Aztec castles and other instances of brane tilings in combinatorics

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International Centre for Mathematical Sciences

Workshop on Gauge theories: quivers, tilings and Calabi-Yaus

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Outline

1. Introduction to Cluster Algebras
2. Markoff Quiver
3. Graphs from Cluster Algebras from Surfaces
4. Brane Tilings associated to the Del Pezzo 3 Quiver
5. Aztec Castles

http://math.umn.edu/~musiker/ICMS.pdf
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Cluster algebras are a certain class of commutative rings which have a distinguished set of generators that are grouped into overlapping subsets, called clusters, each having the same cardinality.
Definition (Sergey Fomin and Andrei Zelevinsky 2001) A cluster algebra $\mathcal{A}$ (of geometric type) is a subalgebra of $k(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m})$ constructed cluster by cluster by certain exchange relations.

Generators: Specify an initial finite set of them, a Cluster, $\{x_1, x_2, \ldots, x_{n+m}\}$.

Construct the rest via Binomial Exchange Relations:

$$x_\alpha x'_\alpha = \prod x_{d+i}^{\gamma_i} \prod x_{d-i}^{-\gamma_i}.$$  

The set of all such generators are known as Cluster Variables, and the initial pattern of exchange relations (described as a valued quiver, i.e. a directed graph) determines the Seed.

Relations: Induced by the Binomial Exchange Relations.
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**Relations:**

Induced by the Binomial Exchange Relations.
Example: Rank 2 Cluster Algebras

Let $B = \begin{bmatrix} 0 & b \\ -c & 0 \end{bmatrix}$, $b, c \in \mathbb{Z}_{>0}$. ($\{x_1, x_2\}, B$) is a seed for a cluster algebra $A(b, c)$ of rank 2.
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Let \( B = \begin{bmatrix} 0 & b \\ -c & 0 \end{bmatrix} \), \( b, c \in \mathbb{Z}_{>0} \). \( \{x_1, x_2\}, B \) is a seed for a cluster algebra \( \mathcal{A}(b, c) \) of rank 2. (E.g. when \( b = c \), \( B = B(Q) \) where \( Q \) is a 2-vertex quiver with \( b \) arrows from \( v_1 \to v_2 \).)

\[
\mu_1(B) = \mu_2(B) = -B \quad \text{and} \quad x_1x'_1 = x_2^c + 1, \quad x_2x'_2 = 1 + x_1^b.
\]

Thus the cluster variables in this case are

\[
\{x_n : n \in \mathbb{Z}\} \quad \text{satisfying} \quad x_nx_{n-2} = \begin{cases} x_{n-1}^b + 1 & \text{if } n \text{ is odd} \\ x_{n-1}^c + 1 & \text{if } n \text{ is even} \end{cases}.
\]
Example: Rank 2 Cluster Algebras

Let \( B = \begin{bmatrix} 0 & b \\ -c & 0 \end{bmatrix} \), \( b, c \in \mathbb{Z}_{>0} \). \( (\{x_1, x_2\}, B) \) is a seed for a cluster algebra \( \mathcal{A}(b, c) \) of rank 2. (E.g. when \( b = c \), \( B = B(Q) \) where \( Q \) is a 2-vertex quiver with \( b \) arrows from \( v_1 \to v_2 \).)

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    x_{n-1}^b + 1 & \text{if } n \text{ is odd} \\
    x_{n-1}^c + 1 & \text{if } n \text{ is even}
\end{cases}.
\]

Example (\( b = c = 1 \)): (Finite Type, of Type \( A_2 \))

\[
\begin{align*}
  x_3 &= \frac{x_2 + 1}{x_1}, &
  x_4 &= \frac{x_3 + 1}{x_2} &= \frac{x_2 + 1}{x_1} + 1 = \frac{x_1 + x_2 + 1}{x_1 x_2}, \\
  x_5 &= \frac{x_4 + 1}{x_3} = \frac{x_1 + x_2 + 1}{(x_2 + 1)/x_1} = \frac{x_1(x_1 + x_2 + 1 + x_1 x_2)}{x_1 x_2(x_2 + 1)} = \frac{x_1 + 1}{x_2}. &
  x_6 = x_1.
\end{align*}
\]
Example: Rank 2 Cluster Algebras

Example \((b = c = 2)\): \((\text{Affine Type, of Type } \tilde{A}_1)\)

\[
x_3 = \frac{x_2^2 + 1}{x_1}, \quad x_4 = \frac{x_3^2 + 1}{x_2} = \frac{x_2^4 + 2x_2^2 + 1 + x_1^2}{x_1^2 x_2}.
\]

\[
x_5 = \frac{x_4^2 + 1}{x_3} = \frac{x_2^6 + 3x_2^4 + 3x_2^2 + 1 + x_1^4 + 2x_1^2 + 2x_1^2 x_2^2}{x_1^3 x_2^2}, \quad \ldots
\]

If we let \(x_1 = x_2 = 1\), we obtain \(\{x_3, x_4, x_5, x_6\} = \{2, 5, 13, 34\}\).
Example (\(b = c = 2\)): (Affine Type, of Type \(\tilde{A}_1\))

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If we let \(x_1 = x_2 = 1\), we obtain \(\{x_3, x_4, x_5, x_6\} = \{2, 5, 13, 34\}\).

The next number in the sequence is \(x_7 = \frac{34^2 + 1}{13} = \ldots\)
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The next number in the sequence is \(x_7 = \frac{34^2 + 1}{13} = \frac{1157}{13} = \)
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\]

If we let $x_1 = x_2 = 1$, we obtain $\{x_3, x_4, x_5, x_6\} = \{2, 5, 13, 34\}$.

