Combinatorial Expansion Formulas for Decorated Super Teichmüller Spaces

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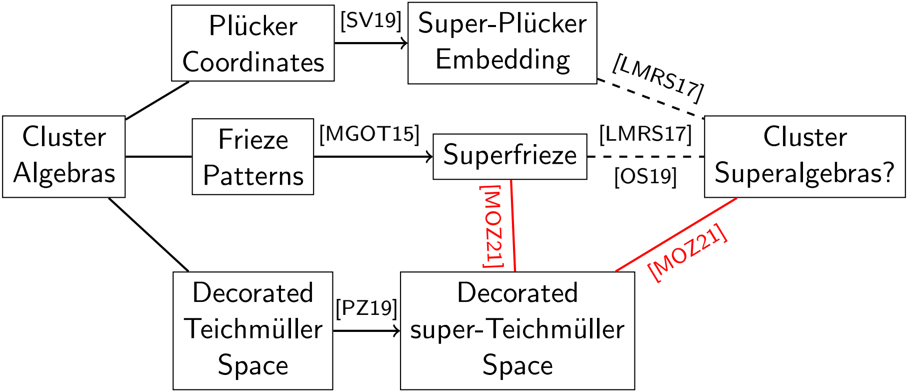
Joint work with Nicholas Ovenhouse and Sylvester Zhang.

http://www-users.math.umn.edu/~musiker/Paris21.pdf

https://arxiv.org/pdf/2102.09143.pdf

Motivation



This Talk

Providing combinatorial formulas for λ-lengths and μ-invariants in decorated super-Teichmüller spaces assoicated to polygons, and their relationship to superfriezes and (steps towards) super cluster algebras.

What is a Cluster Algebra?

Definition (Sergey Fomin and Andrei Zelevinsky 2001)

A cluster algebra $\mathcal{A}$ (of geometric type) is a subalgebra of $k(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m})$ constructed cluster by cluster by certain exchange relations.

Generators:

Specify an initial finite set of them, a Cluster, $\{x_1, x_2, \ldots, x_{n+m}\}$.

Construct the rest via Binomial Exchange Relations:

$$x_\alpha x_\alpha' = \prod x_{\gamma_i}^{d_i^+} + \prod x_{\gamma_i}^{d_i^-}.$$

The set of all such generators are known as Cluster Variables, and the initial pattern B of exchange relations determines the Seed.

Relations:

Induced by the Binomial Exchange Relations.

Example: Coordinate Ring of Grassmannian$(2, n+3)$

Let $Gr_{2,n+3} = \{V | V \subset \mathbb{C}^{n+3},\ \dim V = 2\}$ planes in $(n+3)$-space

Elements of $Gr_{2,n+3}$ represented by 2-by-$(n+3)$ matrices of full rank.

Plücker coordinates $p_{ij}(M) =$ det of 2-by-2 submatrices in columns i and j.

The coordinate ring $\mathbb{C}[Gr_{2,n+3}]$ is generated by all the p_{ij}'s for $1 \leq i < j \leq n+3$ subject to the Plücker relations given by the 4-tuples

$$p_{ik}p_{j\ell} = p_{ij}p_{k\ell} + p_{i\ell}p_{jk} \text{ for } i < j < k < \ell.$$

Claim. $\mathbb{C}[Gr_{2,n+3}]$ has the structure of a cluster algebra. Clusters are each maximal algebraically independent sets of p_{ij}'s.

Each have size $(2n+3)$ where $(n+3)$ of the variables are frozen and n of them are exchangeable.

Example: Coordinate Ring of Grassmannian$(2, n+3)$

Cluster algebra structure of $Gr_{2,n+3}$ as a triangulated $(n+3)$-gon.

Frozen Variables / Coefficients $\longleftrightarrow$ sides of the $(n+3)$-gon

Cluster Variables $\longleftrightarrow \{p_{ij} : |i-j| \neq 1 \mod (n+3)\} \longleftrightarrow$ diagonals

Seeds $\longleftrightarrow$ triangulations of the $(n+3)$-gon

Clusters $\longleftrightarrow$ Set of p_{ij}'s corresponding to a triangulation

Can exchange between various clusters by flipping between triangulations.

This is called mutation, and we will present a detailed example later.

Frieze Patterns

A Conway-Coxeter *frieze* $\mathcal{F} = \{\mathcal{F}_{ij}\}_{i \leq j}$ is an array of rows such that $\mathcal{F}_{i,i} = 0$ and $\mathcal{F}_{i,i+1} = 1$, and, for every diamond

$$\begin{array}{ccc} & B & \\ A & & D \\ & C & \end{array}$$

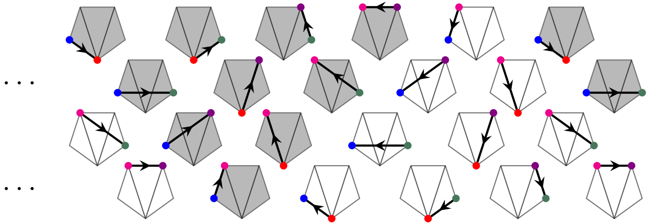
of entries in the the frieze, the equation $AD - BC = 1$ is satisfied.

$$\begin{array}{cccccccccccccccc}
\cdots & & \mathbf{0} & & \mathbf{0} & & \mathbf{0} & & \mathbf{0} & & \mathbf{0} & & \mathbf{0} & & \mathbf{0} & \\
 & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \cdots \\
\cdots & & \mathbf{3} & & \mathbf{1} & & \mathbf{2} & & \mathbf{2} & & \mathbf{1} & & \mathbf{3} & & \mathbf{1} & \\
 & \mathbf{2} & & \mathbf{2} & & \mathbf{1} & & \mathbf{3} & & \mathbf{1} & & \mathbf{2} & & \mathbf{2} & & \cdots \\
\cdots & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} &
\end{array}$$

Frieze Patterns

Diagonals of a polygon correspond to entries of a finite frieze. The diamond condition $AD - BC = 1$ stands in for the Plücker relation $p_{ik}p_{j\ell} = p_{ij}p_{k\ell} + p_{i\ell}p_{jk}$ for $i < j < k < \ell$.

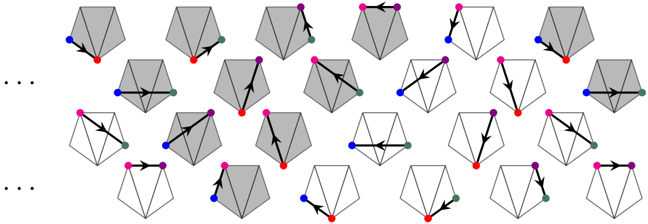
$$\begin{array}{ccccccccccccccc}
 & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \cdots \\
\cdots & & \mathbf{3} & & \mathbf{1} & & \mathbf{2} & & \mathbf{2} & & \mathbf{1} & & \mathbf{3} & \\
 & \mathbf{2} & & \mathbf{2} & & \mathbf{1} & & \mathbf{3} & & \mathbf{1} & & \mathbf{2} & & \cdots \\
\cdots & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} &
\end{array}$$



Frieze Patterns (of Laurent polynomials)

Diagonals of a polygon correspond to entries of a finite frieze. The diamond condition $AD - BC = 1$ stands in for the Plücker relation $p_{ik}p_{j\ell} = p_{ij}p_{k\ell} + p_{i\ell}p_{jk}$ for $i < j < k < \ell$.

