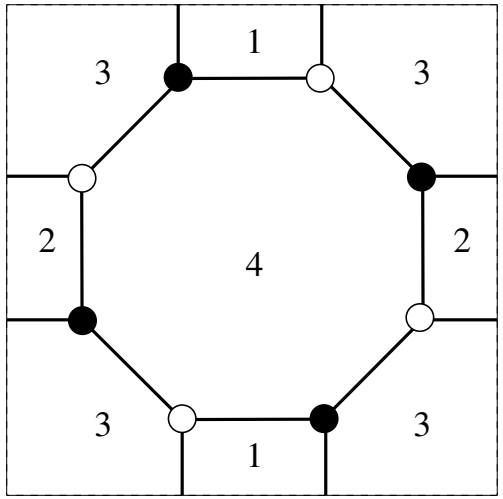
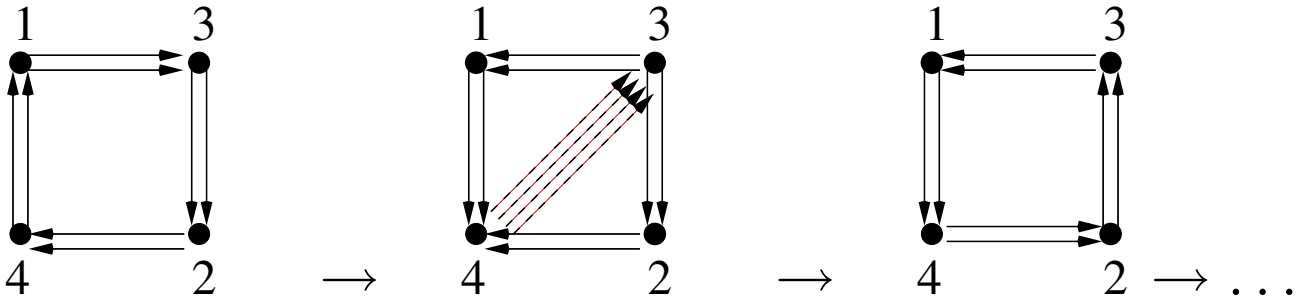
$Q' =$ 



Fortresses from M. Ciucu's work "Perfect matchings and perfect powers".

Non-periodic mutation sequences in F_0

$Q =$ 

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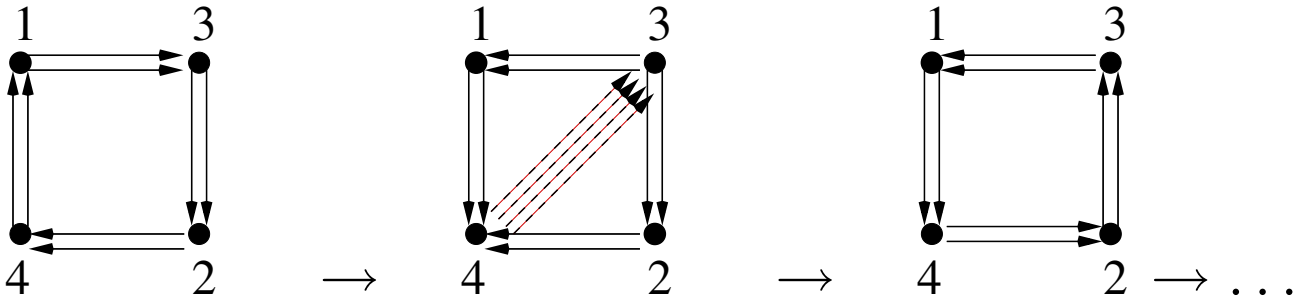
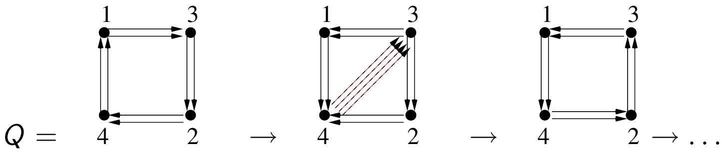
$Q =$ 

If we instead mutate by $1, 2, 3, 2, 3, \ldots$ or $1, 3, 2, 3, 2, \ldots$, we obtain quivers where the number of arrows grows without bound.

Non-periodic mutation sequences in F_0



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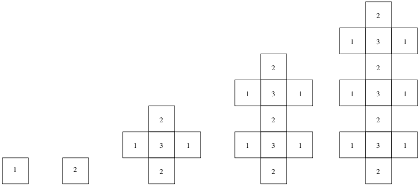
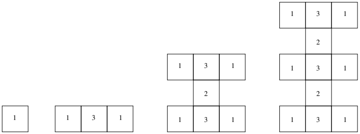
Nonetheless, a combinatorial interpretation for the cluster variables is