The next number in the sequence is $x_7 = \frac{34^2 + 1}{13} = \frac{1157}{13} = 89$, an integer!
Example: The Markoff Quiver and Cluster Algebra

Consider the cyclic quiver

\[
\begin{array}{ccc}
\rightarrow & \rightarrow & \rightarrow \\
1 & 2 & 3 \\
\rightarrow & \rightarrow & \\
\end{array}
\]

Let \( \{x, y, z\} \) be a seed in this cluster algebra.

Mutation by \( z \) leads to \( \{x, y, x^2 + y^2 + z^2\} \) and all mutations in this cluster algebra have the same form.

We will return to this example later.

Called the Markoff quiver since starting with the seed \( \{1, 1, 1\} \), every seed mutation-equivalent is a Markoff triple (i.e. satisfying \( 3xyz = x^2 + y^2 + z^2 \)).

\( \{1, 2, 5\}, \{1, 5, 13\}, \{2, 5, 29\}, \{2, 29, 169\}, \{13, 34, 1325\}, ... \)
Consider the cyclic quiver \[ 1 \xrightarrow{\rightarrow} 3 \xrightarrow{\rightarrow} 2 \xrightarrow{\rightarrow} 1 \].

Let \( \{x, y, z\} \) be a seed in this cluster algebra. **Mutation** by \( z \) leads to \( \{x, y, \frac{x^2+y^2}{z}\} \) and all mutations in this cluster algebra have the **same form**.

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Example: The Markoff Quiver and Cluster Algebra

Consider the cyclic quiver 1

\[ \begin{array}{ccc}
1 & \rightarrow & 2 \\
\downarrow & & \downarrow \\
3 & \rightarrow & 1 \\
\end{array} \]

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\[ \{1, 2, 5\}, \quad \{1, 5, 13\}, \quad \{2, 5, 29\}, \quad \{2, 29, 169\}, \quad \{13, 34, 1325\}, \ldots \]
Theorem (Fomin-Shapiro-Thurston 2006), (Based on earlier work of Fock-Goncharov and Gekhtman-Shapiro-Vainshtein):

Given a Riemann surface with marked points \((S, M)\), one can define a corresponding cluster algebra \(A(S, M)\).

Seed ↔ Triangulation \(T = \{\tau_1, \tau_2, \ldots, \tau_n\}\)

Cluster Variable ↔ Arc \(\gamma (x_i \leftrightarrow \tau_i \in T)\)

Cluster Mutation (Binomial Exchange Relations) ↔ Flipping Diagonals.
Example: Cluster Algebras from Surfaces

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Given a Riemann surface with marked points \((S, M)\), one can define a corresponding cluster algebra \(\mathcal{A}(S, M)\).

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Cluster Mutation (Binomial Exchange Relations) ↔ Flipping Diagonals.

**Theorem.** (M-Schiffler-Williams 2009) Let \(\mathcal{A}(S, M)\) be any cluster algebra arising from a surface (with or without punctures), where the coefficient system is of geometric type, and let \(\Sigma\) be any initial seed.

Then the Laurent expansion of every cluster variable with respect to the seed \(\Sigma\) has non-negative coefficients. Proof via explicit combinatorial formulas in terms of graphs.
Example of Hexagon (Type $A_3$ or $\mathbb{C}[Gr_{2,6}]$)

Consider the triangulated hexagon $(S, M)$ by the triangulation $T$.

\[ x_1 x'_1 = (x_7 x_9) + x_2(x_8) \]
Example of Hexagon (Type $A_3$ or $\mathbb{C}[Gr_{2,6}]$)

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\[
x_1 x'_1 = (x_7 x_9) + x_2 (x_8)
\]
\[
x_2 x''_2 = x_3 (x_9) + x'_1 (x_4)
\]
Example of Hexagon (Type $A_3$ or $\mathbb{C}[Gr_{2,6}]$)

Consider the triangulated hexagon $(S, M)$ by the triangulation $T$. 

$$
\begin{align*}
\tau_1 & = (x_7 x_9) + x_2(x_8) \\
\tau_2 & = x_3(x_9) + x_1'(x_4) \\
\tau_3 & = x_2''(x_6) + x_1'(x_5)
\end{align*}
$$
Example of Hexagon (Type $A_3$ or $\mathbb{C}[Gr_{2,6}]$)

By using the Ptolemy exchange relations on $\tau_1$, $\tau_2$, then $\tau_3$, we obtain

$$x_3''' = x_\gamma = \frac{1}{x_1x_2x_3} \left( x_2^2 (x_5x_8) + x_2 (x_5x_7x_9) + x_2 (x_4x_6x_8) 
+ (x_4x_6x_7x_9) + x_1x_3 (x_6x_9) \right).$$
Example of Hexagon (Type $A_3$ or $\mathbb{C}[Gr_{2,6}]$)

Consider the graph $G_{TH,\gamma} = 1418232935627$.

$G_{TH,\gamma}$ has five perfect matchings $(x_4, x_5, \ldots, x_9 = 1)$:

$$(x_9) x_1 x_3 (x_6), \quad (x_9 x_7 x_4 x_6),$$

$$(x_2 (x_8) (x_4 x_6), \quad (x_9 x_7) x_2 (x_5),$$

$$(x_2 (x_8) x_2 (x_5)).$$

A perfect matching $M \subseteq E$ is a set of distinguished edges so that every vertex of $V$ is covered exactly once. The weight of a matching $M$ is the product of the weights of the constituent edges, i.e. $x(M) = \prod_{e \in M} x(e)$. 
Principal coefficients: for $i \in Q_0$, add $i' \in Q_0$, $i' \rightarrow i \in Q_1$

\[
\frac{x_1 x_3 (x_6 x_9) y_1 y_2 y_3 + (x_4 x_6 x_7 x_9) y_1 y_3 + x_2 (x_4 x_6 x_8) y_3 + x_2 (x_5 x_7 x_9) y_1 + x_2^2 (x_5 x_8)}{x_1 x_2 x_3}
\]

$G_{T_{H, \gamma}}$ has five perfect matchings ($x_4, x_5, \ldots, x_9 = 1$):

- $(x_9) x_1 x_3 (x_6)$,
- $x_2 (x_8) (x_4 x_6)$,
- $x_2 (x_8) x_2 (x_5)$.