1		$\mathbf{1}$		1		1		1		1	
	$\frac{x_1+x_2+1}{x_1x_2}$		$\mathbf{x_1}$		$\frac{x_2+1}{x_1}$		$\frac{x_1+1}{x_2}$		x_2		$\frac{x_1+x_2+1}{x_1x_2}$
$\frac{x_2+1}{x_1}$		$\frac{x_1+1}{x_2}$		$\mathbf{x_2}$		$\frac{x_1+x_2+1}{x_1x_2}$		x_1		$\frac{x_2+1}{x_1}$	
	1		1		1		1		1		1



Teichmüller and Decorated Teichmüller Spaces

Let $S = S_g^n$ be a smooth oriented surface (possibly with boundary) of genus g equipped with a collection of marked points $p_1, p_2, \ldots, p_n$.

Here $n \geq 0$. The marked points either lie on boundary components, or in the interior of S, in which case they are called punctures.

Roughly speaking, the *Teichmüller space* of such a surface is

$$T(S) = \text{the set of hyperbolic structures on S/isotopy.}$$

Definition

Define the *Teichmüller space* of S to be the quotient space

$$T(S) = \mathrm{Hom}\left(\pi_1(S), \mathrm{PSL}(2, \mathbb{R})\right) / \mathrm{PSL}(2, \mathbb{R}).$$

Definition (Penner)

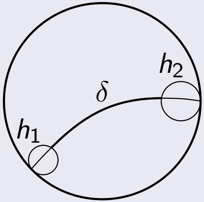
When $n > 0$, any such surface $S = S_g^n$ also admits a *decorated Teichmüller space*, which is a trivial $\mathbb{R}^n_{>0}$-bundle over $T(S)$, denoted $\tilde{T}(S)$.

Decorated Teichmüller Theory

Throughout the rest of the paper, let $S = S_0^n$ be a disk with n marked points on its unique boundary (i.e. a polygon). Such surfaces admit the *Poincaré disk* $\mathbb{D}$ model as a hyperbolic structure.

$\mathbb{D} := \{z = x + yi \in \mathbb{C} : |z| < 1\}$, with metric $ds = 2\frac{\sqrt{dx^2+dy^2}}{1-|z|^2}$.

Definition (λ-length via horocycles)



A *horocycle* is a smooth curve in the hyperbolic plane with constant geodesic curvature 1. In $\mathbb{D}$, it is a Euclidean circle tangent to an infinite point, which is the center.

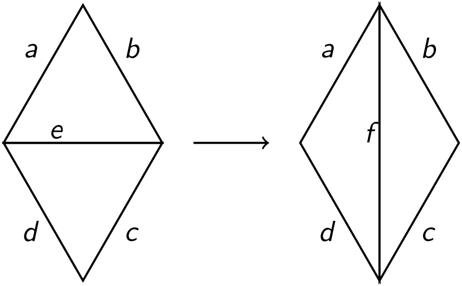
For a pair of horocycles h_1, h_2, the λ*-length* between them is

$$\lambda(h_1, h_2) = e^{\delta/2}$$

where δ is the hyperbolic distance between the two intersections.

Ptolemy Relations

Given a quadruple of horocycles with distinct centers (a decorated ideal quadrilateral), one has the **Ptolemy transformation** induced by flipping the diagonal of the quadrilateral.

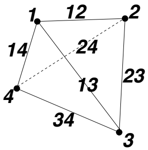


At the level of λ-lengths, this induces the identity

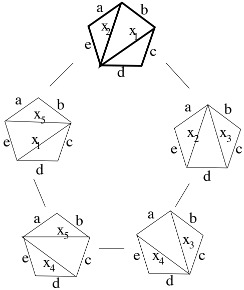
$$\lambda(e)\lambda(f) = \lambda(a)\lambda(c) + \lambda(b)\lambda(d).$$

Note that we will often abbreviate this as $ef = ac + bd$.

Plücker Relations, Frieze Patterns, and Ptolemy Relations



Plücker:
$p_{13}p_{24} = p_{12}p_{34} + p_{14}p_{23}$



1		**1**		1		1		1		1	
	$\frac{x_1+x_2+1}{x_1x_2}$		$\mathbf{x_1}$		$\frac{x_2+1}{x_1}$		$\frac{x_1+1}{x_2}$		x_2		$\frac{x_1+x_2+1}{x_1x_2}$
$\frac{x_2+1}{x_1}$		$\frac{x_1+1}{x_2}$		$\mathbf{x_2}$		$\frac{x_1+x_2+1}{x_1x_2}$		x_1		$\frac{x_2+1}{x_1}$	
	1		1		1		1		1		1

$$a = p_{12},\ b = p_{23},\ c = p_{34},\ d = p_{45},\ e = p_{15},\ x_1 = p_{35},\ x_2 = p_{25},$$

$$x_3 = p_{24} = \frac{x_2+1}{x_1},\ x_4 = p_{14} = \frac{x_1+x_2+1}{x_1x_2},\ x_5 = p_{13} = \frac{x_1+1}{x_2}.$$

Structural Theorems for Cluster Algebras

Theorem (Fomin-Zelevinsky 2001, The Laurent Phenomenon)

For any cluster algebra defined by initial seed $(\{x_1, x_2, \ldots, x_{n+m}\}, B)$, all cluster variables of $\mathcal{A}(B)$ are Laurent polynomials in $\{x_1, x_2, \ldots, x_{n+m}\}$ (with no coefficient $x_{n+1}, \ldots, x_{n+m}$ in the denominator).

Because of the Laurent Phenomenon, any cluster variable x_α can be expressed as $\frac{P_\alpha(x_1,\ldots,x_{n+m})}{x_1^{\alpha_1}\cdots x_n^{\alpha_n}}$ where $P_\alpha \in \mathbb{Z}[x_1, \ldots, x_{n+m}]$ and the α_i's $\in \mathbb{Z}$.

Theorem (Lee-Schiffler 2014, Gross-Hacking-Keel-Kontsevich 2015, Prooof of the Positivity Conjecture)

For any cluster variable x_α and any initial seed (i.e. initial cluster $\{x_1, \ldots, x_{n+m}\}$ and initial exchange pattern B), the polynomial $P_\alpha(x_1, \ldots, x_n)$ has nonnegative integer coefficients.

Cluster Algebras from Surfaces

Theorem (Fomin-Shapiro-Thurston 2006)

Given a Riemann surface with marked points (S, M)*, one can define a corresponding cluster algebra* $\mathcal{A}(S, M)$*.*

$$\text{Seed} \leftrightarrow \text{Triangulation } T = \{\tau_1, \tau_2, \ldots, \tau_n\}$$

$$\text{Cluster Variable} \leftrightarrow \text{Arc } \gamma \ (x_i \leftrightarrow \tau_i \in T)$$

Cluster Mutation (Binomial Exchange Relations) $\leftrightarrow$ *Flipping Diagonals.*

(Based on earlier work of Gekhtman-Shapiro-Vainshtein and Fock-Goncharov.)