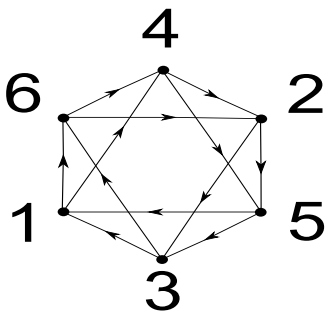


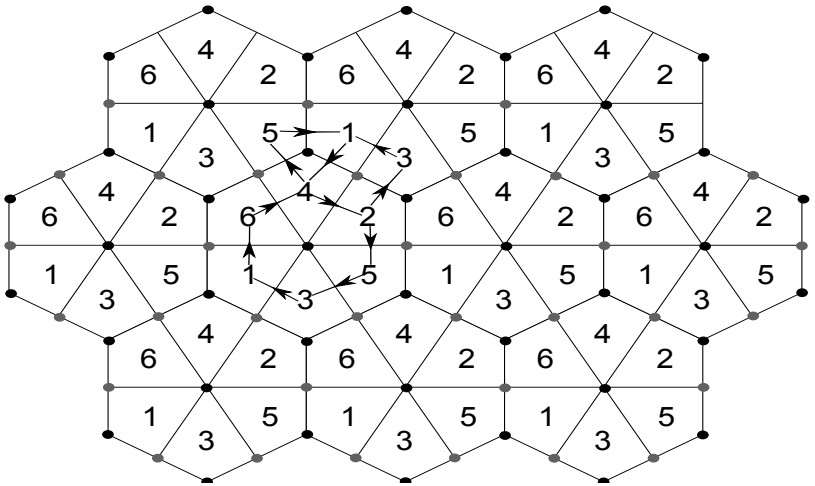
. . . or



. . . .

Non-periodic mutation sequences in the dP_3 Lattice

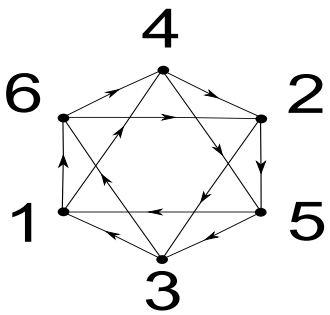


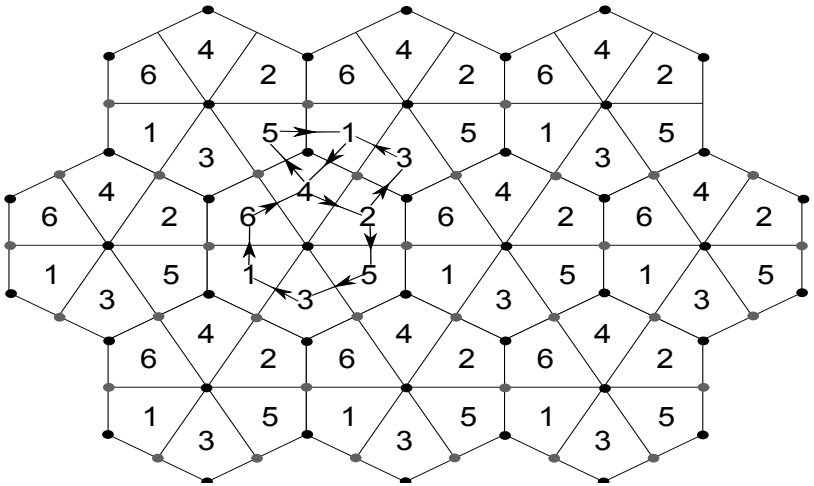


Mutating at a vertex of dP_3 and its antipode commute so a mutation such as $1, 2, 1, 2$ can be reordered to $1, 1, 2, 2$, and hence is the identity.

Letting $\tau_1 = \mu_1 \circ \mu_2$, $\tau_2 = \mu_3 \circ \mu_4$, $\tau_3 = \mu_5 \circ \mu_6$, the above can be written as $\tau_1^2 = \tau_2^2 = \tau_3^2 = 1$.

Non-periodic mutation sequences in the dP_3 Lattice





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As discussed with Pylyavskyy, we see the further relations $(\tau_1\tau_2)^3 = (\tau_1\tau_3)^3 = (\tau_2\tau_3)^3 = 1$, and it can be shown that there are no other relations. Thus $\langle \tau_1, \tau_2, \tau_3 \rangle \cong \tilde{A}_2$.

τ-Mutation-Sequences

We call a mutation sequence built out of a concatenation of τ_1, τ_2, and τ_3 a τ-**mutation sequence**.

As an example, the periodic sequences $1, 2, 3, 4, 5, 6, 1, 2, \ldots$ yielding the Aztec Dragons, were examples of τ-mutation sequences given by $\tau_1, \tau_2, \tau_3, \tau_1, \ldots$.

Up to the $\tilde{A}_2$ relations and relabeling vertices, other τ-mutation sequences do not necessarily give new cluster variables.

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First non-trivial τ-mutation sequence: $\tau_1, \tau_2, \tau_1, \tau_3 = 1, 2, 3, 4, 1, 2, 5, 6$ yields the following as the last cluster variable:

$$\begin{aligned} & (x_1x_2^2x_3^3x_5^4 + x_2^3x_3^2x_4x_5^4 + 2x_1^2x_2x_3^3x_5^3x_6 + 4x_1x_2^2x_3^2x_4x_5^3x_6 + 2x_2^3x_3x_4^2x_5^3x_6 + x_1^3x_3^3x_5^2x_6^2 \\ + \quad & 5x_1^2x_2x_3^2x_4x_5^2x_6^2 + 5x_1x_2^2x_3x_4^2x_5^2x_6^2 + x_2^3x_4^3x_5^2x_6^2 + 2x_1^3x_3^2x_4x_5x_6^3 + 4x_1^2x_2x_3x_4^2x_5x_6^3 \\ + \quad & 2x_1x_2^2x_4^3x_5x_6^3 + x_1^3x_3x_4^2x_6^4 + x_1^2x_2x_4^3x_6^4)/x_1^2x_2^2x_3^2x_4^2x_6 \end{aligned}$$

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What is a combinatorial interpretation of this Laurent polynomial?