These five monomials exactly match those appearing in the numerator of the expansion of $x_\gamma$. The denominator of $x_1 x_2 x_3$ corresponds to the labels of the three tiles. The $y_i$'s correspond to principal coefficients (heights).
Letting $x_4, x_5, \ldots, x_9 = 1$:

$$x_\gamma = \frac{x_1x_3y_1y_2y_3 + y_1y_3 + x_2y_3 + x_2y_1 + x_2^2}{x_1x_2x_3}$$

This matching is $M\_$. 
More on Height Functions

Letting $x_4, x_5, \ldots, x_9 = 1$:

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This matching is $M_-$. 

![Graphical representation of the matching]

$y_1 y_2 y_3$, $y_1 y_3$, $y_3$, $y_1$, 1, and

1 2 3, 1 3, and 3.
Another Height Function Example (Markoff Quiver)

For $T, \gamma = \gamma_1 \gamma_3 \gamma_3 \gamma_1$, the snake graph is $G_{T, \gamma}$. For this graph, $M^- = \gamma_1 \gamma_2 \gamma_1 \gamma_3$, and one matching, $M$, is $\gamma_1 \gamma_2 \gamma_1 \gamma_3$, so $M \ominus M^- = \gamma_1 \gamma_2 \gamma_1 \gamma_3$ which has height $y_1 y_2$.
For $T, \gamma = \begin{bmatrix} 3 \\ 3 \\ 2 \\ 1 \\ 2 \end{bmatrix}$, the snake graph is $G_{T, \gamma} = \begin{array}{cccc} 1 & 2 & 1 & 3 \\ 2 & 1 & 3 & 2 \\ 3 & 1 & 2 & 1 \\ 2 & 3 & 1 & 2 \\ 1 & 2 & 3 & 1 \end{array}$.  

For this graph, $M_-$ is \[
\begin{array}{c} 
1 \\
2 \\
1 \\
2 \\
1 \\
3 \\
1 \\
2 \\
1 \\
\end{array}
\]
and one matching, $M$, is \[
\begin{array}{c} 
1 \\
2 \\
1 \\
3 \\
1 \\
2 \\
1 \\
3 \\
1 \\
\end{array}
\]
so $M \oplus M_- = \begin{array}{c} 
1 \\
2 \\
1 \\
3 \\
1 \\
2 \\
1 \\
3 \\
1 \\
\end{array}$, which has height $y_1 y_2^2$. So one of the 29 terms in the cluster expansion of $x_\gamma$ is $\frac{x_2^2 x_3^6}{x_1^4 x_2 x_3^2} (y_1 y_2^2)$. 


Graph Theoretic Summary

For every triangulation $T$ (in a surface with or without punctures) and an ordinary arc $\gamma$ through ordinary triangles, we construct a snake graph $G_{T,\gamma}$ such that

$$x_\gamma = \sum_{\text{perfect matching } M \text{ of } G_{T,\gamma}} \frac{x(M)y(M)}{x_1^{e_1(T,\gamma)}x_2^{e_2(T,\gamma)}\cdots x_n^{e_n(T,\gamma)}}.$$

$x_\gamma$ is cluster variable (corresp. to $\gamma$ w.r.t. seed given by $T$) with principal coefficients.
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$x_\gamma$ is cluster variable (corresp. to $\gamma$ w.r.t. seed given by $T$) with principal coefficients.

$e_i(T,\gamma)$ is the crossing number of $\tau_i$ and $\gamma$ (min. int. number),

$x(M)$ is the weight of $M$,

$y(M)$ is the height of $M$,

Similar formula holds for non-ordinary arcs (or with self-folded triangles).

Height functions are due to William Thurston, and Conway-Lagarias.
How to build $G_{T,\gamma}$

Using the above construction for
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![Diagram of a geometric construction with labels $\tau_1, \tau_2, \ldots, \tau_9$.]
How to build $G_{T,\gamma}$

Using the above construction for

\[ \begin{array}{c}
\tau_1 \\
\tau_2 \\
\tau_3 \\
\tau_4 \\
\tau_5 \\
\tau_6 \\
\tau_7 \\
\tau_8 \\
\tau_9 \\
\end{array} \]

Thus

\[ G_{T,\gamma} = \begin{array}{c}
\tau_1 \\
\tau_2 \\
\tau_3 \\
\tau_4 \\
\tau_5 \\
\tau_6 \\
\tau_7 \\
\tau_8 \\
\tau_9 \\
\end{array} \]

...
How to build $G_{T, \gamma}$

Using the above construction for

$$G_{T, \gamma} = \frac{x_1 x_3 \pm 1 + 2 x_2 + x_2^2}{x_1 x_2 x_3}.$$
Brane Tilings and Cluster Algebras

Most simply stated, a Brane Tiling is a Bipartite graph on a torus. We view such a tiling as a doubly-periodic tiling of its universal cover.

A Brane Tiling can be associated to a pair \((Q, W)\), where \(Q\) is a quiver and \(W\) is a potential (a.k.a. superpotential).
Most simply stated, a **Brane Tiling** is a **Bipartite graph on a torus**. We view such a tiling as a doubly-periodic tiling of its universal cover.

