Beyond Aztec Dragons and Castles: Toric Cluster Variables for the dP3 Quiver

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MIT Combinatorics Seminar

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http://math.umn.edu/∼musiker/MIT16.pdf

Outline

1. Introduction to Cluster Algebras
2. Aztec Diamonds and Dragons
3. Gale-Robinson Sequences and Pinecones
4. Toric Mutations in an Infinite Mutation Type Cluster Algebra (dP_3)
5. Combinatorial Interpretation
6. Sketch of the Proof

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Special Thanks to Jim Propp and the 2001 REACH Program.

`http://math.umn.edu/~musiker/MIT16.pdf`

What is a Cluster Algebra?

Definition (Sergey Fomin and Andrei Zelevinsky 2001) A cluster algebra $\mathcal{A}$ is a subalgebra of $k(x_1, \ldots, x_n)$ constructed cluster by cluster by certain exchange relations.

Generators:

Specify an initial finite set of them, a Cluster, $\{x_1, x_2, \ldots, x_n\}$.

Construct the rest via Binomial Exchange Relations:

$$x_\alpha x'_\alpha = \prod x_{\gamma_i}^{d_i^+} + \prod x_{\gamma_i}^{d_i^-}.$$

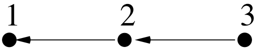
The set of all such generators are known as Cluster Variables, and the initial pattern of exchange relations determines the Seed.

Relations:

Induced by the Binomial Exchange Relations.

Quiver Mutation

We focus on cluster algebras whose initial pattern of exchange relations is determined by a quiver, i.e. a directed graph

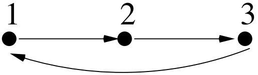


$\{x_1, x_2, x_3\}$

$$x_j x_j' = \prod_{i \to j \in Q} x_i + \prod_{j \to i \in Q} x_i, \quad \left(i.e. \;\; x_j x_j' = \prod x_i^{d_i^+} + \prod x_i^{d_i^-} \right)$$

where d_i^+ is the number of arrows from vertex i to j and
d_j^- is the number of arrow from vertex j to i.

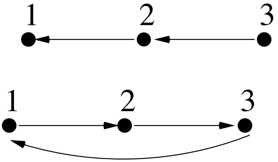
Example: Mutating at vertex 2 yields $x_2' x_2 = x_1 + x_3$



$\{x_1, \frac{x_1+x_3}{x_2}, x_3\}$

Observe: we also mutate the quiver Q and obtain a new exchange pattern.

Quiver Mutation (at vertex j)



$\{x_1, x_2, x_3\}$

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1st) Add an edge $i \to k$ for every 2-path $i \to j \to k$ in Q, the original quiver.

2nd) Reverse all arrows, i.e. directed edges, incident to vertex j.

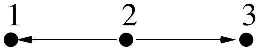
3rd) Lastly, we erase all 2-cycles (that have been created by steps 1 and 2), and denote the resulting quiver as $\mu_j(Q)$.

Basic Example of a Cluster Algebra

Let $\mathcal{A}$ be the cluster algebra defined by the initial cluster $\{x_1, x_2, x_3\}$ and the initial exchange pattern

$$x_1x_1' = 1 + x_2, \quad x_2x_2' = x_1x_3 + 1, \quad x_3x_3' = 1 + x_2.$$

corresponding to the quiver



$\mathcal{A}$ is of finite type, type A_3, generated by the cluster variables

$$\left\{x_1, x_2, x_3, \frac{1+x_2}{x_1}, \frac{x_1x_3+1}{x_2}, \frac{1+x_2}{x_3}, \frac{x_1x_3+1+x_2}{x_1x_2}, \frac{x_1x_3+1+x_2}{x_2x_3}, \frac{x_1x_3+1+x_2+x_2+x_2^2}{x_1x_2x_3}\right\}.$$

Second Example of a Cluster Algebra

Kronecker Quiver, otherwise known as (Affine Type, of Type $\widetilde{A}_1$) or corresponding to an annulus with two marked points.

$$\bullet_1 \Longrightarrow \bullet_2 \qquad \text{yields} \qquad x_n x_{n-2} = x_{n-1}^2 + 1.$$

$$x_3 = \frac{x_2^2 + 1}{x_1}.$$

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$$x_5 = \frac{x_4^2 + 1}{x_3} = \frac{x_2^6 + 3x_2^4 + 3x_2^2 + 1 + x_1^4 + 2x_1^2 + 2x_1^2 x_2^2}{x_1^3 x_2^2}, \quad \ldots$$

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If we let $x_1 = x_2 = 1$, we obtain $\{x_3, x_4, x_5, x_6\} = \{2, 5, 13, 34\}$.

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The next number in the sequence is $x_7 = \frac{34^2+1}{13} = \frac{1157}{13} = 89$, an integer!

This is a example of a cluster algebra of finite mutation type.

Finite, Finite Mutation, and Infinite Mutation Types

A cluster algebra is of finite type if the number of cluster variables and the number of quivers reachable via mutations is finite.

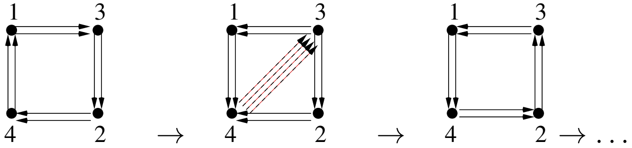
A cluster algebra is of finite mutation type if the number of quivers reachable via mutations is finite (but the number of cluster variables could be infinite).

A cluster algebra is of **infinite mutation type** if both the number of cluster variables and the number of quivers reachable via mutations is **infinite**.

Most cluster algebras of finite mutation type come from a surface (e.g. Kronecker quiver comes from an annulus).

We now shift our focus to cluster algebras of **infinite mutation type**.

Consider the quiver Q (on the left below). Instead of all cluster variables, we focus on those obtained by mutating $1, 2, 3, 4, 1, 2, \ldots$ periodically:



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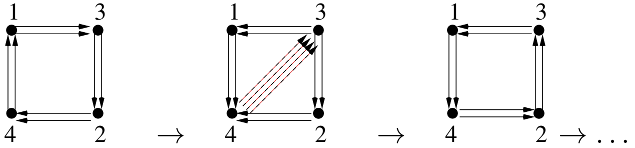


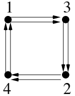
Yields a sequence of cluster variables, with initial cluster variables x_1, x_2, x_3, x_4, with x_{n+4} denoting the nth new cluster variable obtained by this mutation sequence $\{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}, \ldots\}$.

Because of the periodicity, it follows that the x_n's satisfy the recurrences

$$x_n x_{n-4} = \begin{cases} x_{n-1}^2 + x_{n-2}^2 & \text{when } n \text{ is odd, and} \\ x_{n-2}^2 + x_{n-3}^2 & \text{when } n \text{ is even.} \end{cases}$$

For example, $x_5 = \frac{x_3^2 + x_4^2}{x_1}$, $x_6 = \frac{x_3^2 + x_4^2}{x_2}$, $x_7 = \frac{x_5^2 + x_6^2}{x_3}$, and $x_8 = \frac{x_5^2 + x_6^2}{x_4}$.

Aztec Diamonds

Let $Q =$ 

, and mutate periodically at $1, 2, 3, 4, 1, 2, 3, 4, \ldots$.

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By letting $x_1 = x_2$ and $x_3 = x_4$, we get $x_{2n+1} = x_{2n}$ for all n.

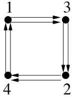
Letting $\{T_n\}$ be the sequence $\{x_{2n}\}_{n \in \mathbb{Z}}$, we obtain a single recurrence.

$$T_n T_{n-2} = 2 T_{n-1}^2.$$

If $T_1 = T_2 = 1$, $\{T_n\} = \{1, 1, 2, 8, 64, 1024, 32768, \ldots\} = \left\{2^{\frac{(n-1)(n-2)}{2}}\right\}$.

For $n \geq 3$, $T_n = \#$ (perfect matchings of the $(n-2)$nd Aztec Diamond).

Aztec Diamonds

Let $Q =$ 

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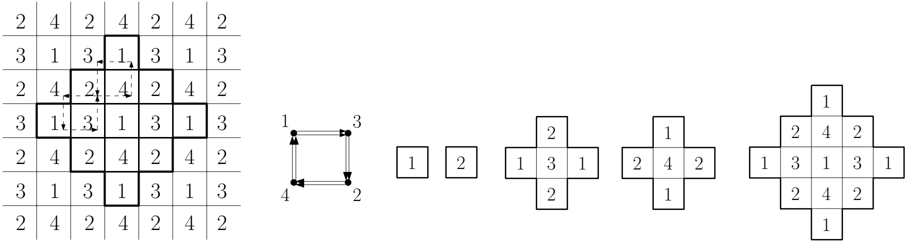
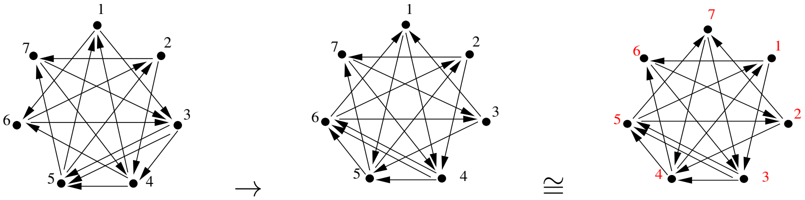


$x_5 = \frac{x_3^2+x_4^2}{x_1}$, $x_6 = \frac{x_3^2+x_4^2}{x_2}$, $x_7 = \frac{(x_3^2+x_4^2)^2(x_1^2+x_2^2)}{x_1^2x_2^2x_3}$, and $x_8 = \frac{(x_3^2+x_4^2)^2(x_1^2+x_2^2)}{x_1^2x_2^2x_4}$.

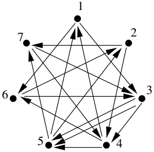
Example ($Q_N^{(r,s)}$): (e.g. $r = 2,\ s = 3,\ N = 7$)

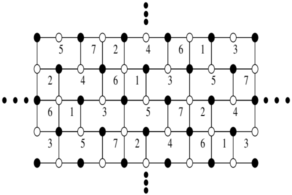


Mutating at $1, 2, 3, \ldots, N, 1, 2, \ldots$ yields the same quiver, up to cyclic permutation, at each step, hence we obtain the infinite sequence of $x_{N+1}, x_{N+2}, \ldots$ satsifying

$$x_n = \left(x_{n-r}x_{n-N+r} + x_{n-s}x_{n-N+s}\right)/x_{n-N} \quad \text{for} \quad n > N.$$

Known as the **Gale-Robinson Sequence** of Laurent polynomials.