Aztec Castles

Investigated this non-periodic mutation sequences in the dP_3 Lattice with UMN 2013 REU Students Megan Leoni, Seth Neel, and Paxton Turner.

Aztec Castles

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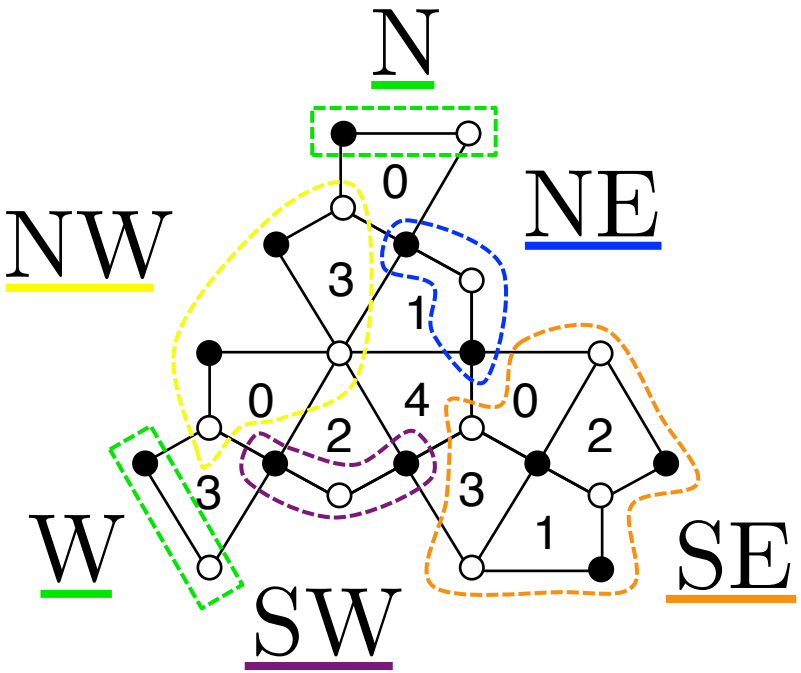
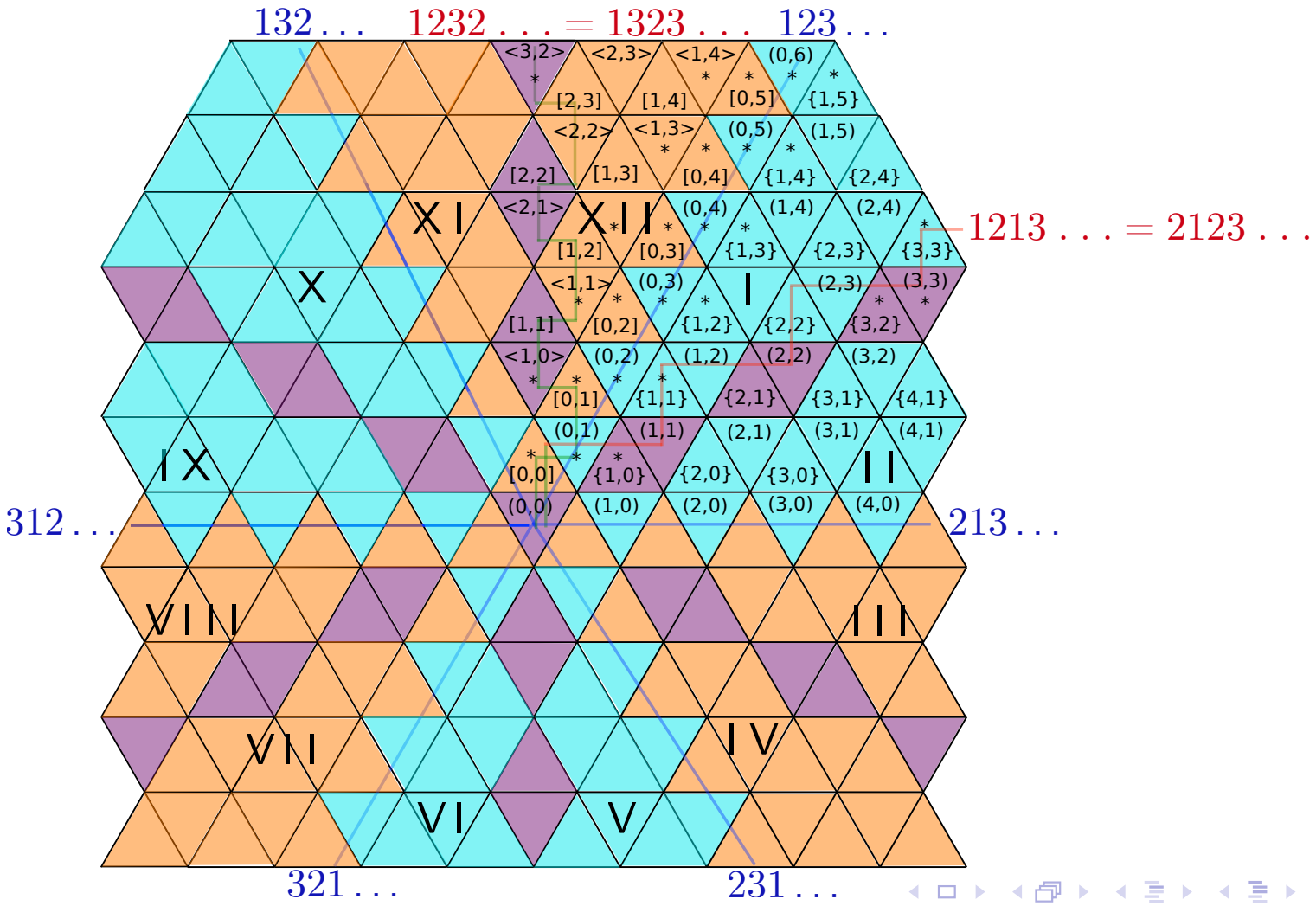
Developed a two-parameter family of graphs, **Aztec Castles**, to encode cluster variables arising from τ-mutation sequences.