A **Brane Tiling** can be associated to a pair \((Q, W)\), where \(Q\) is a quiver and \(W\) is a potential (a.k.a. superpotential).

A quiver \(Q\) is a directed graph where each edge is referred to as an arrow, and multiple edges are allowed.

A potential \(W\) is a linear combination of cyclic paths in \(Q\) (possibly an infinite linear combination).
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A quiver \(Q\) is a directed graph where each edge is referred to as an arrow, and multiple edges are allowed.

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For combinatorial purposes, we assume other conditions on \((Q, W)\):

- Each arrow of \(Q\) appears in one term of \(W\) with a positive sign, and one term with a negative sign.

- The number of terms of \(W\) with a positive sign equals the number with a negative sign. All coefficients in \(W\) are \(\pm 1\).
Example (The $dP_3$ Quiver): $Q_{dP_3} = Q =$

\[ W = A_{16}A_{64}A_{42}A_{25}A_{53}A_{31} + A_{14}A_{45}A_{51} + A_{23}A_{36}A_{62} 
- A_{16}A_{62}A_{25}A_{51} - A_{36}A_{64}A_{45}A_{53} - A_{14}A_{42}A_{23}A_{31}. \]
Brane Tilings and Cluster Algebras

Example (The $dP_3$ Quiver): $Q_{dP_3} = Q = \begin{array}{ccc}
4 & 6 & 1 \\
3 & 5 & 2 \\
\end{array}$,

$W = A_{16}A_{64}A_{42}A_{25}A_{53}A_{31} + A_{14}A_{45}A_{51} + A_{23}A_{36}A_{62} - A_{16}A_{62}A_{25}A_{51} - A_{36}A_{64}A_{45}A_{53} - A_{14}A_{42}A_{23}A_{31}$.

We now unfold $Q$ onto the plane, letting the three positive (resp. negative) terms in $W$ depict clockwise (resp. counter-clockwise) cycles on $\tilde{Q}$. 
Example (continued):

\[ Q = 4 \begin{array}{ccc}
|   |   |
\hline
1 & 3 & 5
\end{array} \]

unfolds to \( \tilde{Q} = \)

\[ W = A_{16} A_{42} A_{25} A_{53} A_{31}(A) + A_{14} A_{45} A_{51}(B) + A_{23} A_{36} A_{62}(C) - A_{16} A_{62} A_{25} A_{51}(D) - A_{36} A_{64} A_{45} A_{53}(E) - A_{14} A_{42} A_{23} A_{31}(F). \]

Locally, the configurations around vertices of \( Q \) and \( \tilde{Q} \) are identical.
Taking the planar dual yields a bipartite graph on a torus (Brane Tiling):

\[ \tilde{Q} \longrightarrow T_Q = \]

Negative Term in \( W \) \( \longleftrightarrow \) Counter-Clockwise cycle in \( \tilde{Q} \) \( \longleftrightarrow \bullet \) in \( T_Q \)

Positive Term in \( W \) \( \longleftrightarrow \) Clockwise cycle in \( \tilde{Q} \) \( \longleftrightarrow \circ \) in \( T_Q \)

(To obtain \( \tilde{Q} \) from \( T_Q \), we dualize edges so that white is on the right.)
Summarizing the $dP_3$ Example:

Negative Term in $W$ $\leftrightarrow$ Counter-Clockwise cycle in $\tilde{Q}$ $\leftrightarrow$ $\bullet$ in $\mathcal{T}_Q$

Positive Term in $W$ $\leftrightarrow$ Clockwise cycle in $\tilde{Q}$ $\leftrightarrow$ $\circ$ in $\mathcal{T}_Q$

(To obtain $\tilde{Q}$ from $\mathcal{T}_Q$, we dualize edges so that white is on the right.)
If we mutate $Q_{dP_3}$ by $1, 2, 3, 4, 5, 6, 1, 2, \ldots$, after the first two mutations, we obtain same quiver back up to cyclically permuting the vertex labels.
If we mutate $Q_{dP_3}$ by 1, 2, 3, 4, 5, 6, 1, 2, \ldots, after the first two mutations, we obtain same quiver back up to cyclically permuting the vertex labels.

**Point:** Mutating once in the $Q_N^{(r,s)}$ case, or twice in the $Q_{dP_3}$ case, yields a quiver with potential that is equivalent up to cyclic rotation. Such quivers are called periodic in the Fordy-Marsh sense.
Cluster Variable Mutation

In addition to the mutation of quivers, there is also a complementary cluster mutation that can be defined.

Cluster mutation yields a sequence of Laurent polynomials in \( \mathbb{Q}(x_1, x_2, \ldots, x_n) \) known as cluster variables.

Given a quiver \( Q \) (the potential is irrelevant here) and an initial cluster \( \{x_1, \ldots, x_N\} \), then mutating at vertex 1 yields a new cluster variable \( x_{N+1} \) defined by

\[
x_{N+1} = \left( \prod_{1 \rightarrow i \in Q} x_i + \prod_{i \rightarrow 1 \in Q} x_i \right) / x_1.
\]
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defined by

\[
x_{N+1} = \left( \prod_{1 \to i \in Q} x_i + \prod_{i \to 1 \in Q} x_i \right) / x_1.
\]

Example (\( dP_3 \)): \( x_{2n+7}x_{2n+1} = x_{2n+3}x_{2n+5} + x_{2n+4}x_{2n+6} \) and

\[
x_{2n+8}x_{2n+2} = x_{2n+3}x_{2n+5} + x_{2n+4}x_{2n+6}.
\]

Example (Gale-Robinson): \( x_{n+N}x_n = x_{n+r}x_{n+N-r} + x_{n+s}x_{n+N-s} \).
Main Theorem (Jeong-M-Zhang)

For certain periodic quivers $Q$, which include the Gale-Robison quiver family, the $dP_3$ quiver, and some other 2-periodic quivers, we can use the Brane Tiling $\mathcal{T}_Q$ to obtain combinatorial formulas for an infinite sequence of cluster variables in $\mathcal{A}_Q$. 
Main Theorem (Jeong-M-Zhang)

For certain periodic quivers $Q$, which include the Gale-Robison quiver family, the $dP_3$ quiver, and some other 2-periodic quivers, we can use the Brane Tiling $\mathcal{T}_Q$ to obtain combinatorial formulas for an infinite sequence of cluster variables in $\mathcal{A}_Q$.