From the perspective of hyperbolic geometry, Laurent expansions of cluster variables may be expressed as λ-lengths of arcs, which can be measured by choosing a point in Penner's decorated Teichmüller space.

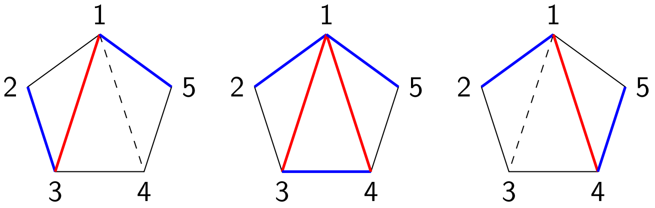
Positivity of Cluster Algebras from Surfaces

Theorem (Schiffler 2006)

Let $\mathcal{A}$ be any cluster algebra of type A_n, i.e. with a seed Σ defined by a triangulation T of an $(n+3)$-gon.

Then the Laurent expansion of every cluster variable with respect to the seed Σ has non-negative coefficients.

Proof via explicit combinatorial formulas in terms of **T-paths**.



$$\lambda_{25} = \frac{x_{23}x_{15}}{x_{13}} + \frac{x_{12}x_{34}x_{15}}{x_{13}x_{14}} + \frac{x_{12}x_{45}}{x_{14}} = \frac{x_{23}x_{14}x_{15} + x_{12}x_{34}x_{15} + x_{12}x_{13}x_{45}}{x_{13}x_{14}}.$$

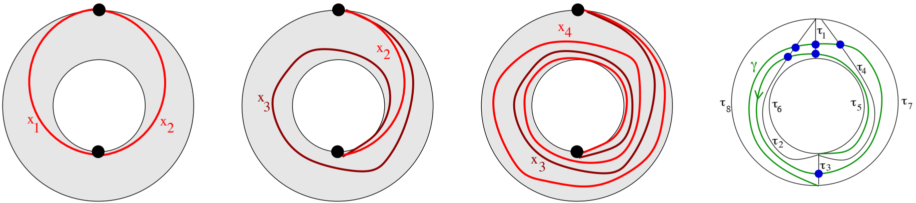
Positivity of Cluster Algebras from Surfaces

Theorem (Schiffler-Thomas 2007, Schiffler 2008)

Let $\mathcal{A}(S, M)$ be any cluster algebra arising from an unpunctured surface S with marked points M, with principal coefficients, and let Σ be any initial seed. Here Σ correponds to a triangulation of S with respect to the marked points M.

Then the Laurent expansion of every cluster variable with respect to the seed Σ has non-negative coefficients.

Proof via explicit combinatorial formulas in terms of **T-paths**.



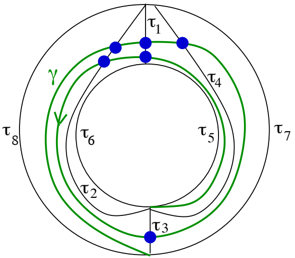
Positivity of Cluster Algebras from Surfaces

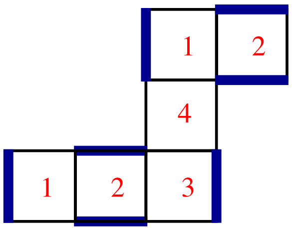
Theorem (M-Schiffler 2008)

Let $\mathcal{A}(S, M)$ be any cluster algebra arising from an unpunctured surface, with principal coefficients, and let Σ be any initial seed.

Then the Laurent expansion of every cluster variable with respect to the seed Σ has non-negative coefficients.

Proof via explicit combinatorial formulas in terms of **snake graphs**.





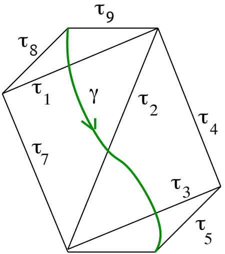
Positivity of Cluster Algebras from Surfaces

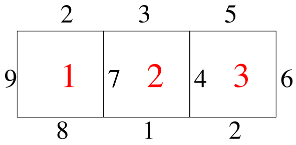
Theorem (M-Schiffler-Williams 2009)

Let $\mathcal{A}(S, M)$ be any cluster algebra arising from a surface (with or without punctures), where the coefficient system is of geometric type, and let Σ be any initial seed.

Then the Laurent expansion of every cluster variable with respect to the seed Σ has non-negative coefficients.

Proof via explicit combinatorial formulas in terms of **snake graphs**.





Superalgebras (and towards Superspace)

A super algebra is a $\mathbb{Z}_2$-graded algebra.

i.e. $A = A_0 \oplus A_1$, (the "*even*" and "*odd*" parts) and

$$A_i A_j \subseteq A_{i+j} \text{ for } i, j \in \{0, 1\} \bmod 2$$

The algebra A generated by $x_1, \cdots, x_n, \theta_1, \cdots, \theta_m$, subject to the following relations

$$x_i x_j = x_j x_i \quad x_i \theta_j = \theta_j x_i \quad \theta_i \theta_j = -\theta_j \theta_i$$

is a superalgebra. In particular $\theta_i^2 = 0$.

Here A_0 is spanned by monomials with an even number of θ's and A_1 is spanned by monomials with an odd number of θ's.

E.g. $x_1 x_2 + x_1 \theta_1 \theta_3 + x_2 \theta_1 \theta_2 \theta_3 \theta_4 \in A_0$, $x_1 \theta_1 \theta_2 \theta_3 + x_1 x_4 \theta_2 + \theta_4 \in A_1$

Decorated Super-Teichmüller Spaces [PZ19]

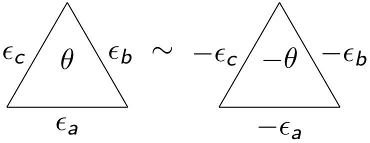
- By replacing $\mathrm{PSL}(2, \mathbb{R})$ with $\mathrm{OSp}(1|2)$, Penner and Zeitlin define the **super-Teichmüller space** of a surface S to be
$$ST(S) = \mathrm{Hom}(\pi_1(S), \mathrm{OSp}(1|2)) / \mathrm{OSp}(1|2)$$
- Similar to the bosonic case, the decorated space is encoded by a collection of horocycles centered at each ideal point, which leads to the definition of super λ-length.
- But unlike the bosonic case, we need additional invariants to accommodate for the extra degree of freedom coming from the odd dimension.
- They associate an odd variable to each triangle (triple of ideal points), and call them the μ-invariants.

Spin Structures

Components of $ST(S)$ are indexed by the set of **spin structures** on S.

Cimasoni-Reshetikhin formulated the set of spin structures of S in terms of the set of isomorphism classes of Kasteleyn orientations of a fatgraph spine of S.

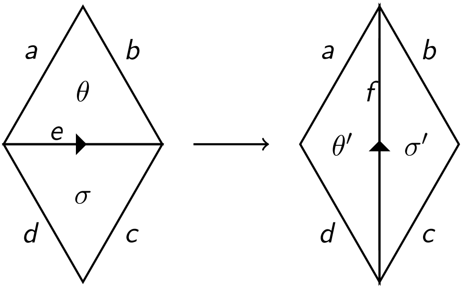
Dual to this formulation, we consider the set of spin structures on S to be the set of equivalence classes of orientations on triangulations of S of the following equivalence relation.



where $\epsilon_a, \epsilon_b, \epsilon_c$ are orientations on the edges, and θ is the μ-invariant associated to the triangle.