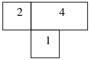
$Q =$ 

$\mathcal{T}_Q =$ 

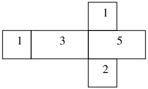
$x_8 \leftrightarrow$ 

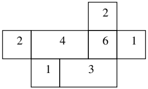
, $x_9 \leftrightarrow$ 

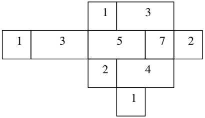
, $x_{10} \leftrightarrow$ 

, $x_{11} \leftrightarrow$ 

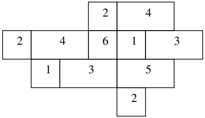
,

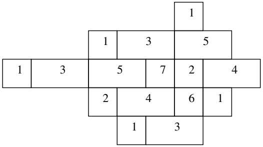
$x_{12} \leftrightarrow$ 

, $x_{13} \leftrightarrow$ 

, $x_{14} \leftrightarrow$ 

,

$x_{15} \leftrightarrow$ 

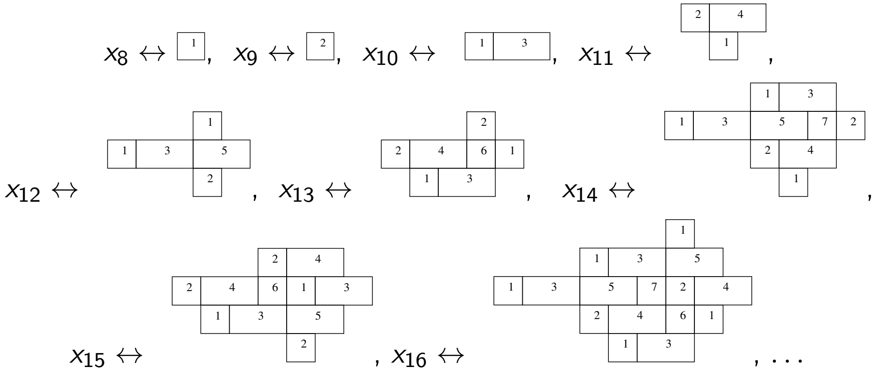
, $x_{16} \leftrightarrow$ 

, . . .

FPSAC Proceedings 2013 (Jeong-M-Zhang)

Obtain **pinecone graphs** from Bousquet-Mélou, Propp, and West in terms of Brane Tilings Terminology.

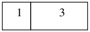
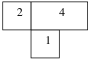
Furthermore, to get cluster variable formulas with coefficients, need only use weights (Goncharov-Kenyon, Speyer) and heights (Kenyon-Propp-...)

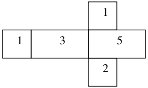
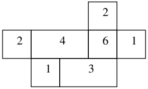
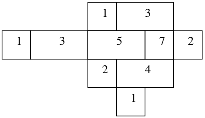


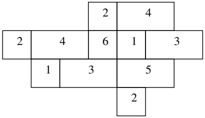
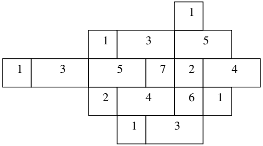
FPSAC Proceedings 2013 (Jeong-M-Zhang)

Similar connections (without principal coefficients) also observed in "Brane tilings and non-commutative geometry" by Richard Eager.

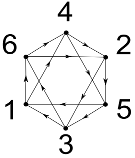
Eager uses physics terminology where he looks at $Y^{p,q}$ and $L^{a,b,c}$ quiver gauge theories, and their periodic Seiberg duality (i.e. quiver mutations).

$x_8 \leftrightarrow$ , $x_9 \leftrightarrow$ , $x_{10} \leftrightarrow$ , $x_{11} \leftrightarrow$ ,

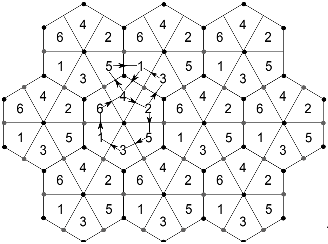
$x_{12} \leftrightarrow$ , $x_{13} \leftrightarrow$ , $x_{14} \leftrightarrow$ ,

$x_{15} \leftrightarrow$ , $x_{16} \leftrightarrow$ , $\ldots$

The Del Pezzo 3 Quiver and Aztec Dragons

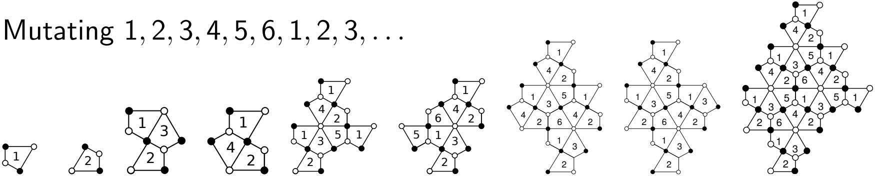


Q



$\mathcal{T}_Q$

Mutating $1, 2, 3, 4, 5, 6, 1, 2, 3, \ldots$



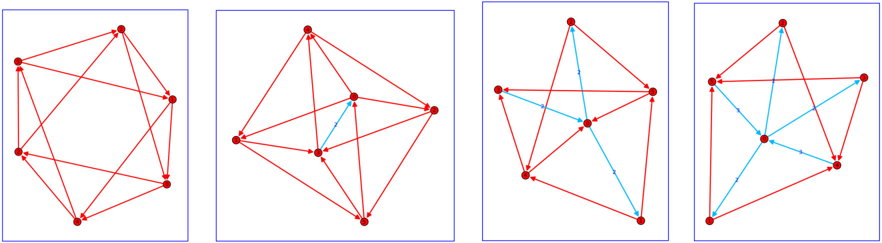
$\ldots$

Introduced by Jim Propp, Ben Wieland, and Mihai Ciucu. Studied further by Cottrell-Young.

$$x_{2n+7}x_{2n+1} = x_{2n+3}x_{2n+5} + x_{2n+4}x_{2n+6} \text{ and}$$
$$x_{2n+8}x_{2n+2} = x_{2n+3}x_{2n+5} + x_{2n+4}x_{2n+6}.$$

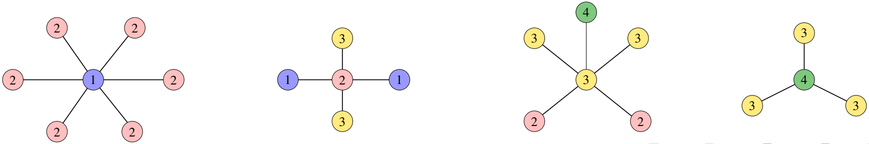
Toric Mutations and Toric Phases of dP_3

Toric mutations take place at vertices with in-degree and out-degree 2.



Starting with any of these four models of the dP_3 quiver, any sequence of toric mutations yields a quiver that is graph isomorphic to one of these.

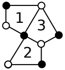
Figure 20 of Eager-Franco (Incidences betweeen these Models):



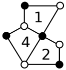
Example from S. Zhang (2012 REU): Periodic mutation $1, 2, 3, 4, 5, 6, 1, 2, \ldots$ yields partition functions for Aztec Dragons (as studied by Ciucu, Cottrell-Young, and Propp) under appropriate weighted enumeration of perfect matchings.

$$\frac{x_3x_5+x_4x_6}{x_1}$$

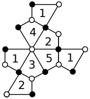
$$\frac{x_4x_6+x_3x_5}{x_2}$$



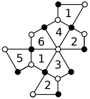
$$\frac{x_2x_3x_5^2+x_1x_3x_5x_6+x_2x_4x_5x_6+x_1x_4x_6^2}{x_1x_2x_3}$$



$$\frac{x_2x_3x_5^2+x_1x_3x_5x_6+x_2x_4x_5x_6+x_1x_4x_6^2}{x_1x_2x_4}$$



$$\frac{(x_2x_5+x_1x_6)(x_1x_3+x_2x_4)(x_3x_5+x_4x_6)^2}{x_1^2x_2^2x_3x_4x_5}$$



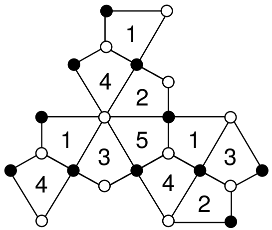
$$\frac{(x_2x_5+x_1x_6)(x_1x_3+x_2x_4)(x_3x_5+x_4x_6)^2}{x_1^2x_2^2x_3x_4x_6}$$

Goal: Combinatorial Formula for Toric Cluster Variables

Example from M. Leoni, S. Neel, and P. Turner (2013 REU):
Mutations at antipodal vertices of dP_3 quiver yield τ-mutation sequences. Resulting Laurent polynomials correspond to Aztec Castles under appropriate weighted enumeration of perfect matchings.

e.g. $1, 2, 3, 4, 1, 2, 5, 6$ yields cluster variable

$$\begin{aligned} & (x_1x_2^2x_3^3x_5^4 + x_2^3x_3^2x_4x_5^4 + 2x_1^2x_2x_3^3x_5^3x_6 + 4x_1x_2^2x_3^2x_4x_5^3x_6 + 2x_2^3x_3x_4^2x_5^3x_6 + x_1^3x_3^3x_5^2x_6^2 \\ + \quad & 5x_1^2x_2x_3^2x_4x_5^2x_6^2 + 5x_1x_2^2x_3x_4^2x_5^2x_6^2 + x_2^3x_4^3x_5^2x_6^2 + 2x_1^3x_3^2x_4x_5x_6^3 + 4x_1^2x_2x_3x_4^2x_5x_6^3 \\ + \quad & 2x_1x_2^2x_4^3x_5x_6^3 + x_1^3x_3x_4^2x_6^4 + x_1^2x_2x_4^3x_6^4)/x_1^2x_2^2x_3^2x_4^2x_6 = \frac{(x_1x_3 + x_2x_4)(x_4x_6 + x_3x_5)^2(x_1x_6 + x_2x_5)^2}{x_1^2x_2^2x_3^2x_4^2x_6} \end{aligned}$$



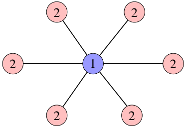
Theorem 1 [Lai-M 2015] Starting from the initial cluster $\{x_1, x_2, \ldots, x_6\}$, the set of cluster variables reachable via toric mutations can be parameterized by $\mathbb{Z}^3$.