For $n > N$, $x_n = cm(G_n) \sum_{M=\text{perfect matching of } G_n} x(M)y(M)$, where

$\{G_n : n > N\}'s$ are a collection of subgraphs of $\mathcal{T}_Q$, $x(M) = \prod_{\text{edge } e \in M} \frac{1}{x_i x_j}$ (for edge $e$ straddling faces $i$ and $j$), $y(M) = \text{height of } M$ (recording what faces need to be twisted to obtain matching $M$ starting from the minimal matching, and $cm(G_n) = \text{the covering monomial of the graph } G_n$ (which records what face labels are contained in $G_n$ and along its boundary).

**Remark:** This weighting scheme is a reformulation of schemes appearing in works of Speyer ("Octahedron Recurrence") and Goncharov-Kenyon.
Gale-Robinson Example \((Q_7^{(2,3)}, \text{Mutating } 1, 2, \ldots, 7, \ldots)\)

\[ Q = \]

\[ T_Q = \]

\[ x_8 \leftrightarrow \boxed{1}, \ x_9 \leftrightarrow \boxed{2}, \ x_{10} \leftrightarrow \boxed{1 \ 3}, \ x_{11} \leftrightarrow \]

\[ x_{12} \leftrightarrow \]

\[ x_{13} \leftrightarrow \]

\[ x_{14} \leftrightarrow \]

\[ x_{15} \leftrightarrow \]

\[ x_{16} \leftrightarrow \]

\( \ldots \)
Gale-Robinson Example \((Q^{(2,3)}_7, \text{Mutating } 1, 2, \ldots, 7, \ldots)\)

Obtain **pinecone graphs** from Bousquet-Mélou, Propp, and West in terms of Brane Tilings Terminology.

Furthermore, to get **cluster variable formulas with coefficients**, need only use weights (Goncharov-Kenyon, Speyer) and heights (Kenyon-Propp-...)

\[
\begin{align*}
X_8 & \leftrightarrow \begin{array}{c}
1 \downarrow \\
1 & 3 & 5
\end{array} , \\
X_9 & \leftrightarrow \begin{array}{c}
2 \downarrow \\
2 & 4 & 6 & 1
\end{array} , \\
X_{10} & \leftrightarrow \begin{array}{c}
1 \downarrow \\
1 & 3 & 5
\end{array} , \\
X_{11} & \leftrightarrow \begin{array}{c}
2 \downarrow \\
2 & 4 & 6 & 1
\end{array} , \\
X_{12} & \leftrightarrow \begin{array}{c}
2 \downarrow \\
2 & 4 & 6 & 1
\end{array} , \\
X_{13} & \leftrightarrow \begin{array}{c}
1 \downarrow \\
1 & 3 & 5
\end{array} , \\
X_{14} & \leftrightarrow \begin{array}{c}
2 \downarrow \\
2 & 4 & 6 & 1
\end{array} , \\
X_{15} & \leftrightarrow \begin{array}{c}
2 \downarrow \\
2 & 4 & 6 & 1
\end{array} , \\
X_{16} & \leftrightarrow \begin{array}{c}
1 \downarrow \\
1 & 3
\end{array} , \\
\ldots
\end{align*}
\]
Gale-Robinson Example \((Q^{(2,3)}_7, \text{Mutating } 1, 2, \ldots, 7, \ldots)\)

Similar connections (without principal coefficients) also observed in “Brane tilings and non-commutative geometry” by Richard Eager.

Eager uses physics terminology where he looks at \(Y^{p,q}\) and \(L^{a,b,c}\) quiver gauge theories, and their periodic Seiberg duality (i.e. quiver mutations).

\(x_8 \leftrightarrow \begin{array}{c} 1 \\ 1 \\ 1 \\ 2 \\ 2 \\ 1 \\ 1 \\ 1 \\ 2 \\ 1 \end{array},\  x_9 \leftrightarrow \begin{array}{c} 2 \\ 2 \\ 2 \\ 1 \\ 2 \\ 2 \\ 2 \\ 2 \\ 1 \\ 2 \end{array},\ x_{10} \leftrightarrow \begin{array}{c} 1 \\ 1 \\ 3 \\ 1 \\ 3 \\ 1 \\ 1 \\ 3 \\ 1 \\ 3 \\ 1 \\ 3 \\ 1 \\ 1 \end{array},\ x_{11} \leftrightarrow \begin{array}{c} 1 \\ 1 \\ 3 \\ 1 \\ 3 \\ 1 \\ 1 \\ 3 \\ 1 \\ 3 \\ 1 \\ 3 \\ 1 \\ 1 \end{array},\ x_{12} \leftrightarrow \begin{array}{c} 2 \\ 2 \\ 2 \\ 1 \\ 2 \\ 2 \\ 2 \\ 2 \\ 1 \\ 2 \end{array},\ x_{13} \leftrightarrow \begin{array}{c} 1 \\ 1 \\ 3 \\ 1 \\ 3 \\ 1 \\ 1 \\ 3 \\ 1 \\ 3 \\ 1 \\ 3 \\ 1 \\ 1 \end{array},\ x_{14} \leftrightarrow \begin{array}{c} 1 \\ 1 \\ 3 \\ 1 \\ 3 \\ 1 \\ 1 \\ 3 \\ 1 \\ 3 \\ 1 \\ 3 \\ 1 \\ 1 \end{array},\ x_{15} \leftrightarrow \begin{array}{c} 2 \\ 2 \\ 2 \\ 1 \\ 2 \\ 2 \\ 2 \\ 2 \\ 1 \\ 2 \end{array},\ x_{16} \leftrightarrow \begin{array}{c} 1 \\ 1 \\ 3 \\ 1 \\ 3 \\ 1 \\ 1 \\ 3 \\ 1 \\ 3 \\ 1 \\ 3 \\ 1 \\ 1 \end{array},\ \ldots
$dP_3$ Example (Mutating 1, 2, 3, 4, 5, 6, 1, 2, ...)