Super Ptolemy Relation

The Ptolemy transformation on super λ-length coordinates is given as follows.

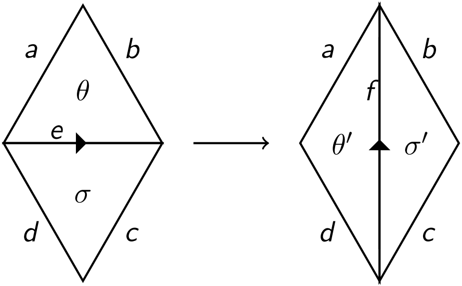


$$ef = (ac + bd)\left(1 + \frac{\sigma\theta\sqrt{\chi}}{1+\chi}\right), \quad \chi = \frac{ac}{bd}$$

$$\sigma' = \frac{\sigma - \sqrt{\chi}\theta}{\sqrt{1+\chi}} \quad \text{and} \quad \theta' = \frac{\theta + \sqrt{\chi}\sigma}{\sqrt{1+\chi}}$$

Super Ptolemy Relation

The Ptolemy transformation on super λ-length coordinates is given as follows.



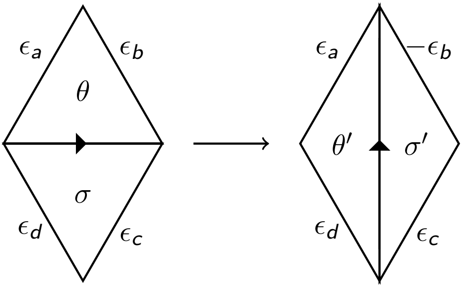
$$ef = ac + bd + \sqrt{abcd}\,\sigma\theta$$

$$\sigma' = \frac{\sigma\sqrt{bd} - \theta\sqrt{ac}}{\sqrt{ac+bd}} \quad \text{and} \quad \theta' = \frac{\theta\sqrt{bd} + \sigma\sqrt{ac}}{\sqrt{ac+bd}}$$

$$\sigma\theta = \sigma'\theta'$$

Super Ptolemy Relation

Super-flip also reverses the orientation of the edge b.

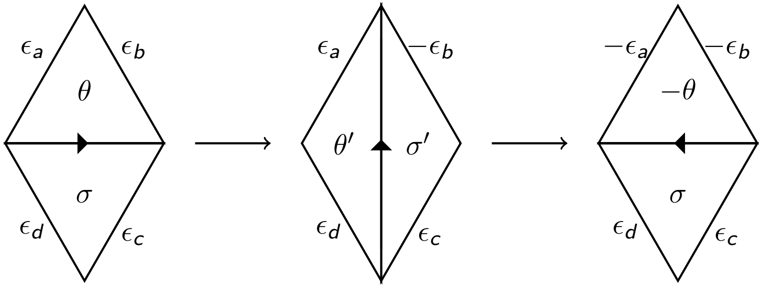


Remark

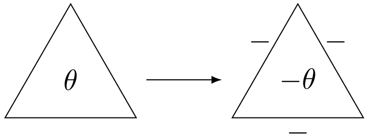
- Super Ptolemy moves are not involutions: $\mu_i^8 = I$.
- The even-degree-0 terms of a super λ-length are exactly the (ordinary) λ-length in the bosonic decorated space.

Super Ptolemy Relation

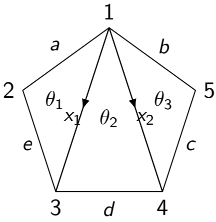
If we flip a diagonal twice



the orientations of the triangle θ are reversed and θ is changed to $-\theta$.



This orientation is equivalent to the original one, i.e. both the first and third pictures represent the same spin structure.



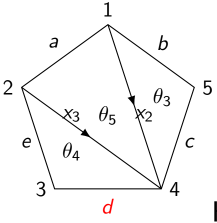
Start with a Pentagon with given orientation.

The boundary orientations are ignored, because they are irrelevant in the calculations.

What are λ_{24}, λ_{25}, and λ_{35}?

We first flip the edge x_1.

Super Ptolemy Relation - Example



After flipping x_1 to x_3, we get:

$$x_3 = \frac{ad + ex_2}{x_1} + \frac{\sqrt{adex_2}}{x_1}\theta_1\theta_2$$

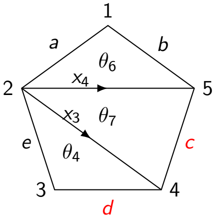
$$\theta_4 = \frac{\sqrt{ad}\,\theta_1 - \sqrt{ex_2}\,\theta_2}{\sqrt{x_1x_3}}$$

$$\theta_5 = \frac{\sqrt{ad}\,\theta_2 + \sqrt{ex_2}\,\theta_1}{\sqrt{x_1x_3}}$$

Here the red color indicates that the orientation on the boundary edge has been reversed.

Next we flip x_2.

Super Ptolemy Relation - Example



After flipping x_2 to x_4, we have:

$$x_4 = \frac{ac + bx_3}{x_2} + \frac{\sqrt{acbx_3}}{x_2}\,\theta_5\theta_3$$

$$= \frac{acx_1 + abd + bex_2}{x_1x_2} + \frac{b\sqrt{adex_2}}{x_1x_2}\theta_1\theta_2 + \frac{\sqrt{acb\left(\frac{ad+ex_2}{x_1} + \frac{\sqrt{adex_2}}{x_1}\theta_1\theta_2\right)}}{x_2}\left(\frac{\sqrt{ad}\,\theta_2 + \sqrt{ex_2}\,\theta_1}{\sqrt{x_1x_3}}\right)\theta_3$$

$$= \frac{acx_1}{x_1x_2} + \frac{abd}{x_1x_2} + \frac{bex_2}{x_1x_2} + \frac{b\sqrt{ade}}{x_1\sqrt{x_2}}\theta_1\theta_2 + \frac{a\sqrt{bcd}}{\sqrt{x_1}x_2}\theta_2\theta_3 + \frac{\sqrt{abce}}{\sqrt{x_1x_2}}\theta_1\theta_3$$

Question: If we now flip x_3 to x_5, what do we expect x_5 to look like?

Main Question

In a cluster algebra A, any cluster variable can be expressed as a positive Laurent polynomial in the initial cluster, i.e.

$$A \subset \mathbb{R}[x_1^{\pm 1}, \cdots, x_n^{\pm 1}].$$

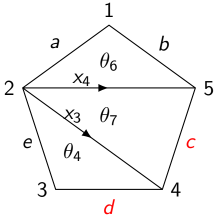
Questions

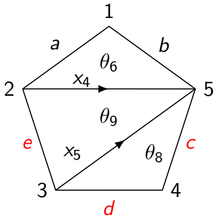
- Does the super λ-length satisfy some Laurent phenomenon?
- Is there a "positivity" for terms with anti-commuting variables?

Answers (Spoiler Alert)

- Super λ-lengths live in $\mathbb{R}[x_1^{\pm\frac{1}{2}}, \cdots, x_1^{\pm\frac{1}{2}} | \theta_1, \cdots, \theta_{n+1}]$.
- There exists an ordering on the odd variables, called *positive ordering*, such that if we multiply θ's in the positive ordering then the coefficients are positive.