Example: $x_{\tau_1,\tau_2,\tau_1,\tau_3}$ corresponds to .
(Up to a shift in indexing on faces by one.)

Aztec Castles (cont.)

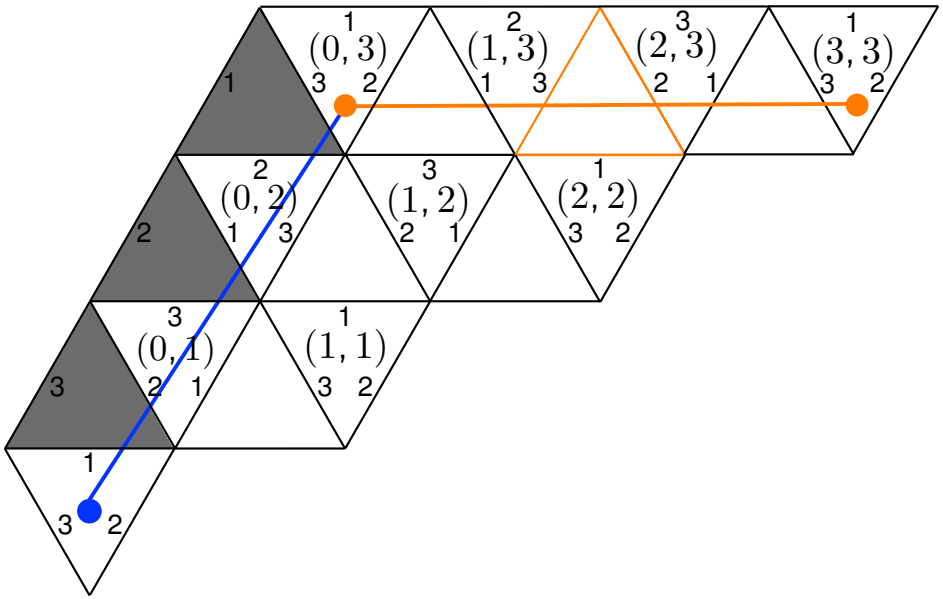
For a general τ-mutation sequence, the last cluster variable associated to an alcove in the $\tilde{A}_2$ Coxeter Lattice, indexed by (i,j), $[i,j]$, $\langle i,j\rangle$ or $\{i,j\}$.

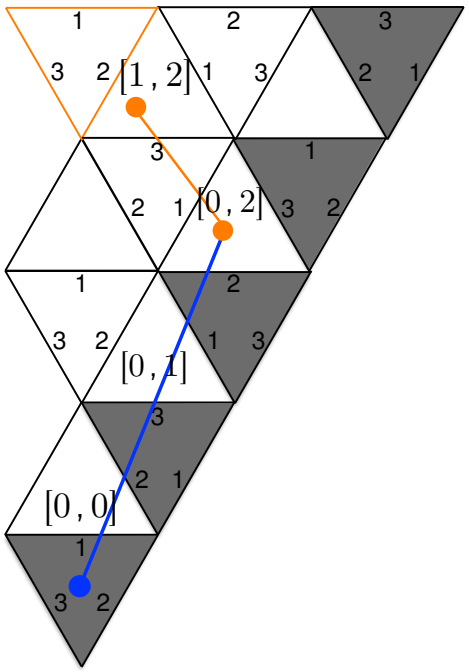


Aztec Castles (cont.)

For a general τ-mutation sequence, the last cluster variable associated to an alcove in the $\tilde{A}_2$ Coxeter Lattice, indexed by (i,j), $[i,j]$, $\langle i,j \rangle$ or $\{i,j\}$.

More precisely, we cut the lattice into twelve regions (cones) and by symmetry it suffices to describe the families of graphs for two adjacent cones.