These subgraphs appear in work by Cottrell-Young and a subsequence of them appear in M. Ciucu’s work “Perfect matchings and perfect powers”, where they are called **Aztec Dragons**.

More on Aztec Dragons and the $dP_3$ lattice shortly.
The quiver $Q$ below is 2-periodic, as illustrated by mutating in order 1, 2, 3, 4, 1, 2, …
The quiver $Q$ below is 2-periodic, as illustrated by mutating in order 1, 2, 3, 4, 1, 2, ...
The quiver \( Q' \) below is \textit{2-periodic}, as illustrated by mutating in order \( 1, 2, 3, 4, 1, 2, \ldots \)

\[
Q' = \begin{array}{ccc}
1 & 3 & \\
4 & 2 & \\
\end{array} \rightarrow \begin{array}{ccc}
1 & 3 & \\
4 & 2 & \\
\end{array} \rightarrow \ldots
\]
Hirzebruch Quiver $F_0$: Aztec Diamonds and **Fortresses**

The quiver $Q'$ below is **2-periodic**, as illustrated by mutating in order $1, 2, 3, 4, 1, 2, \ldots$

$Q' = \begin{array}{c}
1 & 3 \\
\downarrow & \downarrow \\
4 & 2
\end{array} \rightarrow \begin{array}{c}
1 & 3 \\
\uparrow & \uparrow \\
4 & 2
\end{array} \rightarrow \ldots$

**Fortresses** from M. Ciucu's work “Perfect matchings and perfect powers”.
Non-periodic mutation sequences in $F_0$

$Q =$

![Diagram of quivers]

... or

![Another diagram of quivers]
Non-periodic mutation sequences in $F_0$

$$Q = \begin{array}{ccc}
1 & 3 & \\
\downarrow & \uparrow & \\
4 & 2 & \\
\end{array} \rightarrow \begin{array}{ccc}
1 & 3 & \\
\downarrow & \uparrow & \\
4 & 2 & \\
\end{array} \rightarrow \begin{array}{ccc}
1 & 3 & \\
\downarrow & \uparrow & \\
4 & 2 & \\
\end{array} \rightarrow \ldots$$

If we instead mutate by $1, 2, 3, 2, 3, \ldots$ or $1, 3, 2, 3, 2, \ldots$, we obtain quivers where the number of arrows grows without bound.
Non-periodic mutation sequences in $F_0$

$Q = \begin{array}{c}
1 & 3 \\
4 & 2
\end{array} \rightarrow \begin{array}{c}
1 & 3 \\
4 & 2
\end{array} \rightarrow \begin{array}{c}
1 & 3 \\
4 & 2
\end{array} \rightarrow \ldots$

If we instead mutate by $1, 2, 3, 2, 3, \ldots$ or $1, 3, 2, 3, 2, \ldots$, we obtain quivers where the number of arrows grows without bound.

Nonetheless, a combinatorial interpretation for the cluster variables is

\[ \begin{array}{c}
1 & 3 & 1 \\
2 & 2 & 2
\end{array} \quad \begin{array}{c}
1 & 3 & 1 \\
2 & 2 & 2
\end{array} \quad \begin{array}{c}
1 & 3 & 1 \\
2 & 2 & 2
\end{array} \quad \ldots \quad \text{or} \quad \begin{array}{c}
1 & 3 & 1 \\
2 & 2 & 2
\end{array} \quad \begin{array}{c}
1 & 3 & 1 \\
2 & 2 & 2
\end{array} \quad \begin{array}{c}
1 & 3 & 1 \\
2 & 2 & 2
\end{array} \quad \ldots \quad \ldots \]
Non-periodic mutation sequences in the $dP_3$ Lattice

Mutating at a vertex of $dP_3$ and its antipode commute so a mutation such as $1, 2, 1, 2$ can be reordered to $1, 1, 2, 2$, and hence is the identity.

Letting $\tau_1 = \mu_1 \circ \mu_2$, $\tau_2 = \mu_3 \circ \mu_4$, $\tau_3 = \mu_5 \circ \mu_6$, the above can be written as $\tau_1^2 = \tau_2^2 = \tau_3^2 = 1$. 
Non-periodic mutation sequences in the $dP_3$ Lattice

Mutating at a vertex of $dP_3$ and its antipode commute so a mutation such as $1, 2, 1, 2$ can be reordered to $1, 1, 2, 2$, and hence is the identity.