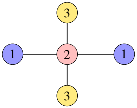
Under this correspondence, the initial cluster bijects to

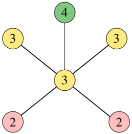
$$[(0,-1,1),(0,-1,0),(-1,0,0),(-1,0,0),(-1,0,1),(0,0,1),(0,0,0)]$$

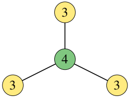
and toric mutations transform the six-tuple in $\mathbb{Z}^3$ as we will illustrate.

Up to symmetry, enough to consider $\mu_1\mu_2$, $\mu_1\mu_4\mu_1\mu_5\mu_1$, and $\mu_1\mu_4\mu_3$.



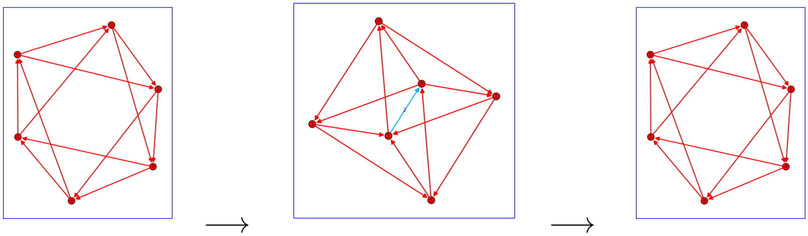




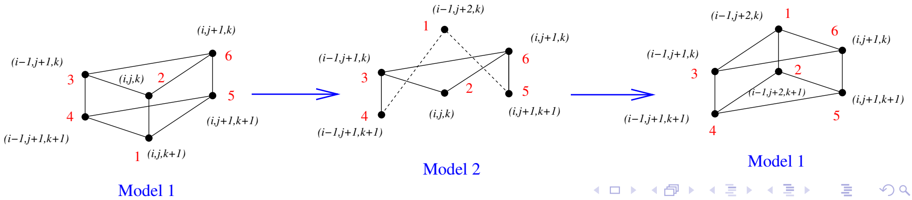


Mutating Model I to Model II and back to Model I

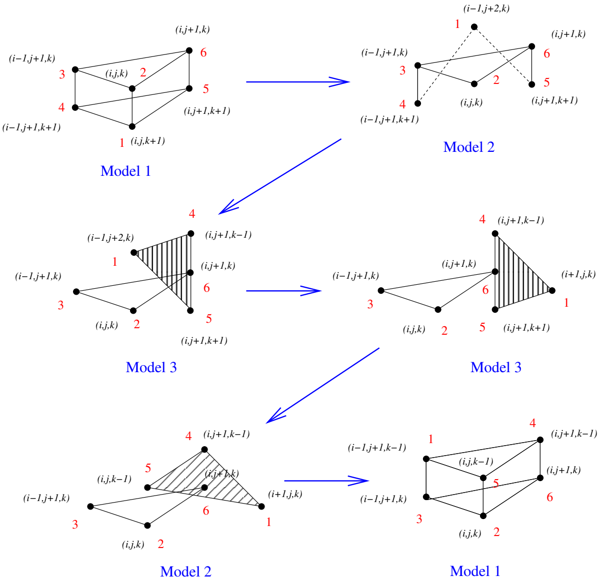
By applying $\mu_1 \circ \mu_2$, $\mu_3 \circ \mu_4$, or $\mu_5 \circ \mu_6$, we mutate the quiver (up to graph isomorphism):



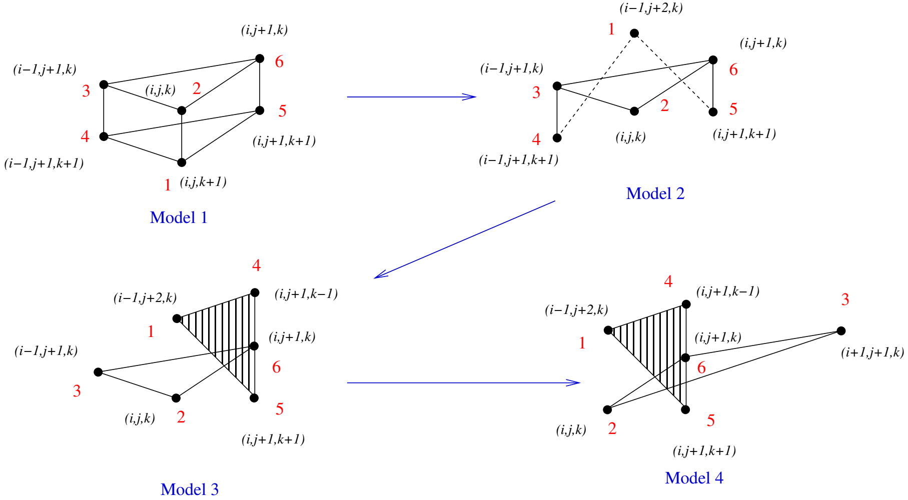
Corresponding action in $\mathbb{Z}^3$ (on triangular prisms):



Illustrating the mutation sequence $\mu_1\mu_4\mu_1\mu_5\mu_1$



Illustrating the mutation sequence $\mu_1\mu_4\mu_3$



Segway: Algebraic Formula for Toric Cluster Variables

$$\text{Let} \quad A = \frac{x_3x_5 + x_4x_6}{x_1x_2}, \quad B = \frac{x_1x_6 + x_2x_5}{x_3x_4}, \quad C = \frac{x_1x_3 + x_2x_4}{x_5x_6},$$

$$D = \frac{x_1x_3x_6 + x_2x_3x_5 + x_2x_4x_6}{x_1x_4x_5}, \text{ and } E = \frac{x_2x_4x_5 + x_1x_3x_5 + x_1x_4x_6}{x_2x_3x_6}.$$

Let $z_i^{j,k}$ be the cluster variable corresponding to $(i,j,k) \in \mathbb{Z}^3$

Theorem 2 [Lai-M 2015] (Extension of [LMNT 2013] and [Lai 2014]):

$$z_i^{j,k} = x_r \;\; A^{\lfloor \frac{(i^2+ij+j^2+1)+i+2j}{3} \rfloor} \; B^{\lfloor \frac{(i^2+ij+j^2+1)+2i+j}{3} \rfloor} \; C^{\lfloor \frac{i^2+ij+j^2+1}{3} \rfloor} \; D^{\lfloor \frac{(k-1)^2}{4} \rfloor} \; E^{\lfloor \frac{k^2}{4} \rfloor}$$

where, working **modulo** 6, we have (cyclically around the dP_3 Quiver)

$$\begin{aligned} r = 6 &\text{ if } 2(i-j)+3k \equiv 0, \quad & r = 4 &\text{ if } 2(i-j)+3k \equiv 1, \\ r = 2 &\text{ if } 2(i-j)+3k \equiv 2, \quad & r = 5 &\text{ if } 2(i-j)+3k \equiv 3, \\ r = 3 &\text{ if } 2(i-j)+3k \equiv 4, \quad & r = 1 &\text{ if } 2(i-j)+3k \equiv 5. \end{aligned}$$

i.e. we **determine** x_r by looking at $(i-j)$ modulo 3 and k modulo 2.

Segway: Algebraic Formula for Toric Cluster Variables

Consequences:

(1) Every cluster variable reachable by toric mutations resides in a cluster

$$\left\{x_1\mathcal{A}_i^j\mathcal{D}^{k+1},\ x_2\mathcal{A}_i^j\mathcal{D}^k,\ x_3\mathcal{A}_{i-1}^{j+1}\mathcal{D}^k,\ x_4\mathcal{A}_{i-1}^{j+1}\mathcal{D}^{k+1},\ x_5\mathcal{A}_i^{j+1}\mathcal{D}^{k+1},\ x_6\mathcal{A}_i^{j+1}\mathcal{D}^k\right\}$$

where $\mathcal{A}_i^j = A^{\lfloor\frac{(i^2+ij+j^2+1)+i+2j}{3}\rfloor}B^{\lfloor\frac{(i^2+ij+j^2+1)+2i+j}{3}\rfloor}C^{\lfloor\frac{i^2+ij+j^2+1}{3}\rfloor}$ and

$$\mathcal{D}^k = D^{\lfloor\frac{(k-1)^2}{4}\rfloor}E^{\lfloor\frac{k^2}{4}\rfloor}.$$

(2) We have conserved quantities

$$A = \frac{x_3x_5+x_4x_6}{x_1x_2} = \frac{\mathcal{A}_{i-1}^{j+2}\ \mathcal{A}_i^j}{\mathcal{A}_{i-1}^{j+1}\ \mathcal{A}_i^{j+1}},\ D = \frac{x_2^2A+x_3x_6}{x_4x_5} = \frac{x_3^2B+x_2x_6}{x_1x_5} = \frac{x_6^2C+x_2x_3}{x_1x_4} = \frac{\mathcal{D}^{k+1}\mathcal{D}^{k-1}}{\mathcal{D}^k\mathcal{D}^k}$$

$$B = \frac{x_2x_5+x_1x_6}{x_3x_4} = \frac{\mathcal{A}_{i+1}^j\ \mathcal{A}_{i-1}^{j+1}}{\mathcal{A}_i^j\ \mathcal{A}_i^{j+1}},\ E = \frac{x_1^2A+x_4x_5}{x_3x_6} = \frac{x_4^2B+x_1x_5}{x_2x_6} = \frac{x_5^2C+x_1x_4}{x_2x_3} \text{ and}$$

$$C = \frac{x_1x_3+x_2x_4}{x_1x_2} = \frac{\mathcal{A}_{i-1}^j\ \mathcal{A}_i^{j+1}}{\mathcal{A}_i^j\ \mathcal{A}_{i-1}^{j+1}} \qquad \text{coming from the cluster mutation relations}$$

Segway: Algebraic Formula for Toric Cluster Variables

$$z_{i-1}^{j+2,k} z_i^{j,k+1} =^{(R4)} z_{i-1}^{j+1,k} z_i^{j+1,k+1} + z_{i-1}^{j+1,k+1} z_i^{j+1,k}$$