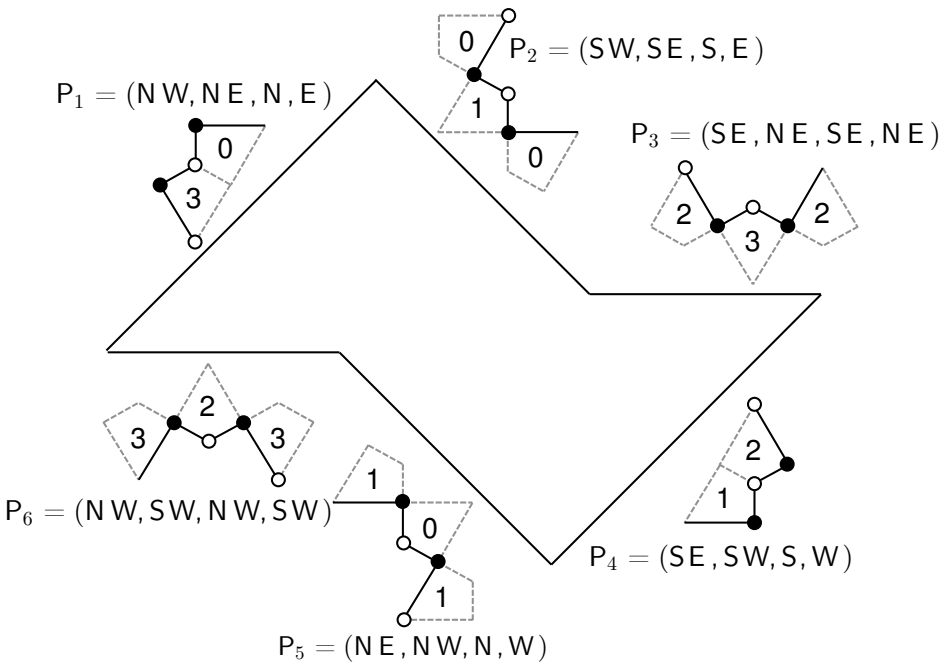


Aztec Castles in Cone I (Integer Cone)

For a general τ-mutation sequence, the last cluster variable associated to an alcove in the $\tilde{A}_2$ Coxeter Lattice, indexed by (i,j), $[i,j]$, $\langle i,j \rangle$ or $\{i,j\}$.

For (i,j), we let γ_i^j be the Aztec Castle defined by the subgraph cut out by

$$P_1^{i+j} P_2^j P_3^{i-1} P_4^{i+j-1} P_5^{j-1} P_6^i \quad \text{in}$$



.

Example $(i = 3, j = 3)$

For a general τ-mutation sequence, the last cluster variable associated to an alcove in the $\tilde{A}_2$ Coxeter Lattice, indexed by (i,j), $[i,j]$, $\langle i,j \rangle$ or $\{i,j\}$.

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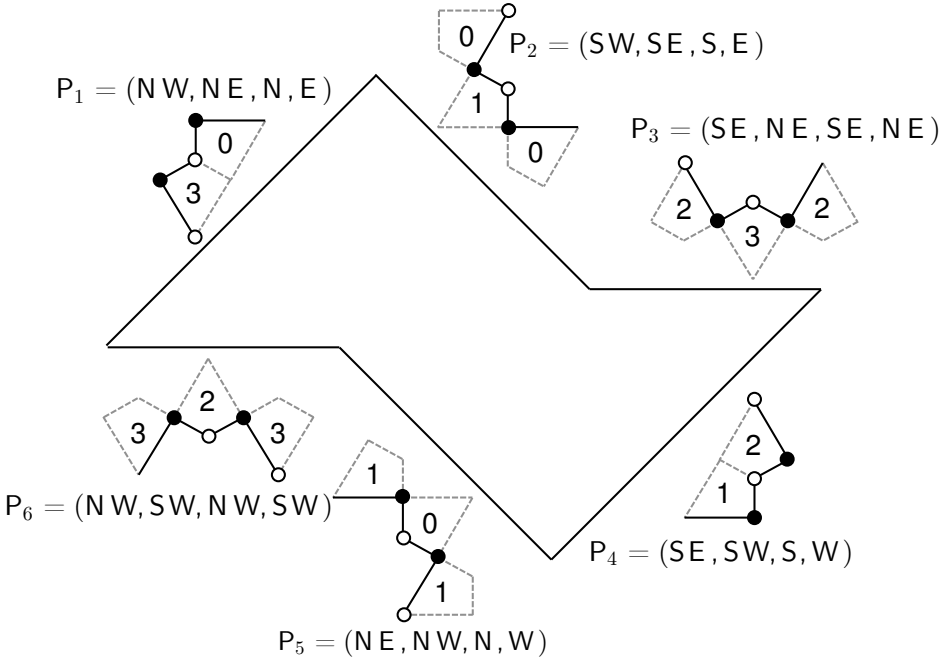
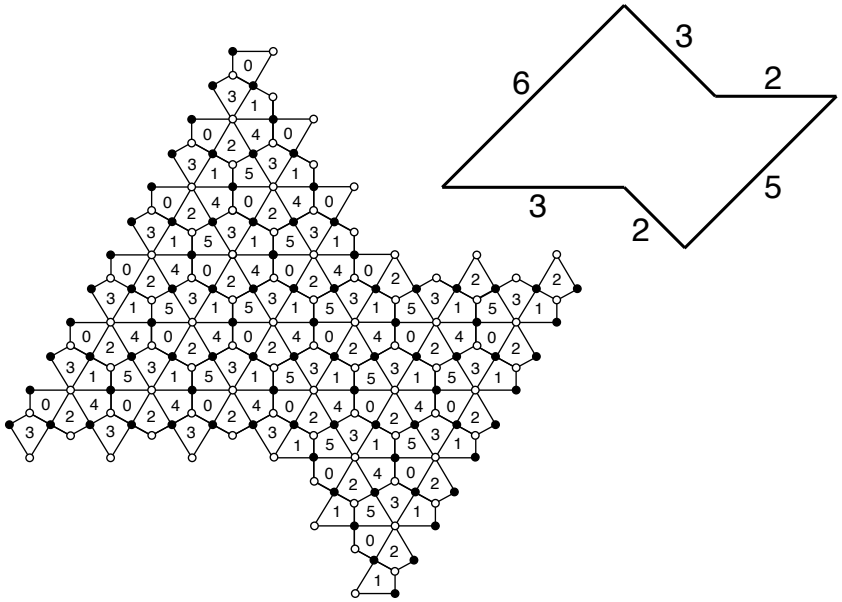