Super Ptolemy Relation - Example Continued





Before giving the general answer, we illustrate the result of flipping x_3 to x_5: We first recall

that $\theta_4 = \dfrac{\sqrt{ad}\,\theta_1 - \sqrt{ex_2}\,\theta_2}{\sqrt{x_1x_3}}$ and note that

$$\theta_7 = \frac{\sqrt{ac}\,\theta_5 - \sqrt{bx_3}\,\theta_3}{\sqrt{x_2x_4}} =$$

$$\frac{1}{\sqrt{cx_3x_4}}\left(c\sqrt{\frac{ae}{x_1}}\theta_1 + ac\sqrt{\frac{d}{x_1x_2}}\theta_2 - x_3\sqrt{\frac{bc}{x_2}}\theta_3\right).$$

We then proceed to obtain

$$x_5 = \frac{ce+dx_4}{x_3} + \frac{\sqrt{cdex_4}}{x_3}\theta_4\theta_7 = \cdots =$$

$$\frac{bd+cx_1}{x_2} + \frac{\sqrt{bcdx_1}}{x_2}\theta_2\theta_3.$$

Continuing with super-flips of x_4 and x_5, in order, yields x_1 and x_2, respectively.

Schiffler's T-paths [Sch08]

Let T be a triangulation of a polygon, thought of as a graph of vertices and edges.

A *T-path* from i to j is a path in T starting at vertex i, ending at j, such that

(T1) the path does not use any edge twice

(T2) the path has an odd number of edges

(T3) the even-numbered edges cross the diagonal (i, j)

(T4) The intersections of the path and (i, j) move from progressively i to j.

Let T_{ij} denote the set of T-paths from i to j.

For a T-path $\gamma = (x_1, x_2, \cdots)$, define it's weight to be

$$\mathrm{wt}(\gamma) = \prod_{i \text{ odd}} \lambda(x_i) \prod_{i \text{ even}} \lambda(x_i)^{-1}$$

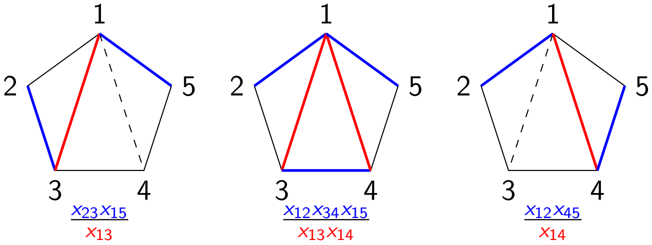
where $\lambda(x_i)$ denote the λ-length of the edge x_i.

Schiffler's T-paths [Sch08]

Theorem (Schiffler)

$$\lambda(x_{i,j}) = \sum_{t \in T_{i,j}} \mathrm{wt}(t)$$

Here are the T-paths in T_{25}. (odd steps are blue and even steps are red)

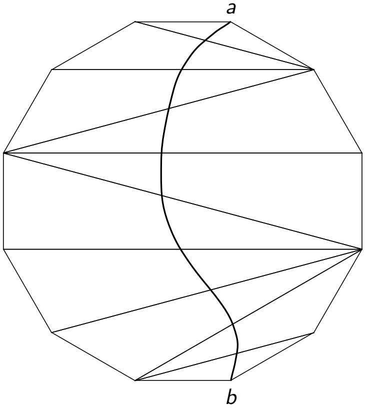


$$\lambda(x_{2,5}) = \sum_{t \in T_{25}} \mathrm{wt}(t) = \frac{x_{23}x_{15}}{x_{13}} + \frac{x_{12}x_{34}x_{15}}{x_{13}x_{14}} + \frac{x_{12}x_{45}}{x_{14}}$$

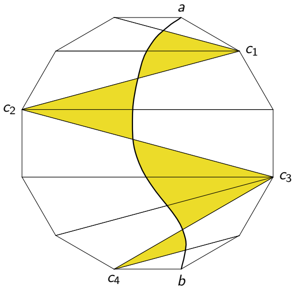
Main Result: Super T-paths

From now on we only consider triangulations with a longest arc crossing all internal diagonals.

In other words, every triangle has a boundary edge. Call the end points of the longest arc a and b.



Fan Decomposition



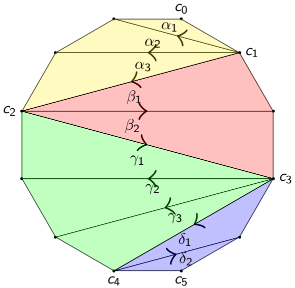
For a triangulation T, we will define a canonical fan decomposition.

The arc (a, b) intersect with internal diagonals, and create smaller triangles (colored yellow).

Vertices of these yellow triangles are called fan centers, denoted $c_1, \cdots, c_n$, ordered by their distance from a. And we further denote $a = c_0$ and $b = c_{n+1}$.

The sub-triangulation bounded by c_{i-1}, c_i, c_{i+1} is called the i-th fan segment of T.

Default Orientation and Positive Ordering



We define a default orientation on the interior diagonals.

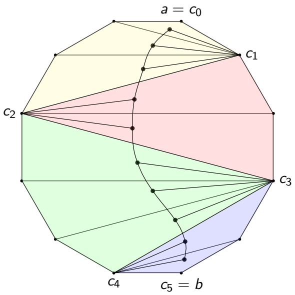
- Edges inside each fan segment are directed away from the center.
- Others are oriented as $c_1 \to c_2 \to \cdots \to c_n$.

We define a positive ordering on μ-invariants.

- μ-invariants in a fan are ordered counterclockwise around the center.
- "Alternate" across the fans.

$$\alpha_1 > \alpha_2 > \alpha_3 > \gamma_1 > \gamma_2 > \gamma_3 > \delta_2 > \delta_1 > \beta_2 > \beta_1$$

The Auxiliary Graph



For each triangle in T, we place an internal vertex.

The internal vertices are connected to the nearest fan centers by σ-edges. The σ-edges are considered to cross the arc (a, b).

Every pair of internal vertices are connected by a teleportation, called a τ-edge. (Note that the τ-edges are drawn to be overlapping.)

The resulting graph $\Gamma_T^{a,b}$ is the **auxiliary graph** associated to $\{T, a, b\}$.

Super T-paths

Finally, we define **super T-paths** to be paths on the auxiliary graph such that:

(T1) the path does not use any edge twice.

(T2) the path has an odd number of edges.

(T3) the even-numbered edges cross the diagonal (a, b).

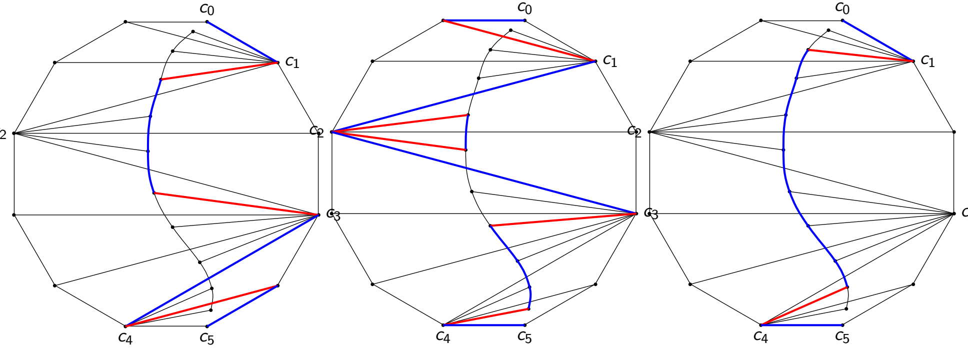
(T4) The intersections of the path and (a, b) move from progressively a to b.