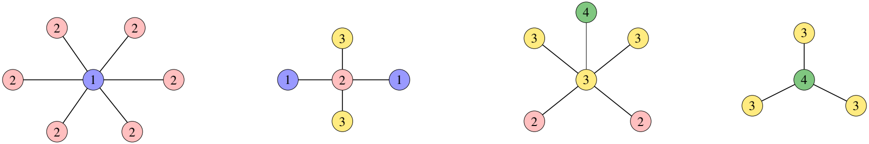
$$z_{i-1}^{j+1,k+1} z_i^{j+1,k-1} =^{(R1)} z_{i-1}^{j+2,k} z_i^{j,k} + z_{i-1}^{j+1,k} z_i^{j+1,k}$$

$$z_{i-1}^{j+2,k} z_{i+1}^{j,k} =^{(R2)} z_i^{j+1,k-1} z_i^{j+1,k+1} + (z_i^{j+1,k})^2$$

$$z_i^{j+1,k+1} z_i^{j,k-1} =^{(R1)} z_{i+1}^{j,k} z_{i-1}^{j+1,k} + z_i^{j,k} z_i^{j+1,k}$$

$$z_{i-1}^{j+1,k-1} z_{i+1}^{j,k} =^{(R4)} z_i^{j,k} z_i^{j+1,k-1} + z_i^{j,k-1} z_i^{j+1,k}$$

$$(\mathrm{I}) \longleftrightarrow (\mathrm{II}) \longleftrightarrow (\mathrm{III}) \longleftrightarrow (\mathrm{III}) \longleftrightarrow (\mathrm{II}) \longleftrightarrow (\mathrm{I})$$

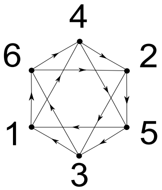


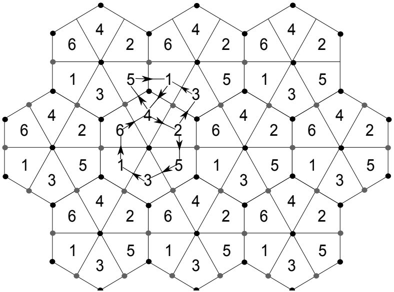
From work of Lai "A generalization of Aztec Dragons": Unweighted versions of these recurrences called type (R1), (R2), or (R4) recurrences.

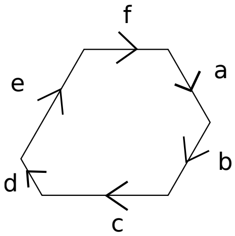
Introducing Contours on the del Pezzo 3 Lattice

We wish to understand combinatorial interpretations for more general **toric** mutation sequences, not necessarily periodic or coming from mutating at antipodes.

To this end, we cut out subgraphs of the dP_3 lattice by using six-sided contours



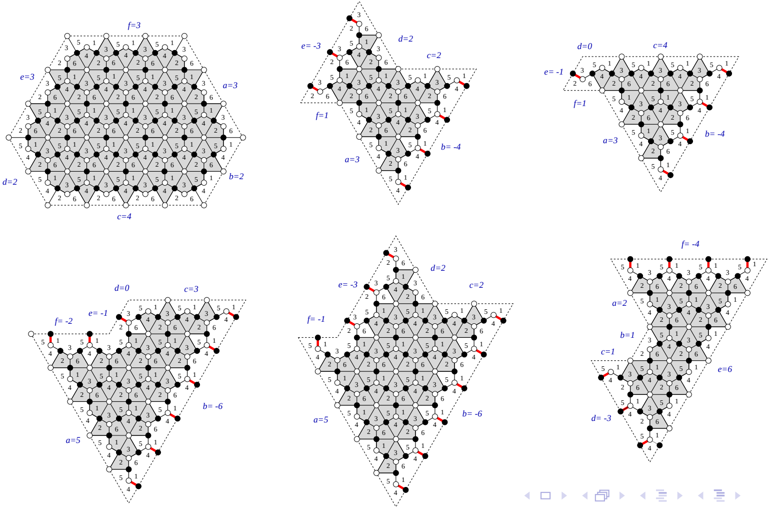




indexed as (a, b, c, d, e, f) with $a, b, c, d, e, f \in \mathbb{Z}$.

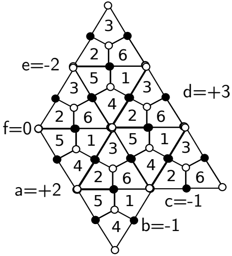
Sign determines direction & Magnitude determines length

(1) $\mathcal{G}(3,2,4,2,3,3)$, (2) $\mathcal{G}(3,-4,2,2,-3,1)$, (3) $\mathcal{G}(3,-4,4,0,-1,1)$,
(4) $\mathcal{G}(5,-6,3,0,-1,-2)$, (5) $\mathcal{G}(5,-6,2,2,-3,-1)$, (6) $\mathcal{G}(2,1,1,-3,6,-4)$



Turning a Contour $\mathcal{C}$ into the Subgraph $\mathcal{G}(\mathcal{C})$

1) Draw the contour $\mathcal{C}$ on top of the dP_3 lattice starting from a degree 6 white vertex.
2) For all sides of "positive" length, we erase all the **black** vertices.
3) For all sides of "negative" length, we erase all the **white** vertices.
For sides of "zero length" (between two sides of positive length), we erase the **white corner** or keep it depending on convexity.
4) After removing "dangling" edges and their incident faces, remaining subgraph inside contour is $\mathcal{G}(\mathcal{C})$.



e.g.

Turning a Contour $\mathcal{C}$ into the Subgraph $\mathcal{G}(\mathcal{C})$

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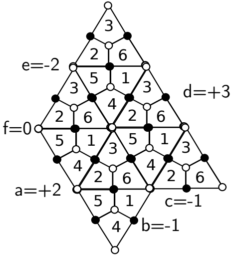

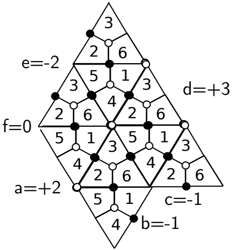

Turning a Contour $\mathcal{C}$ into the Subgraph $\mathcal{G}(\mathcal{C})$

1) Draw the contour $\mathcal{C}$ on top of the dP_3 lattice starting from a degree 6 white vertex.
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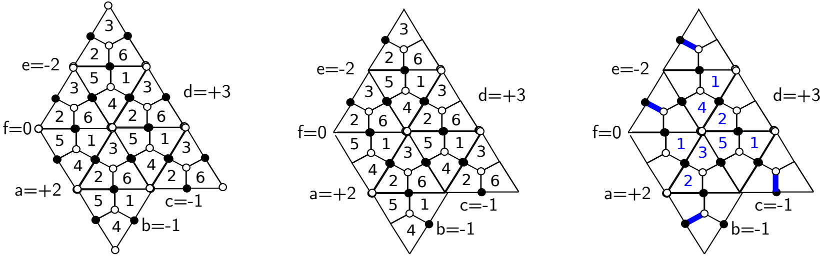
e.g.



Further Examples of Subgraphs $\mathcal{G}(\mathcal{C})$ from Contours $\mathcal{C}$

1) Draw the contour $\mathcal{C}$ on top of the dP_3 lattice starting from a degree 6 white vertex.
2) For all sides of "positive" length, we erase all the **black** vertices.
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For sides of "zero length" (between two sides of positive length), we erase the **white corner** or keep it depending on convexity.
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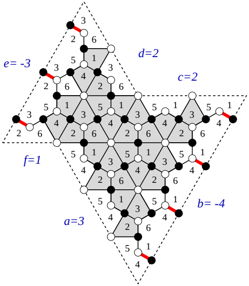
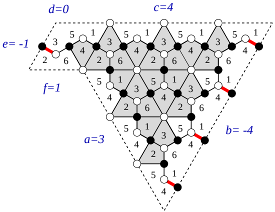
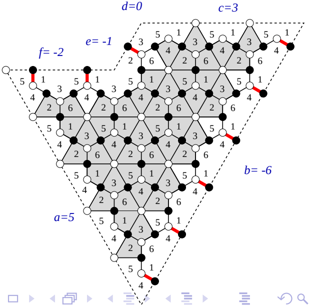
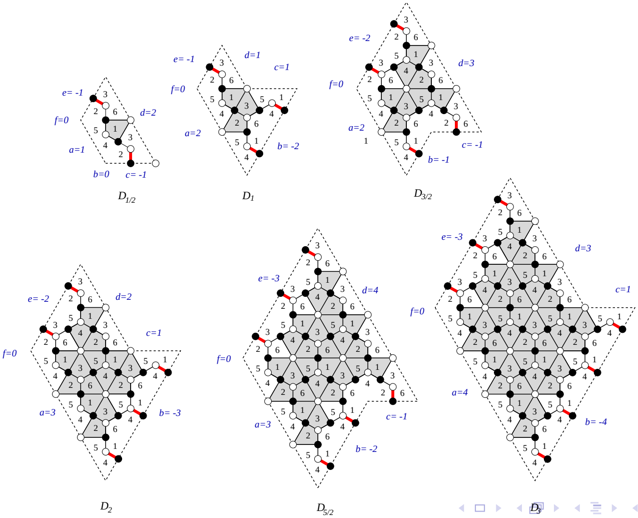






Aztec Dragons (Ciucu, Cottrell-Young, Propp) Revisited

$$D_{n+1/2} = \mathcal{G}(n+1, -n, -1, n+2, -n-1, 0), \quad D_n = \mathcal{G}(n+1, -n-1, 1, n, -n, 0).$$



Turning Subgraphs into Laurent Polynomials

$$G \longrightarrow cm(G) \sum_{M = \text{a perfect matching of } G} x(M), \text{ where}$$

$x(M) = \prod_{\text{edge } e \in M} \frac{1}{x_i x_j}$ (for edge e straddling faces i and j),

$cm(G)$ = the covering monomial of the graph G_n (which records what face labels are contained in G and along its boundary).

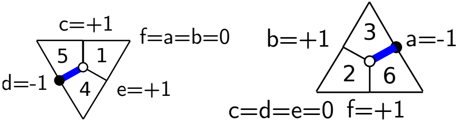
Remark: This is a reformulation of weighting schemes appearing in works such as Speyer ("Perfect Matchings and the Octahedron Recurrence"), Goncharov-Kenyon ("Dimers and cluster integrable systems"), and Di Francesco ("T-systems, networks and dimers").

Alternative definition of cm(G): We record all face labels inside contour and then divide by face labels straddling dangling edges.