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As discussed with Pylyavskyy, we see the further relations $(\tau_1 \tau_2)^3 = (\tau_1 \tau_3)^3 = (\tau_2 \tau_3)^3 = 1$, and it can be shown that there are no other relations. Thus $\langle \tau_1, \tau_2, \tau_3 \rangle \cong \tilde{A}_2$. 

(ICMS, Gauge theories) 

Brane Tilings in Combinatorics 

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We call a mutation sequence built out of a concatenation of $\tau_1$, $\tau_2$, and $\tau_3$ a $\tau$-mutation sequence.

As an example, the periodic sequences $1, 2, 3, 4, 5, 6, 1, 2, \ldots$ yielding the Aztec Dragons, were examples of $\tau$-mutation sequences given by $\tau_1, \tau_2, \tau_3, \tau_1, \ldots$.

Up to the $\tilde{\mathbb{A}}_2$ relations and relabeling vertices, other $\tau$-mutation sequences do not necessarily give new cluster variables.
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Up to the $\tilde{A}_2$ relations and relabeling vertices, other $\tau$-mutation sequences do not necessarily give new cluster variables.

**First non-trivial $\tau$-mutation sequence**: $\tau_1, \tau_2, \tau_1, \tau_3 = 1, 2, 3, 4, 1, 2, 5, 6$ yields the following as the last cluster variable:

\[
\begin{align*}
(x_1 x_2^2 x_3^3 x_5^4 + x_2^3 x_3^2 x_4 x_5^4 + 2x_1^2 x_2 x_3^2 x_4 x_5^3 x_6 + 4x_1 x_2^2 x_3 x_4 x_5^3 x_6 + 2x_2 x_3 x_4 x_5^3 x_6 + x_1^3 x_3 x_5^2 x_6
+ 5x_1 x_2 x_3 x_4 x_5^2 x_6^2 + 5x_1 x_2 x_3 x_4 x_5^2 x_6^2 + x_2^3 x_4 x_5^2 x_6^2 + 2x_1^3 x_3 x_4 x_5 x_6^3 + 4x_1^2 x_2 x_3 x_4^2 x_5 x_6^3 + 2x_1 x_2 x_4 x_5 x_6^3 + x_1^3 x_3 x_4 x_6^4 + x_1^2 x_2 x_4^3 x_6^4) / x_1^2 x_2^2 x_3 x_4 x_6
\end{align*}
\]
We call a mutation sequence built out of a concatenation of τ₁, τ₂, and τ₃ a τ-mutation sequence.

As an example, the periodic sequences 1, 2, 3, 4, 5, 6, 1, 2, ... yielding the Aztec Dragons, were examples of τ-mutation sequences given by τ₁, τ₂, τ₃, τ₁, ... .

Up to the ˜A₂ relations and relabeling vertices, other τ-mutation sequences do not necessarily give new cluster variables.

First non-trivial τ-mutation sequence: τ₁, τ₂, τ₁, τ₃ = 1, 2, 3, 4, 1, 2, 5, 6 yields the following as the last cluster variable:

\[
\begin{align*}
(x_1 x_2^2 x_3^3 x_5^4 + x_2^3 x_3^2 x_4 x_5^4 + 2x_1^2 x_2 x_3^3 x_5 x_6 + 4x_1 x_2^2 x_3^2 x_4 x_5^3 x_6 + 2x_2^3 x_3 x_4^2 x_5^3 x_6 + x_1^3 x_3^3 x_5^2 x_6 \\
+ 5x_1^2 x_2 x_3^2 x_4 x_5^2 x_6 + 5x_1 x_2^2 x_3 x_4^2 x_5^2 x_6 + x_2^3 x_4^3 x_5 x_6 + 2x_1^3 x_3^2 x_4 x_5 x_6 + 4x_1^2 x_2 x_3 x_4^2 x_5 x_6 \\
+ 2x_1 x_2^2 x_4 x_5^3 x_6 + x_1^3 x_3 x_4^2 x_6 + x_2^3 x_4 x_5^3 x_6) / x_1^2 x_2 x_3 x_4 x_6
\end{align*}
\]

What is a combinatorial interpretation of this Laurent polynomial?
Aztec Castles

Investigated this non-periodic mutation sequences in the $dP_3$ Lattice with UMN 2013 REU Students Megan Leoni, Seth Neel, and Paxton Turner.
Aztec Castles

Investigated this non-periodic mutation sequences in the $dP_3$ Lattice with UMN 2013 REU Students Megan Leoni, Seth Neel, and Paxton Turner.

Developed a two-parameter family of graphs, Aztec Castles, to encode cluster variables arising from $\tau$-mutation sequences.

Example: $x_{\tau_1,\tau_2,\tau_1,\tau_3}$ corresponds to (Up to a shift in indexing on faces by one.)
For a general $\tau$-mutation sequence, the last cluster variable associated to an alcove in the $\tilde{A}_2$ Coxeter Lattice, indexed by $(i, j)$, $[i, j]$, $\langle i, j \rangle$ or $\{i, j\}$. 

$132\ldots 1232\ldots = 1323\ldots 123\ldots$ 

$1213\ldots = 2123\ldots$ 

$312\ldots 213\ldots$ 

$321\ldots 231\ldots$
Aztec Castles (cont.)

For a general $\tau$-mutation sequence, the last cluster variable associated to an alcove in the $\tilde{A}_2$ Coxeter Lattice, indexed by $(i, j)$, $[i, j]$, $\langle i, j \rangle$ or $\{i, j\}$.

More precisely, we cut the lattice into twelve regions (cones) and by symmetry it suffices to describe the families of graphs for two adjacent cones.
Aztec Castles in Cone I (Integer Cone)

For a general $\tau$-mutation sequence, the last cluster variable associated to an alcove in the $\tilde{A}_2$ Coxeter Lattice, indexed by $(i, j)$, $[i, j]$, $\langle i, j \rangle$ or $\{i, j\}$.