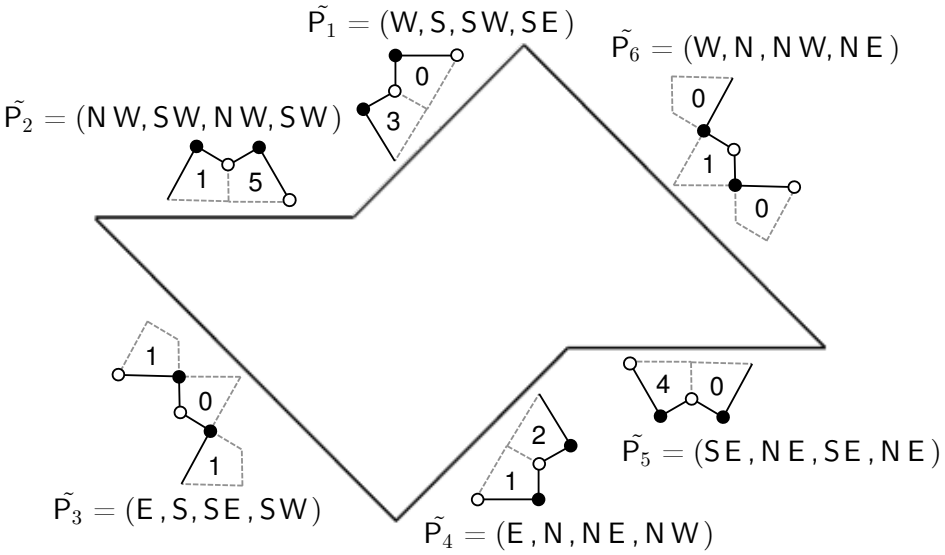


Aztec Castles in Cone XII (Half-Integer Cone)

For a general τ-mutation sequence, the last cluster variable associated to an alcove in the $\tilde{A}_2$ Coxeter Lattice, indexed by (i,j), $[i,j]$, $\langle i,j \rangle$ or $\{i,j\}$.

For $[i,j]$, we let $\tilde{\gamma}_i^j$ be the Aztec Castle defined by the subgraph cut out by

$$\tilde{P}_1^j \tilde{P}_2^{i+1} \tilde{P}_3^{i+j} \tilde{P}_4^{j+1} \tilde{P}_5^i \tilde{P}_6^{i+j-1} \quad \text{in}$$



Example $[i = 2, j = 2]$

For a general τ-mutation sequence, the last cluster variable associated to an alcove in the $\tilde{A}_2$ Coxeter Lattice, indexed by (i, j), $[i, j]$, $\langle i, j \rangle$ or $\{i, j\}$.