Initial cluster $\{x_1, x_2, \ldots, x_6\}$ in terms of contours

Consider the following six special contours

$$C_1 = (0,0,1,-1,1,0), \quad C_2 = (-1,1,0,0,0,1),$$
$$C_3 = (0,1,-1,1,0,0), \quad C_4 = (1,0,0,0,1,-1),$$
$$C_5 = (1,-1,1,0,0,0), \quad C_6 = (0,0,0,1,-1,1).$$



Applying our general algorithm, $\mathcal{G}(C_i)$'s correspond to graphs consisting of a single edge and a triangle of faces.

Initial cluster $\{x_1, x_2, \ldots, x_6\}$ in terms of contours

Consider the following six special contours

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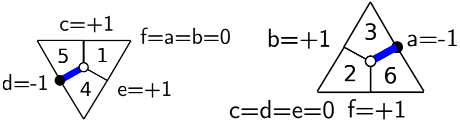
$$C_3 = (0,1,-1,1,0,0), \quad C_4 = (1,0,0,0,1,-1),$$
$$C_5 = (1,-1,1,0,0,0), \quad C_6 = (0,0,0,1,-1,1).$$



Applying our general algorithm, $\mathcal{G}(C_i)$'s correspond to graphs consisting of a single edge and a triangle of faces.

Using $G \longrightarrow cm(G) \sum_{M \text{ = a perfect matching of } G} x(M)$, we see

$$cm(\mathcal{G}(C_1)) = x_1x_4x_5 \text{ and } x(M) = \frac{1}{x_4x_5}, \text{ hence } G \longrightarrow \frac{x_1x_4x_5}{x_4x_5} = x_1$$

Similar calculations show $\mathcal{G}(C_i)) \longleftrightarrow x_i$ for $i \in \{1, 2, \ldots, 6\}$.

Theorem 3 [Lai-M 2015]

Theorem (Reformulation of [Leoni-M-Neel-Turner 2014]): Let $Z^S = [z_1, z_2, \ldots, z_6]$ be the cluster obtained after applying a toric mutation sequence S to the initial cluster $\{x_1, x_2, \ldots, x_6\}$.

Let $w(G) = cm(G) \sum_{M \text{ a perfect matching of } G} x(M)$.

Let $\mathcal{G}(\mathcal{C}_i)$ be the subgraph cut out by the contour $\mathcal{C}_i$.

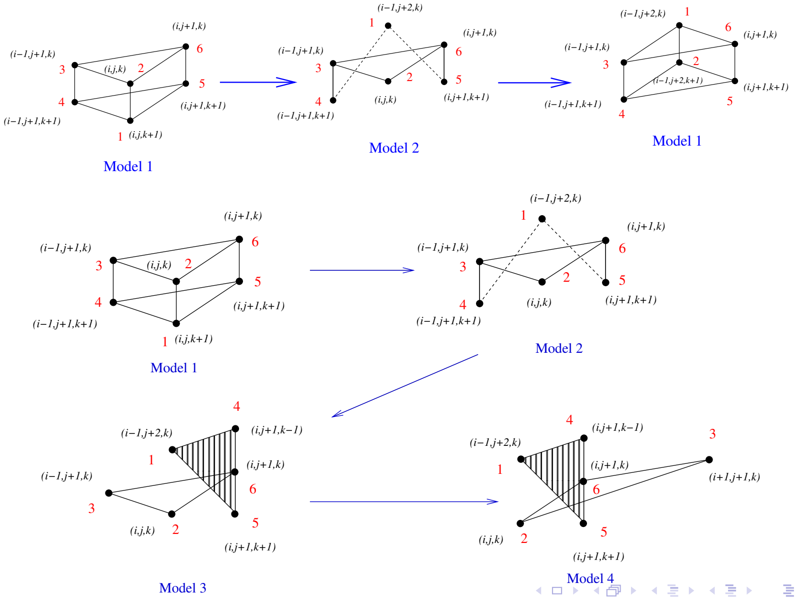
Then $\mathbf{Z^S} = [\mathbf{w}(\mathcal{G}(\mathcal{C}^{\mathbf{S}}_{\mathbf{1}}), \mathbf{w}(\mathcal{G}(\mathcal{C}^{\mathbf{S}}_{\mathbf{2}}), \ldots, \mathbf{w}(\mathcal{G}(\mathcal{C}^{\mathbf{S}}_{\mathbf{6}})]$ where $\mathcal{C}^{S_1}, \mathcal{C}^{S_2}, \ldots, \mathcal{C}^{S_6}$ are defined as follows:

1) Start with the six-tuple
$[(0, -1, 1), (0, -1, 0), (-1, 0, 0), (-1, 0, 0), (-1, 0, 1), (0, 0, 1), (0, 0, 0)]$ in $\mathbb{Z}^3$.
2) Toric Mutations transform this six-tuple as illustrated earlier.
3) Map from $\mathbb{Z}^3$ to $\mathbb{Z}^6$:

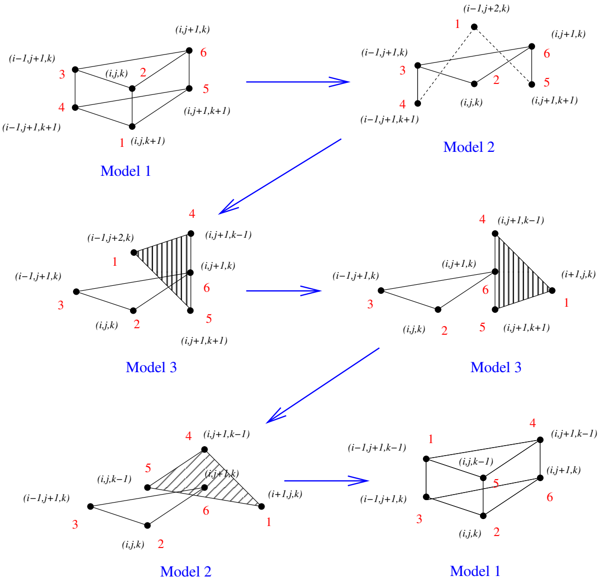
$$(i, j, k) \to (a, b, c, d, e, f) = (j + k, -i - j - k, i + k, j - k + 1, -i - j + k - 1, i - k + 1)$$

and use these six six-tuples to define the contours $\mathcal{C}^{S_1}, \mathcal{C}^{S_2}, \ldots, \mathcal{C}^{S_6}$.

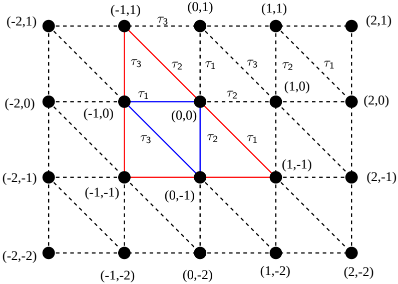
Reminder of $\mathbb{Z}^3$ transformations



Reminder of $\mathbb{Z}^3$ transformations



Example 1: mutation sequence $\mu_1\mu_2\mu_3\mu_4\mu_5\mu_6$



We start at the initial prism $[(0,-1,1),(0,-1,0),(-1,0,0),(-1,0,0),(-1,0,1),(0,0,1),(0,0,0)]$. Applying the mutation sequence $\mu_1, \mu_2, \mu_3\mu_4\mu_5\mu_6$ corresponds to the walk

$$\{(0,-1),(-1,0),(0,0)\} \to \{(-1,1),(-1,0),(0,0)\} \to \{(-1,1),(0,1),(0,0)\} \to \{(-1,1),(0,1),(-1,2)\}$$

Projecting to $\mathbb{Z}^2$ **using** $(i,j) \leftrightarrow (j,-i-j,i,j+1,-i-j-1,i+1)$ and $(j+1,-i-j-1,i+1,j,-i-j,i)$.

$C_1 = (0,0,1,-1,1,0)$, $C_2 = (-1,1,0,0,0,1)$, $C_3 = (0,1,-1,1,0,0)$,
$C_4 = (1,0,0,0,1,-1)$, $C_5 = (1,-1,1,0,0,0)$, $C_6 = (0,0,0,1,-1,1)$.

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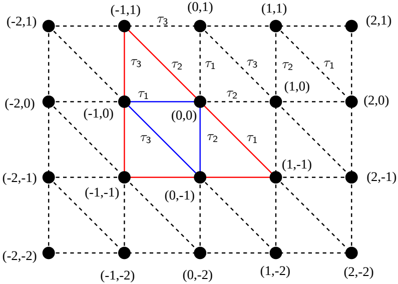

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$$\{(0,-1),(-1,0),(0,0)\} \to \{(-1,1),(-1,0),(0,0)\} \to \{(-1,1),(0,1),(0,0)\} \to \{(-1,1),(0,1),(-1,2)\}$$

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$C_1' = (2,-1,0,1,0,-1)$, $C_2' = (1,0,-1,2,-1,0)$, $C_3 = (0,1,-1,1,0,0)$, $C_4 = (1,0,0,0,1,-1)$, $C_5 = (1,-1,1,0,0,0)$, $C_6 = (0,0,0,1,-1,1)$.

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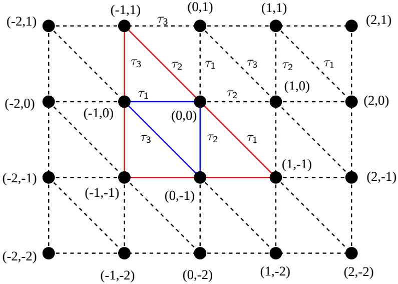

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$$\{(0,-1),(-1,0),(0,0)\} \to \{(-1,1),(-1,0),(0,0)\} \to \{(-1,1),(0,1),(0,0)\} \to \{(-1,1),(0,1),(-1,2)\}$$

Projecting to $\mathbb{Z}^2$ using $(i,j) \leftrightarrow (j,-i-j,i,j+1,-i-j-1,i+1)$ and $(j+1,-i-j-1,i+1,j,-i-j,i)$.

$C_1' = (2,-1,0,1,0,-1)$, $C_2' = (1,0,-1,2,-1,0)$, $C_3' = (1,-1,0,2,-2,1)$, $C_4' = (2,-2,1,1,-1,0)$, $C_5 = (1,-1,1,0,0,0)$, $C_6 = (0,0,0,1,-1,1)$.