(T5) σ-edges must be even and τ-edges must be odd.

Let $\tilde{T}_{a,b}$ denote the set of **super T-paths** on $\Gamma_T^{a,b}$.

Note that, every ordinary T-path is also a super T-path: $T_{a,b} \subset \tilde{T}_{a,b}$

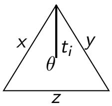
Super T-paths: Examples



Weights of Super T-paths

If a super T-path uses edges $t_1, t_2, \ldots,$ we define its weight as follows.

- If t_i is a diagonal in the triangulation, then:
  $\mathrm{wt}(t_i) = \lambda(t_i)$ if i odd, and
  $\mathrm{wt}(t_i) = \lambda(t_i)^{-1}$ if t is even.
- If t_i is a τ-edge, then $\mathrm{wt}(t_i) = 1$ (teleportation)
- If t_i is a σ-edge, then $\mathrm{wt}(t_i) = \tilde{\theta} := \sqrt{\frac{z}{xy}}\,\theta$. Here x, y, z are λ-lengths and θ is the μ-invariant.



If t is a super T-path with edges $t_1, t_2, \ldots,$ define $\mathrm{wt}(t) = \prod_i \mathrm{wt}(t_i)$.
Here the product is taken under the positive ordering.

Main Theorem

Theorem (M-Ovenhouse-Zhang 2021)

Under default orientation, the super λ-length of the arc (a, b) (assuming to be the longest arc in T) is given by:

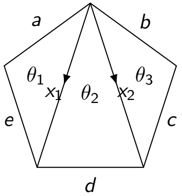
$$\lambda(a, b) = \sum_{t \in \tilde{T}_{a,b}} \mathrm{wt}(t)$$

With the following lemma, we can apply the main theorem for triangulations with arbitrary orientation.

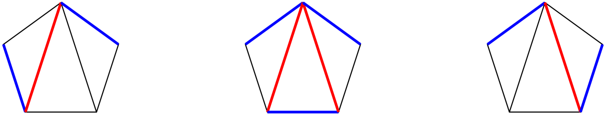
Lemma (M-Ovenhouse-Zhang 2021)

In the equivalence class of any spin structure, there exists (at least) one default orientation. (In other words, up to possibly negating boundary edges, or negating a μ-invariant and its three incident edges, we can transform any orientation on T into the default orientation.)

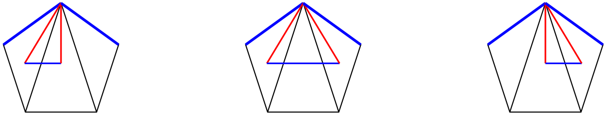
Formula for λ-lengths: Example



$\theta_1 > \theta_2 > \theta_3$



$\frac{be}{x_1}$ $\quad$ $\frac{abd}{x_1 x_2}$ $\quad$ $\frac{ac}{x_2}$



$ab\sqrt{\frac{e}{ax_1}}\theta_1\sqrt{\frac{d}{x_1x_2}}\theta_2$ $\quad$ $ab\sqrt{\frac{e}{ax_1}}\theta_1\sqrt{\frac{c}{bx_2}}\theta_3$ $\quad$ $ab\sqrt{\frac{d}{x_1x_2}}\theta_2\sqrt{\frac{c}{bx_2}}\theta_3$

Formula for μ-invariants

Theorem (M-Ovenhouse-Zhang 2021)

Let T be a triangulation with $a = c_0, c_1, \cdots, c_{n+1} = b$ its fan centers. Let Θ denote the set of all internal vertices in $\Gamma_T^{a,b}$. Then

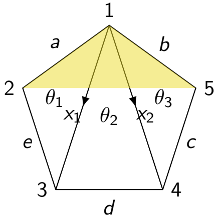
$$\sqrt{\frac{\lambda(a,b)\lambda(b,c_1)}{\lambda(a,c_1)}}\boxed{abc_1} = \sum_{\theta\in\Theta} \mathrm{wt}\{\text{‘partial’ super } T\text{-path from } a \text{ to } \theta\}$$

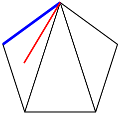
Here wt *means the weighted sum, and a partial super T-path satisfies all axioms except that they have an even number of edges.*

Remark

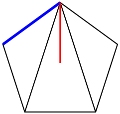
Note that the above theorem only covers a special family of triangles. The μ-invariants themselves don't have simple expansions, because the λ-lengths in the term $\sqrt{\frac{\lambda(a,b)\lambda(b,c_1)}{\lambda(a,c_1)}}$ are not always in the triangulation.

Formula for μ-invariants: Example

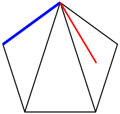




$a\sqrt{\frac{e}{ax_1}}\theta_1$



$a\sqrt{\frac{d}{x_1x_2}}\theta_2$

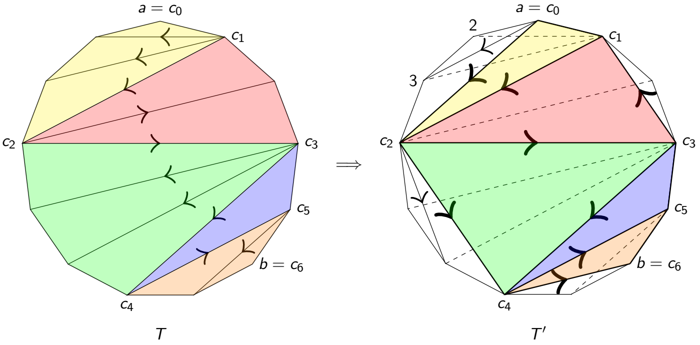


$a\sqrt{\frac{c}{bx_2}}\theta_3$

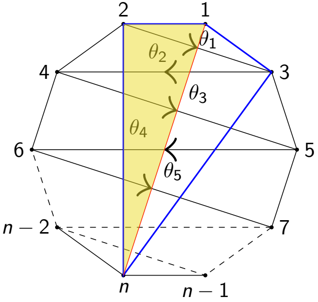
$$\sqrt{\frac{b\lambda_{25}}{a}}\boxed{125} = \sqrt{\frac{ae}{x_1}}\theta_1 + a\sqrt{\frac{d}{x_1x_2}}\theta_2 + a\sqrt{\frac{c}{bx_2}}\theta_3$$

Proof Sketch - Three Steps

- We first prove our theorems for triangulations that consist of a single fan.
- Next, we prove them for triangulations consisting exclusively of a zig-zag. (We call this a zig-zag triangulation.)
- Finally, we prove in full generality by combining the above two cases. We flip the interior edges of each fan (counter-clockwise around fan centers) to reduce to a zig-zag triangulation.