For $(i, j)$, we let $\gamma^j_i$ be the Aztec Castle defined by the subgraph cut out by

$$P_1^{i+j} P_2^j P_3^{i-1} P_4^{i+j-1} P_5^{j-1} P_6^i$$

in

![Diagram of Aztec Castles and related paths](image-url)
Example \((i = 3, j = 3)\)

For a general \(\tau\)-mutation sequence, the last cluster variable associated to an alcove in the \(\tilde{A}_2\) Coxeter Lattice, indexed by \((i, j), [i, j], \langle i, j \rangle\) or \(\{i, j\}\).

For \((i, j)\), we let \(\gamma_i^j\) be the Aztec Castle defined by the subgraph cut out by

\[
P_1^{i+j} P_2^j P_3^{i-1} P_4^{i+j-1} P_5^{j-1} P_6^i \text{ in}
\]

\[
P_1 = (N W, N E, N, E) \quad P_2 = (S W, S E, S, E) \quad P_3 = (S E, N E, S E, N E)
\]

\[
P_4 = (S E, S W, S, W) \quad P_5 = (N E, N W, N, W) \quad P_6 = (N W, S W, N W, S W)
\]
Aztec Castles in Cone XII (Half-Integer Cone)

For a general $\tau$-mutation sequence, the last cluster variable associated to an alcove in the $\tilde{A}_2$ Coxeter Lattice, indexed by $(i, j)$, $[i, j]$, $\langle i, j \rangle$ or $\{i, j\}$.

For $[i, j]$, we let $\tilde{\gamma}^j_i$ be the Aztec Castle defined by the subgraph cut out by

$$\tilde{p}_1^j \tilde{p}_{i+1}^j \tilde{p}_{i+j}^j \tilde{p}_{i+1}^j \tilde{p}_i^j \tilde{p}_{i+j-1}^j$$

in

$$\tilde{P}_1 = (W, S, SW, SE)$$
$$\tilde{P}_2 = (NW, SW, NW, SW)$$
$$\tilde{P}_3 = (E, S, SE, SW)$$
$$\tilde{P}_4 = (E, N, SE, SE, NE)$$
$$\tilde{P}_5 = (SE, NE, SE, NE)$$
$$\tilde{P}_6 = (W, N, NW, NE)$$
Example \([i = 2, j = 2]\)

For a general \(\tau\)-mutation sequence, the last cluster variable associated to an alcove in the \(\tilde{A}_2\) Coxeter Lattice, indexed by \((i, j)\), \([i, j]\), \(\langle i, j \rangle\) or \(\{i, j\}\).

For \([i, j]\), we let \(\tilde{\gamma}_i^j\) be the Aztec Castle defined by the subgraph cut out by

\[
\tilde{p}_1 \tilde{p}_2 \tilde{p}_3 \tilde{p}_4 \tilde{p}_5 \tilde{p}_6^{-1}
\]

in

\[\tilde{P}_1 = (W, S, SW, SE)\]
\[\tilde{P}_2 = (NW, SW, NW, SW)\]
\[\tilde{P}_3 = (E, S, SE, SW)\]
\[\tilde{P}_4 = (E, N, NE, SE)\]
\[\tilde{P}_5 = (SE, NE, SE, NE)\]
\[\tilde{P}_6 = (W, N, NW, NE)\]
Aztec Dragons as Special Case of Castles

\[ D_n = \sigma \gamma_0^n = P_1^{n-1} P_2^{n-1} P_3 P_4^n P_5 P_6^{n-1} \] (\( \sigma \) is reflection by 180°)

(Example: \( D_2 = \sigma \gamma_0^2 \))
Similarly, \( D_{n+1/2} = \sigma \tilde{\gamma}_0^n = \tilde{p}_{1}^{n+1} \tilde{p}_{2}^{n} \tilde{p}_{1}^{n-1} \tilde{p}_{4}^{n} \tilde{p}_{5}^{1} \tilde{p}_{6}^{n} \) (from top white vertex)

(Example: \( D_{5/2} = \sigma \tilde{\gamma}_0^2 \))
If we unfold we get.
If we unfold, we get the snake graphs from earlier: e.g.,

\begin{align*}
\text{Subgraphs yield} & \quad \begin{array}{c}
1 \quad 2 \\
\end{array}
\end{align*}
Open Problems and Work in Progress

- \( \tau \)-mutation sequences and the periodic mutation sequences discussed are examples of toric mutation sequences. Mutation only performed at vertices with at most two incoming and two outgoing arrows.

- Extend definition of heights to other cases to obtain cluster variables with principal coefficients.

- Obtain more combinatorial interpretations for non-toric mutation sequences (like the \( F_0 \) case).

- Understand better the relationship between snake graphs corresponding to arcs in a surface and subgraphs of a brane tiling. (e.g. Markoff quiver)
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“Colored BPS Pyramid Partition Functions, Quivers, and Cluster Transformations” by Richard Eager and Sebastian Franco gives a recipe for getting combinatorial interpretations for toric mutation sequences but fails for examples like \( \tau_1 \tau_2 \tau_1 \tau_3 \).

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Thank You For Listening

- **Aztec Castles and the dP3 Quiver** (with Megan Leoni, Seth Neel, and Paxton Turner, arXiv:1308.3926.
- **Gale-Robinson Sequences and Brane Tilings** (with In-Jee Jeong and and Sicong Zhang), *Discrete Mathematics and Theoretical Computer Science Proc. AS* (2013), 737-748. (Longer version in preparation.)

Slides Available at http://math.umn.edu/~musiker/ICMS.pdf