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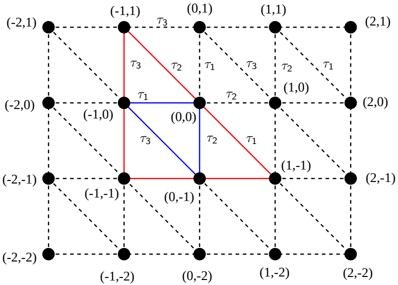

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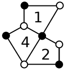
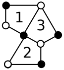
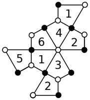
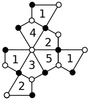
$$\frac{x_4x_6+x_3x_5}{x_2}$$

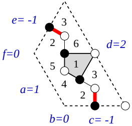

$$\frac{x_3x_5+x_4x_6}{x_1}$$

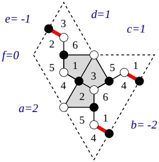

$$\frac{x_2x_3x_5^2+x_1x_3x_5x_6+x_2x_4x_5x_6+x_1x_4x_6^2}{x_1x_2x_4}$$

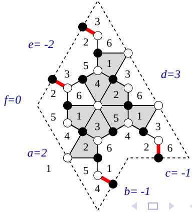

$$\frac{x_2x_3x_5^2+x_1x_3x_5x_6+x_2x_4x_5x_6+x_1x_4x_6^2}{x_1x_2x_3}$$


$$\frac{(x_2x_5+x_1x_6)(x_1x_3+x_2x_4)(x_3x_5+x_4x_6)^2}{x_1^2x_2^2x_3x_4x_6}$$


$$\frac{(x_2x_5+x_1x_6)(x_1x_3+x_2x_4)(x_3x_5+x_4x_6)^2}{x_1^2x_2^2x_3x_4x_5}$$





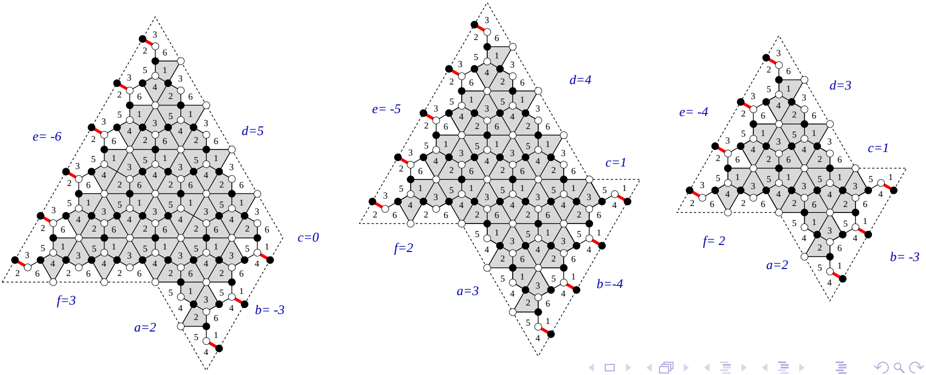


Example 2: $S = \tau_1\tau_2\tau_3\tau_1\tau_2\tau_3\tau_2\tau_1\tau_4$

We reach $\{(1,3),(1,2),(0,3)\}$ from applying $\tau_1\tau_2\tau_3\tau_1\tau_2\tau_3\tau_2\tau_1$ ($\tau_1 = \mu_1\mu_2$, $\tau_2 = \mu_3\mu_4$, and $\tau_3 = \mu_5\mu_6$) and then $\tau_4 = \mu_1\mu_4\mu_1\mu_5\mu_1$ yields $\mathcal{C}^S = [\sigma^{-1}\mathcal{C}_1^3, \mathcal{C}_1^3, \mathcal{C}_1^2, \sigma^{-1}\mathcal{C}_1^2, \sigma^{-1}\mathcal{C}_0^3, \mathcal{C}_0^3] =$

$$[(2,-3,0,5,-6,3),(3,-4,1,4,-5,2),(2,-3,1,3,-4,2),$$

$$(1,-2,0,4,-5,3),(2,-2,-1,5,-5,2),(3,-3,0,4,-4,1)].$$



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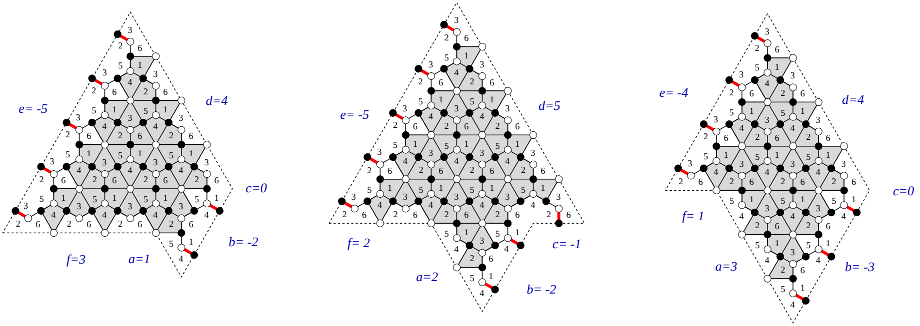
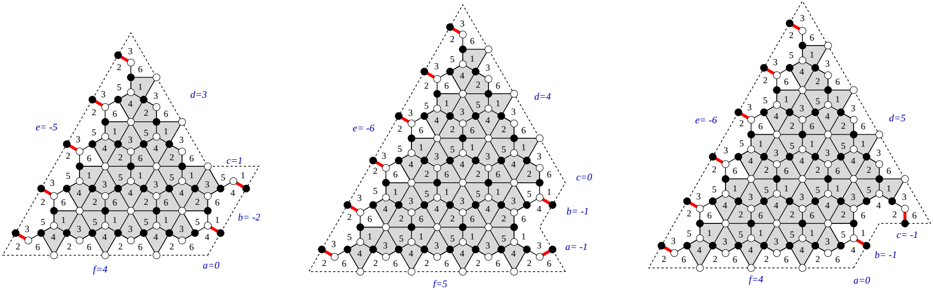


Example 3: $S = \tau_1\tau_2\tau_3\tau_1\tau_3\tau_2\tau_1\tau_4\tau_5$

$$[(0,-2,1,3,-5,4),(-1,-1,0,4,-6,5),(0,-1,-1,5,-6,4),$$

$$(1,-2,0,4,-5,3),(0,-1,0,3,-4,3),(-1,0,-1,4,-5,4)].$$



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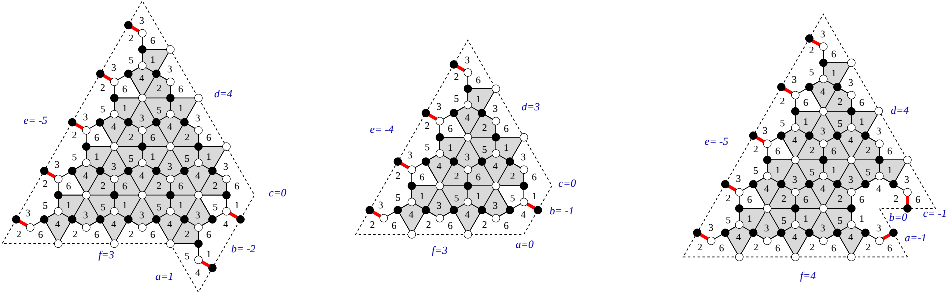
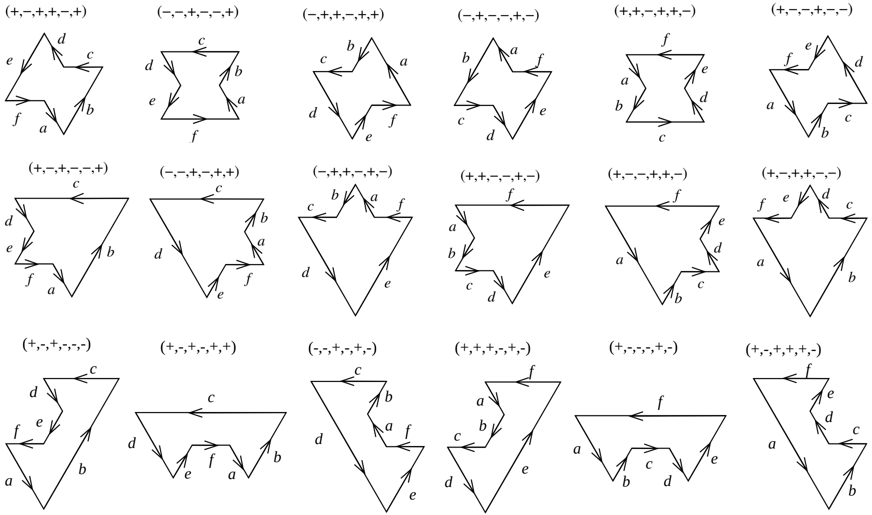
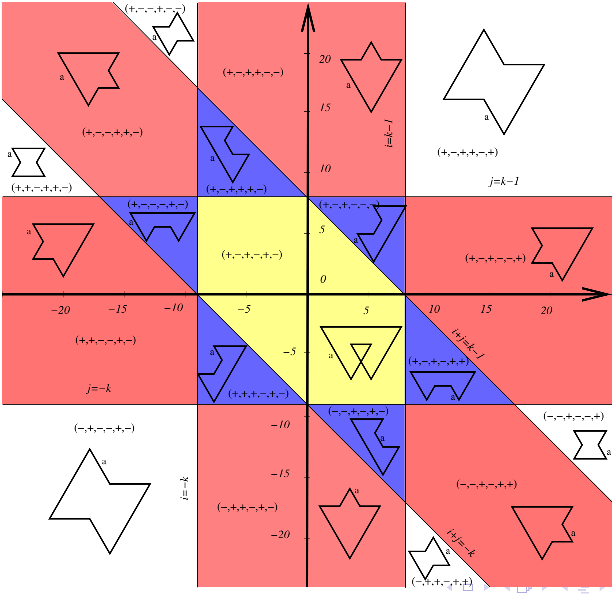
$$(1,-2,0,4,-5,3),(0,-1,0,3,-4,3),(-1,0,-1,4,-5,4)].$$