Proof Sketch - Double Helix Induction



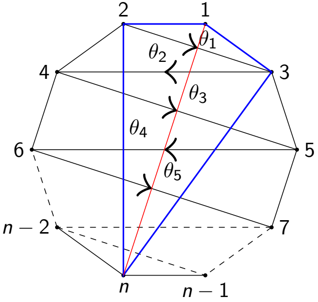
$$\boxed{12n}\sqrt{\frac{\lambda_{1n}\lambda_{2n}}{\lambda_{12}}} = \underbrace{\boxed{23n}\sqrt{\frac{\lambda_{3n}\lambda_{2n}}{\lambda_{23}}}}_{\text{1st term}} + \underbrace{\boxed{123}\sqrt{\frac{\lambda_{13}}{\lambda_{12}\lambda_{23}}}\lambda_{2n}}_{\text{2nd term}}$$

1st term: (by induction hypothesis) all partial super T-paths starting from n and ending at one of $\theta_2, \theta_3, \cdots$.

2nd term: all complete super T-paths from n to 2 plus an σ-edge to θ_1.

1st + 2nd: partial super T-paths from n to one of $\theta_1, \theta_2, \theta_3, \cdots$.

Proof Sketch - Double Helix Induction



$$\lambda_{1n} = \underbrace{\frac{\lambda_{12}\lambda_{3n}}{\lambda_{23}}}_{\text{part 1}} + \underbrace{\frac{\lambda_{13}\lambda_{2n}}{\lambda_{23}}}_{\text{part 2}} - \underbrace{\sqrt{\frac{\lambda_{12}\lambda_{13}}{\lambda_{23}}}\boxed{123} \cdot \sqrt{\frac{\lambda_{2n}\lambda_{3n}}{\lambda_{23}}}\boxed{23n}}_{\text{part 3}}$$

part 1: $\tilde{T}_{1,n}$ whose first two steps are $(1,2)$ and $(2,3)$.

part 2: $\tilde{T}_{1,n}$ whose first step is $(1,3)$.

part 1+2: $\tilde{T}_{1,n}$ without using $\theta_1 = \boxed{123}$.

part 3: By the induction hypothesis for $\boxed{23n}$, part 3 has all super T-paths from 1 to n which used θ_1.

part 1+2+3: Together gives all super T-paths from 1 to n.

Superfriezes

Supersymmetric frieze patterns are introduced by Morier-Genoud, Ovsienko, and Tabachnikov. They are the following array of numbers

$$
\begin{array}{cccccccccccc}
 & \cdots & 0 & & & & 0 & & & & 0 & \\
\cdots & {\color{red}0} & & {\color{red}0} & & {\color{red}0} & & {\color{red}0} & & {\color{red}0} & & \cdots \\
1 & & & & 1 & & & & 1 & & & \cdots \\
 & {\color{red}\varphi_{0,0}} & & {\color{red}\varphi_{\frac{1}{2},\frac{1}{2}}} & & {\color{red}\varphi_{1,1}} & & {\color{red}\varphi_{\frac{3}{2},\frac{3}{2}}} & & {\color{red}\varphi_{2,2}} & & \cdots \\
 & & f_{0,0} & & & & f_{1,1} & & & & f_{2,2} & \\
 & {\color{red}\varphi_{-\frac{1}{2},\frac{1}{2}}} & & {\color{red}\varphi_{0,1}} & & {\color{red}\varphi_{\frac{1}{2},\frac{3}{2}}} & & {\color{red}\varphi_{1,2}} & & {\color{red}\varphi_{\frac{3}{2},\frac{5}{2}}} & & \cdots \\
f_{-1,0} & & & & f_{0,1} & & & & f_{1,2} & & & \\
 & \iddots & & \iddots & & \ddots & & \ddots & & \ddots & & \ddots \\
 & & f_{2-m,1} & & & & f_{0,m-1} & & & & f_{1,m} & \\
\cdots & {\color{red}\varphi_{\frac{3}{2}-m,\frac{3}{2}}} & & {\color{red}\varphi_{2-m,2}} & & \cdots & & {\color{red}\varphi_{0,m}} & & {\color{red}\varphi_{\frac{1}{2},m+\frac{1}{2}}} & & {\color{red}\varphi_{1,m+1}} \\
1 & & & & 1 & & & & 1 & & & \\
\cdots & {\color{red}0} & & {\color{red}0} & & {\color{red}0} & & {\color{red}0} & & {\color{red}0} & & {\color{red}0} \\
 & \cdots & 0 & & & & 0 & & & & 0 & \cdots
\end{array}
$$

Super Diamond

A super frieze is built up out of *super diamonds*.

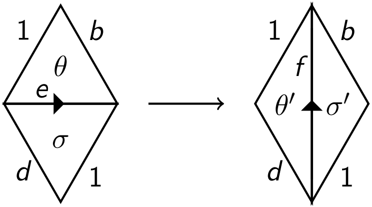
$$\begin{array}{ccccc} & & B & & \\ & \Xi & & \Psi & \\ A & & & & D \\ & \Phi & & \Sigma & \\ & & C & & \end{array}$$

Every super diamond is a matrix in OSp$(1|2)$, satisfying the following frieze rules:

$$\begin{aligned} AD - BC &= 1 + \Sigma\Xi \\ A\Sigma - C\Xi &= \Phi \\ B\Sigma - D\Xi &= \Psi \\ B\Phi - A\Psi &= \Xi \\ D\Phi - C\Psi &= \Sigma \\ \Sigma\Xi &= \Psi\Phi \end{aligned}$$

Super Diamonds as Ptolemy Relations

Consider quadrilateral flips as follows where two of the edges have length 1.



The Ptolemy relation is equivalent to the superfrieze relation of the following diamond:

$$\begin{array}{ccccc} & & b & & \\ & \theta\sqrt{be} & & \sigma'\sqrt{bf} & \\ e & & & & f \\ & \sigma\sqrt{ed} & & \theta'\sqrt{df} & \\ & & d & & \end{array}$$

Set $\tilde{\theta} = \theta\sqrt{be}$, $\tilde{\sigma} = \sigma\sqrt{ed}$, $\tilde{\theta}' = \theta'\sqrt{df}$, and $\tilde{\sigma}' = \sigma'\sqrt{bf}$.

Superfriezes from a marked disk

As a corollary of the previous slide, we have

Theorem (M-Ovenhouse-Zhang 2021)

Every (finite) superfrieze pattern come from the super λ-lengths and μ-invariants of a marked disk.

1

ξ_1

x_1

ξ_2

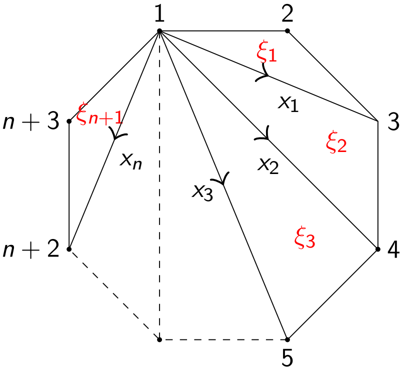
x_2

$\ddots$

x_n

ξ_{n+1}

1



Superfriezes from a marked disk

By Proposition 2.3.1 of [MGOT15], the first non-trivial row of μ-invariants repeats every-other entry.