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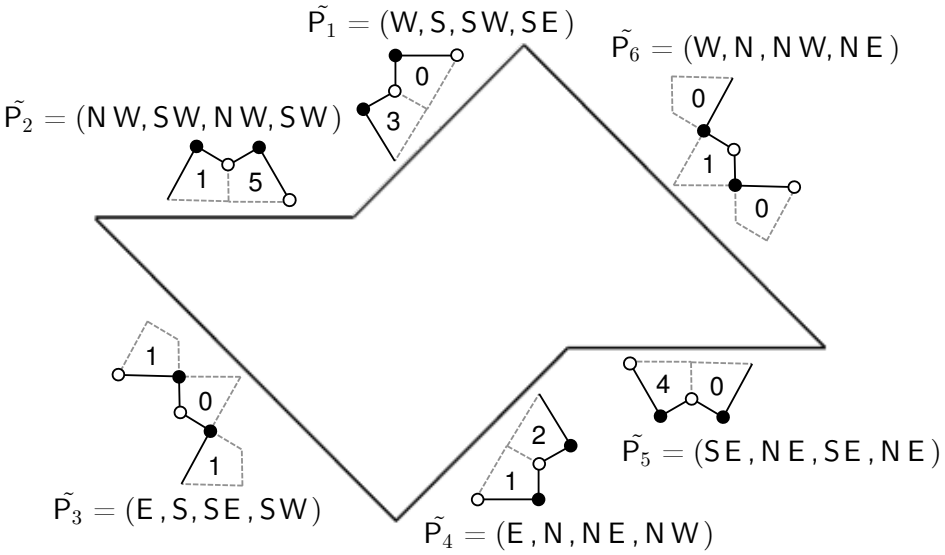
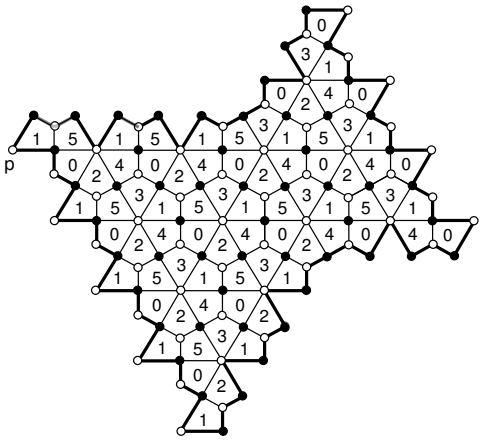
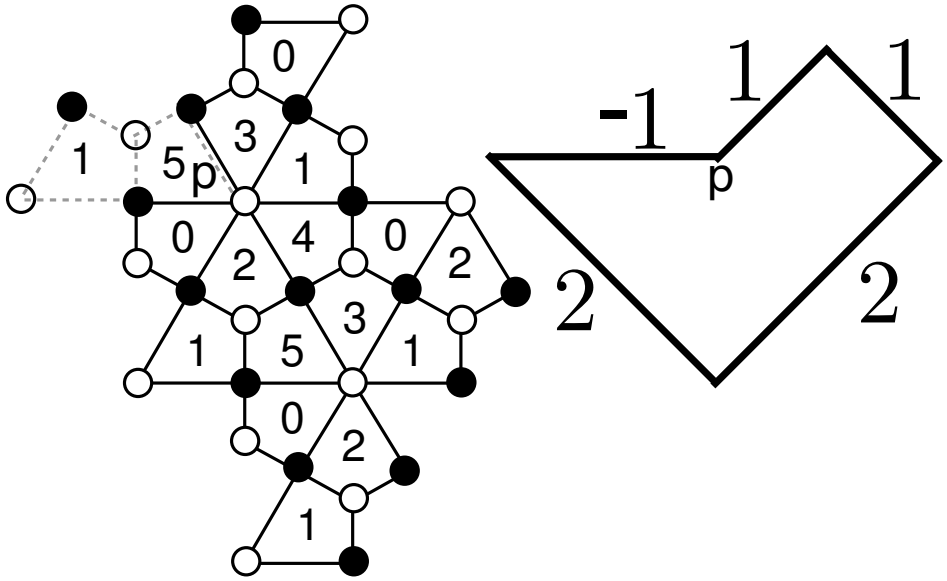




Aztec Dragons as Special Case of Castles

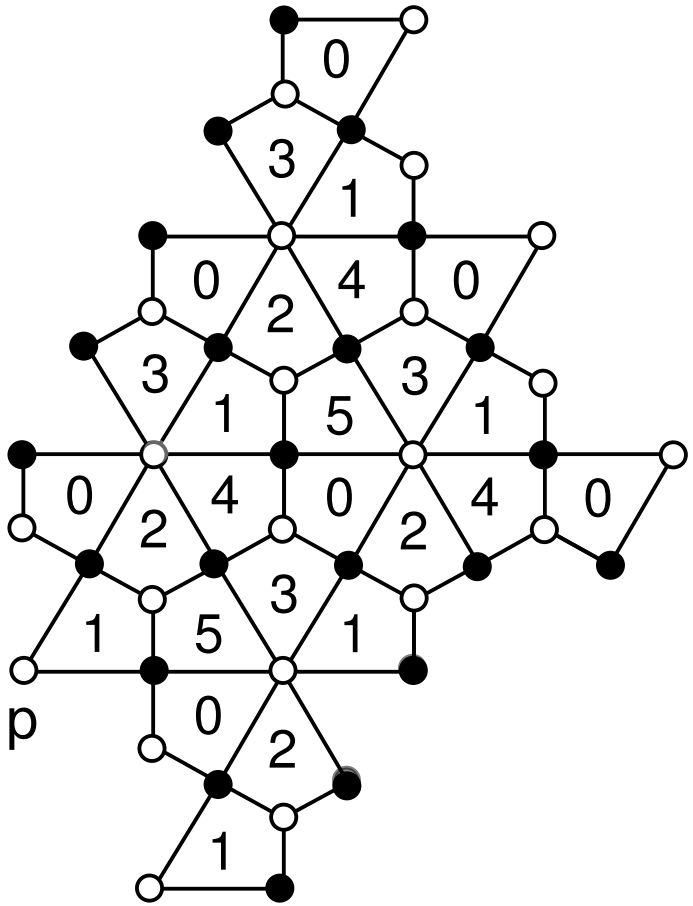
$D_n = \sigma\gamma_0^n = P_1^{n-1} P_2^{n-1} P_3^0 P_4^n P_5^n P_6^{-1}$ (σ is reflection by 180°)



(Example: $D_2 = \sigma\gamma_0^2$)

Aztec Dragons as Special Case of Castles

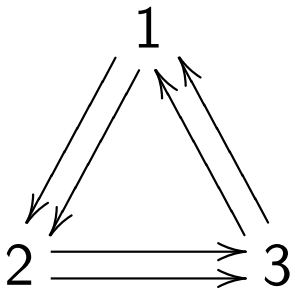
Similarly, $D_{n+1/2} = \sigma\tilde{\gamma}_0^n = \tilde{P}_1^{n+1}\tilde{P}_2^0\tilde{P}_3^{n-1}\tilde{P}_4^n\tilde{P}_5^1\tilde{P}_6^n$ (from top white vertex)