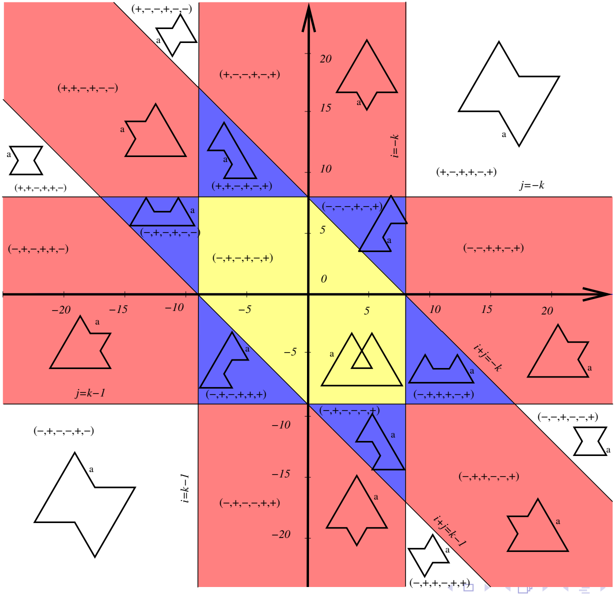
Possible Shapes of Aztec Castles



Cross-section when k positive



Cross-section when k negative

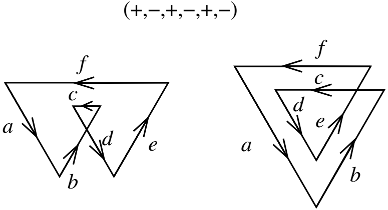


Self-intersecting Contours

Algebraic formula

$$z_i^{j,k} = x_r \quad A^{\lfloor \frac{(i^2+ij+j^2+1)+i+2j}{3} \rfloor} \; B^{\lfloor \frac{(i^2+ij+j^2+1)+2i+j}{3} \rfloor} \; C^{\lfloor \frac{i^2+ij+j^2+1}{3} \rfloor} \; D^{\lfloor \frac{(k-1)^2}{4} \rfloor} \; E^{\lfloor \frac{k^2}{4} \rfloor}$$

still works for (a, b, c, d, e, f) when alternating in signs but combinatorial formula for such cases open.



Work in progress (with David Speyer): Conjectural Double-Dimer combinatorial interpretation for self-intersecting contours.

Sketching the Proof of Theorem 3 [Lai-M 2015]

We use **Kuo's Method of Graphical Condensation** for counting Perfect Matchings. We isolate four vertices $\{a, b, c, d\}$ in our graph on the boundary of the contour.

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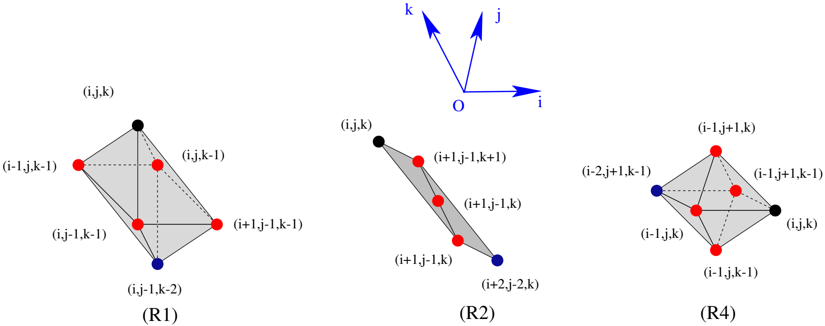
Remark: Further, we will define 15 different types of condensations by choosing 4 out of 6 points. These 15 condensations correspond to the 15 possible **toric mutations**, up to symmetry.

Degenerate Octahedra projected from $\mathbb{Z}^6 \to \mathbb{Z}^3$



Crash Course on Kuo Condensation

Let $G = (V_1, V_2, E)$ be a (weighted) planar bipartite graph and p_1, p_2, p_3, p_4 are four vertices appearing in cyclic order on a face of G.

Theorem (Balanced Kuo Condensation) [Theorem 5.1 in [Kuo]]
Let $|V_1| = |V_2|$ with $p_1, p_3 \in V_1$ and $p_2, p_4 \in V_2$. Then

$$\begin{aligned} w(G)w(G - \{p_1, p_2, p_3, p_4\}) &= w(G - \{p_1, p_2\})w(G - \{p_3, p_4\}) \\ &+ w(G - \{p_1, p_4\})w(G - \{p_2, p_3\}). \end{aligned}$$

Theorem (Unbalanced Kuo Condensation) [Theorem 5.3 in [Kuo]]
Let $|V_1| = |V_2| + 1$ with $p_1, p_2, p_3 \in V_1$ and $p_4 \in V_2$. Then

$$\begin{aligned} w(G - \{p_2\})w(G - \{p_1, p_3, p_4\}) &= w(G - \{p_1\})w(G - \{p_2, p_3, p_4\}) \\ &+ w(G - \{p_3\})w(G - \{p_1, p_2, p_4\}). \end{aligned}$$

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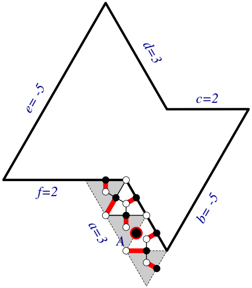
Theorem **(Non-alternating Balanced) [Theorem 5.2 in [Kuo]]**
Let $|V_1| = |V_2|$ with $p_1, p_2 \in V_1$ and $p_3, p_4 \in V_2$. Then

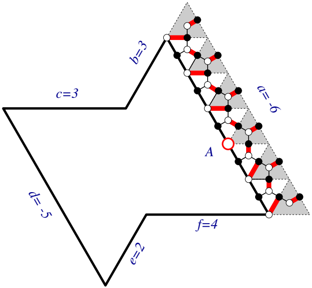
$$\begin{aligned} w(G - \{p_1, p_4\})w(G - \{p_2, p_3\}) &= w(G)w(G - \{p_1, p_2, p_3, p_4\}) \\ &+ w(G - \{p_1, p_3\})w(G - \{p_2, p_4\}). \end{aligned}$$

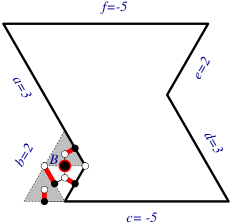
Theorem **(Monochromatic Condensation) [Theorem 5.4 in [Kuo]]**
Let $|V_1| = |V_2| + 2$ with $p_1, p_2, p_3, p_4 \in V_1$. Then

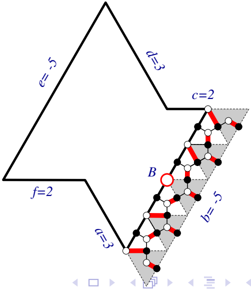
$$\begin{aligned} w(G - \{p_1, p_3\})w(G - \{p_2, p_4\}) &= w(G - \{p_1, p_2\})w(G - \{p_3, p_4\}) \\ &+ w(G - \{p_1, p_4\})w(G - \{p_2, p_3\}). \end{aligned}$$

How we pick vertices $A, B, \ldots, F$









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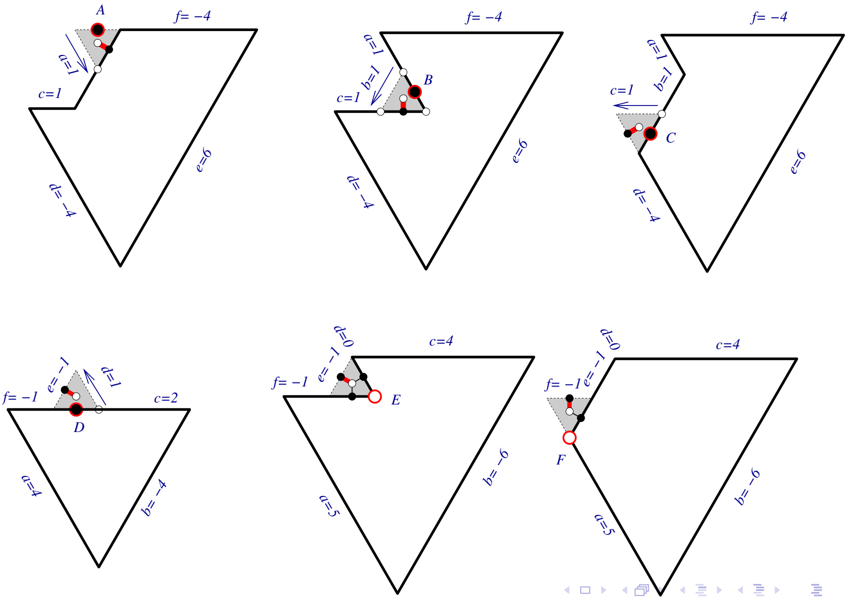
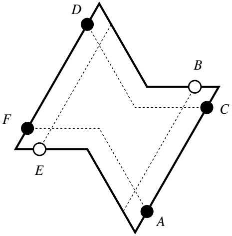


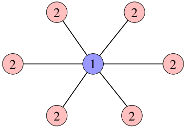
Removing points distance 2 apart, recurrence (R4) in [Lai]

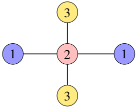


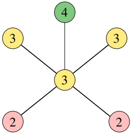
Consider the graph whose shape corresponds to the $i \geq 1$, $j \geq -1$, small $|k|$ case. Removing

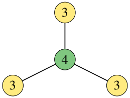
$$\{A\bullet, C\bullet\},\ \{B\circ, D\bullet\},\ \{C\bullet, E\circ\},\ \{D\bullet, F\bullet\},\ \{A\bullet, E\circ\},\ \text{or } \{B\circ, F\bullet\}$$

correspond to mutations between Model I and Model II.

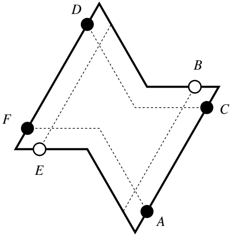








Removing points distance 1 apart, recurrence (R1) in [Lai]



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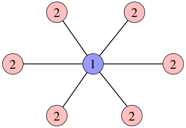
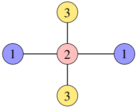
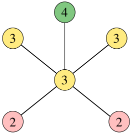
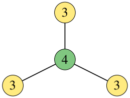
$$\{A\bullet, B\circ\},\ \{B\circ, C\bullet\},\ \{C\bullet, D\bullet\},\ \{D\bullet, E\circ\},\ \{E\circ, F\bullet\},\ \text{or}\ \{A\bullet, F\bullet\}$$

correspond to mutations between Model II and Model III.

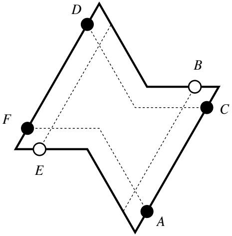








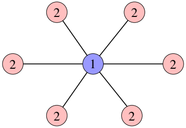
Removing points distance 3 apart, recurrence (R2) in [Lai]

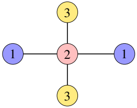
Consider the graph 

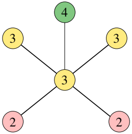
whose shape corresponds to the $i \geq 1$, $j \geq -1$, small $|k|$ case. Removing

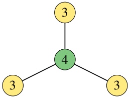
$$\{A\bullet, C\bullet\},\ \{B\circ, E\circ\},\ \text{or}\ \{C\bullet, F\bullet\}$$

correspond to mutations between Model III and itself or Model IV.

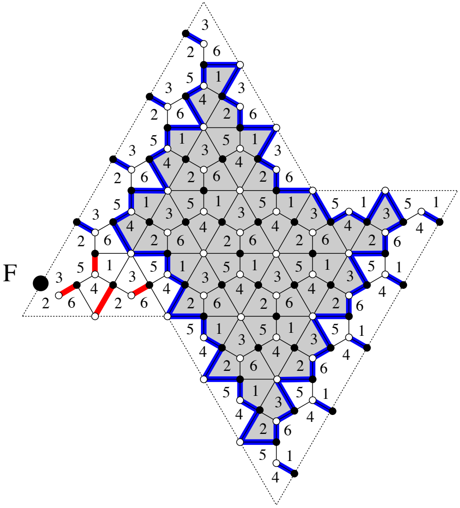


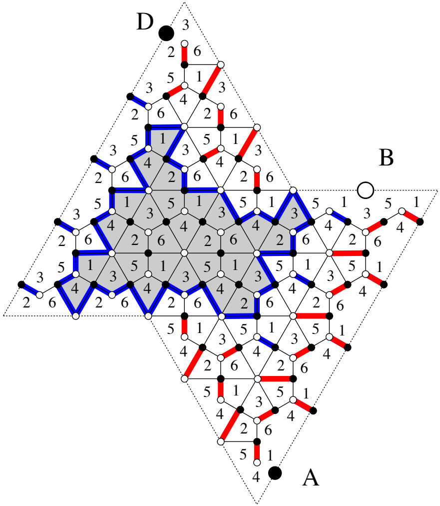




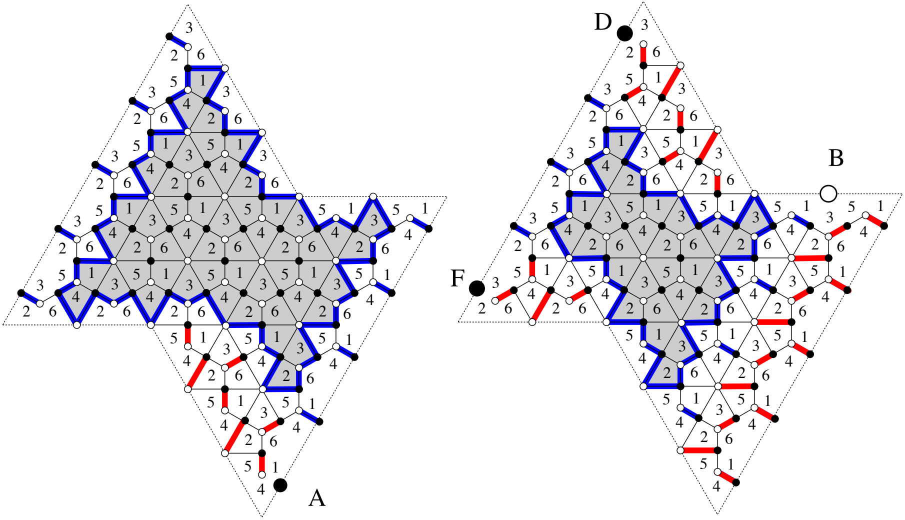


Unbalanced Case: $w(\sigma\mathcal{C}_i^j)w(\mathcal{C}_{i-1}^{j+2}) = \cdots + \ldots$

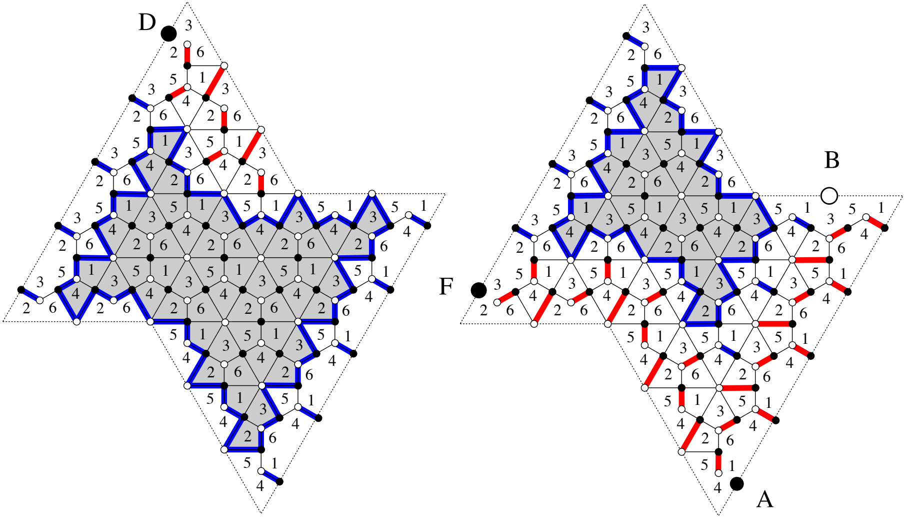




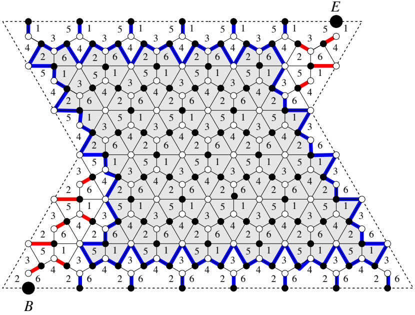
Unbalanced Case: $\cdots = w(\mathcal{C}_{i-1}^{j+1})w(\sigma\mathcal{C}_{i}^{j+1}) + \ldots$

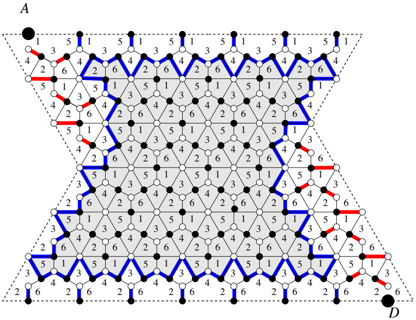


Unbalanced Case: $\cdots = \cdots + w(\sigma\mathcal{C}_{i-1}^{j+1})w(\mathcal{C}_{i}^{j+1})$

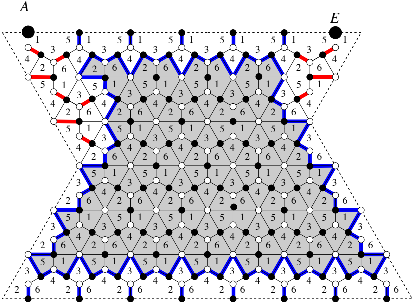


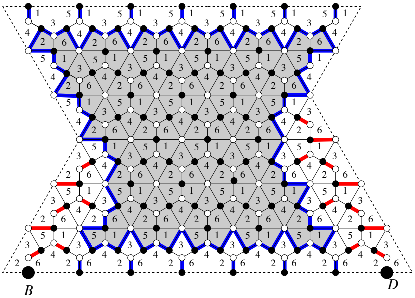
Monochromatic Case $z_{i-1}^{j+2,k} z_{i+1}^{j,k} =^{(R2)} \ldots + \ldots$



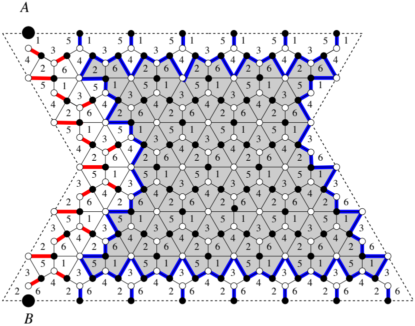


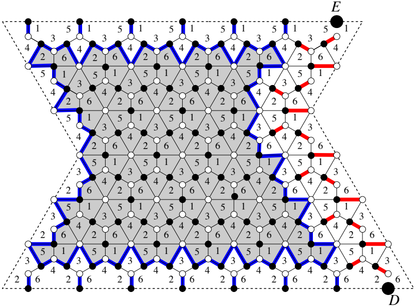
Monochromatic Case $\cdots =^{(R2)} z_i^{j+1,k-1} z_i^{j+1,k+1} + \ldots$





Monochromatic Case $\cdots =^{(R2)} \cdots + (z_i^{j+1,k})^2$





Additional Open Questions

Question: Work of Di Francesco and Soto-Garrido studied arctic curves from T-systems. Can we adapt these methods to obtain Limit Shapes for the graphs arising from toric mutations sequences for the dP_3 quiver?

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Question: Finally, we focused on cluster expansions assuming the initial cluster was Model I. What if we start from a different model. It appears that it the initial cluster is of Model IV that one gets Hexagonal dungeons. T. Lai and I plan to do further work on **Dungeons and Dragons**.

Thanks for Coming (Slides at http://math.umn.edu/∼musiker/MIT16.pdf)

- Richard Eager and Sebastian Franco, *Colored BPS Pyramid Partition Functions, Quivers and Cluster Transformations*, arXiv:1112.1132.
- Eric Kuo, *Applications of Graphical Condensation for Enumerating Matchings and Tilings*, *Theoretical Computer Science*, 319:29–57.
- Sicong Zhang, *Cluster Variables and Perfect Matchings of Subgraphs of the dP_3 Lattice*, 2012 REU Report, arXiv:1511.06055.
- Tri Lai, *A Generalization of Aztec Dragons*, arXiv:1504.00303, to appear in *Graphs and Combinatorics*.

- *Gale-Robinson Sequences and Brane Tilings* (with In-Jee Jeong and and Sicong Zhang), *Discrete Mathematics and Theoretical Computer Science Proc.* **AS** (2013), 737-748.
- *Aztec Castles and the dP3 Quiver* (with Megan Leoni, Seth Neel, and Paxton Turner), *Journal of Physics A*: Math. Theor. 47 474011, arXiv:1308.3926.
- *Beyond Aztec Castles: Toric Cascades in the dP_3 Quiver* (with Tri Lai), arXiv:1512.00507.