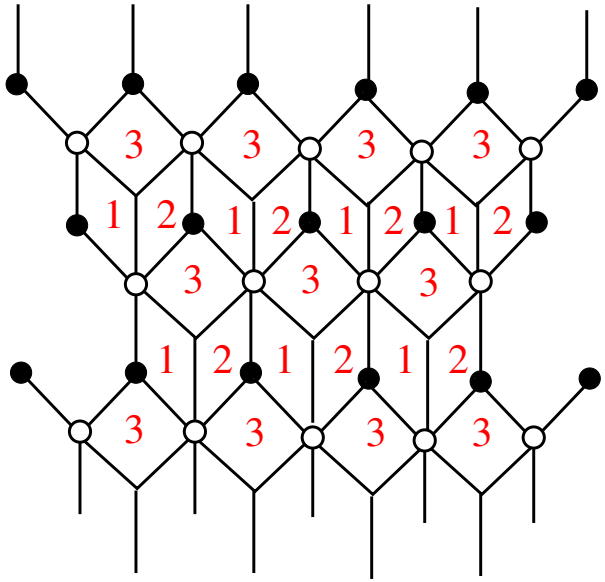
(Example: $D_{5/2} = \sigma\tilde{\gamma}_0^2$)

Markoff Quiver via Brane Tilings

If we unfold



we get



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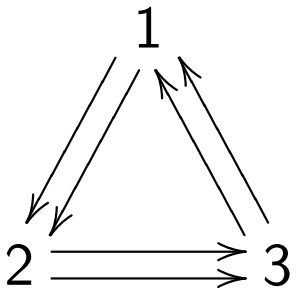
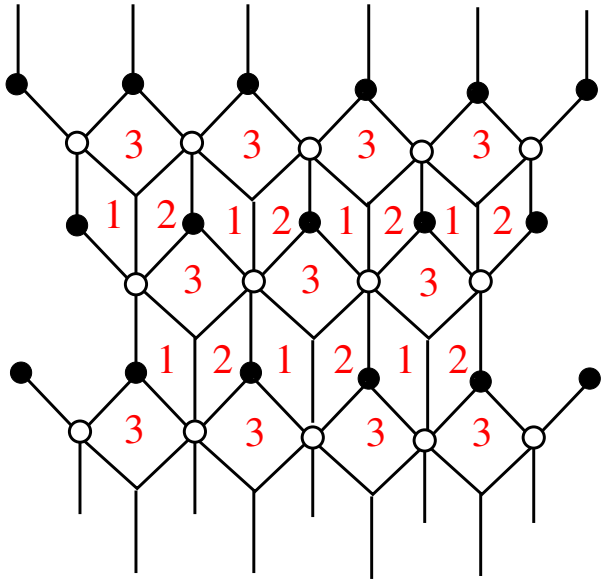
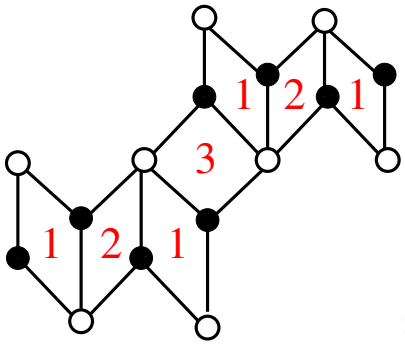


we get



. Subgraphs yield the snake graphs from earlier: e.g.



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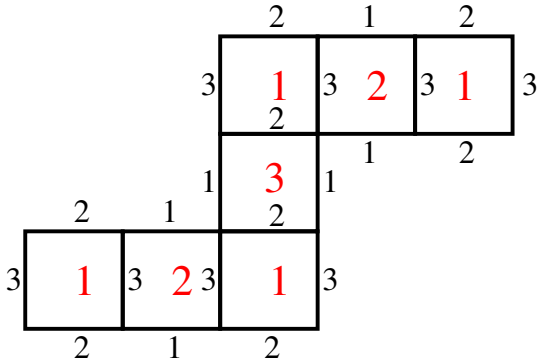

Open Problems and Work in Progress

- τ-mutation sequences and the periodic mutation sequences discussed are examples of **toric** mutation sequences. Mutation only performed at vertices with at most two incoming and two outgoing arrows.

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• Understand better the relationship between snake graphs corresponding to arcs in a surface and subgraphs of a brane tiling. (e.g. Markoff quiver)

Thank You For Listening

• *Aztec Castles and the dP3 Quiver* (with Megan Leoni, Seth Neel, and Paxton Turner, `arXiv:1308.3926`.

• *Positivity for Cluster Algebras from Surfaces* (with Ralf Schiffler and Lauren Williams), *Adv. Math.*, **227** (2011), Issue 6, 2241–2308 `arXiv:math.CO/0906.0748`

• In-Jee Jeong, *Bipartite Graphs, Quivers, and Cluster Variables*, REU Report, `http://www.math.umn.edu/~reiner/REU/Jeong2011.pdf`

• Sicong Zhang, *Cluster Variables and Perfect Matchings of Subgraphs of the dP_3 Lattice*, REU Report, `http://www.math.umn.edu/~reiner/REU/Zhang2012.pdf`

• *Gale-Robinson Sequences and Brane Tilings* (with In-Jee Jeong and and Sicong Zhang), *Discrete Mathematics and Theoretical Computer Science Proc.* **AS** (2013), 737-748. (Longer version in preparation.)

Slides Available at `http//math.umn.edu/~musiker/ICMS.pdf`