$1 \qquad 1$

$\xi_1 \qquad \xi_1$

x_1

ξ_2

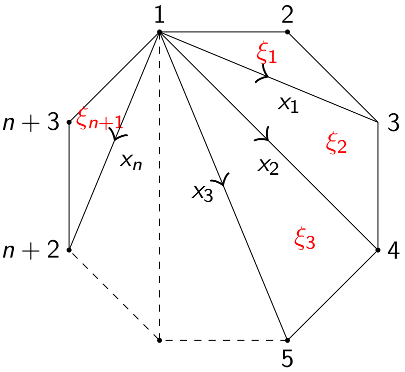
x_2

$\ddots$

x_n

ξ_{n+1}

1



Superfriezes from a marked disk

By Proposition 2.3.1 of [MGOT15], the first non-trivial row of μ-invariants repeats every-other entry. Using this, as well a super-flip of x_1, we can extend the superfrieze (as below):

$1 \quad\quad\quad 1$

$\xi_1 \quad \xi_1 \quad \theta_1$

$x_1 \quad\quad y_1$

$\xi_2 \quad \xi_2'$

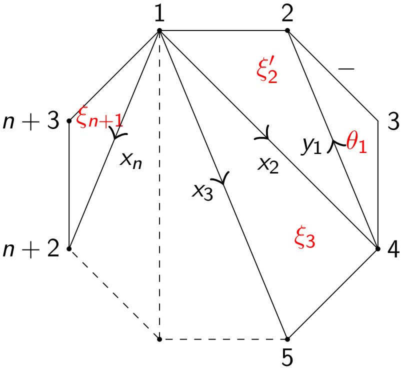
x_2

$\ddots$

x_n

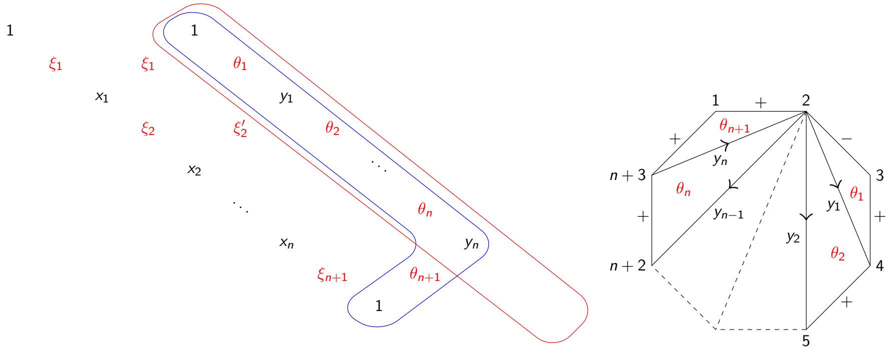
ξ_{n+1}

1



Superfriezes from a marked disk

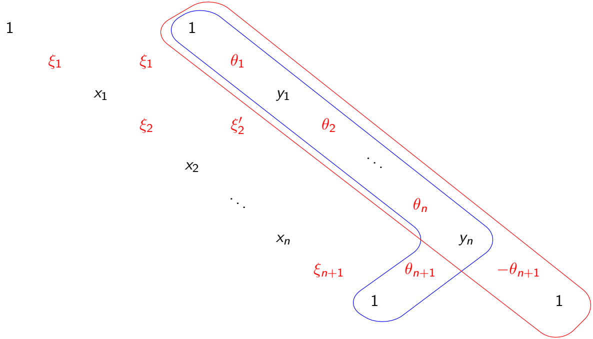
Continuing with super-flips of $x_2 \to y_2$, $x_3 \to y_3$, ..., $x_n \to y_n$, in order:

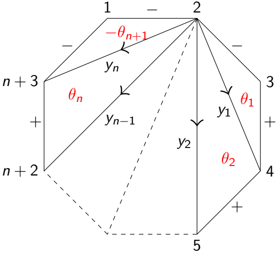


Observe, for $2 \leq k \leq n$, that each super-flip of x_k also negates the orientation on edge y_{k-1}.

Superfriezes from a marked disk

Continuing with super-flips of $x_2 \to y_2$, $x_3 \to y_3$, $\ldots$, $x_n \to y_n$, in order:





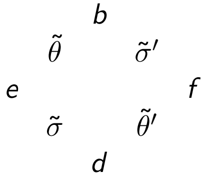
By Proposition 2.3.1 of [MGOT15], the last non-trivial of μ-invariants alternates every-other entry. This agrees with negating the μ-invariant θ_{n+1}, and its incident edges, without changing the spin structure.

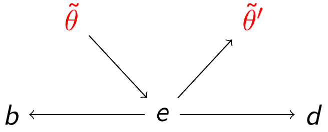
Furthermore, we have simply rotated our fan triangulation clockwise such that the orientation on internal edges has stayed the same.

Relation to Ovsienko-Shapiro Cluster Algebra

Ovsienko and Shapiro [OS19] proposed a Cluster superalgebra using *extended quivers*.

For every super diamond, associate an extended quiver:



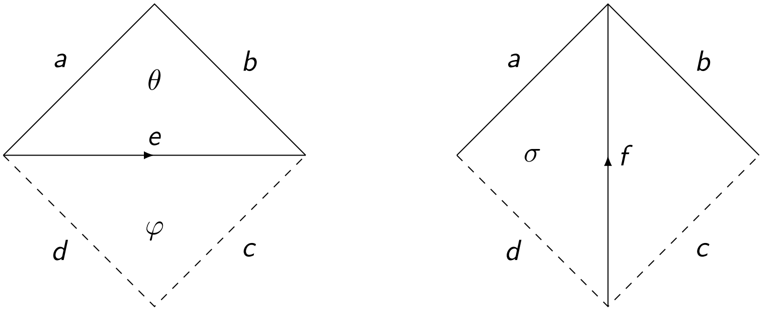


Note that $\tilde{\theta}$ and $\tilde{\theta}'$ are not in the same triangulation!

Question

Can we add odd mutations $\tilde{\sigma} \to \tilde{\theta}'$ and $\tilde{\theta} \to \tilde{\sigma}'$, turning the extended quiver mutation into Ptolemy transformation?

Work in Progress: A Second Combinatorial Interpretation



Theorem (M-Ovenhouse-Zhang 2021+)

Consider a triangulation as pictured as above, where f is the longest edge, and edges c, d are not necessarily in the triangulation. In particular, a and b are assumed to be boundary edges. We build the snake graph G corresponding to the arc f (following [M-Schiffler-Williams]). Then $f = \frac{1}{\mathrm{cross}(f)} \sum_{M \in D(G)} \mathrm{wt}(M)$ *where $D(G)$ is the set of double-dimers on G.*

Work in Progress: A Second Combinatorial Interpretation

Theorem (M-Ovenhouse-Zhang 2021+)

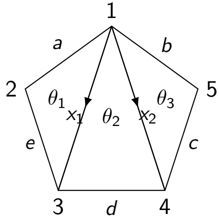
We build the snake graph G corresponding to the arc f. Then the super-λ length for f is given as follows: $\frac{1}{\mathrm{cross}(f)} \sum_{M \in D(G)} \mathrm{wt}(M)$ where $D(G)$ is the set of double-dimers on G.

$\mathrm{cross}(f)$ *denotes the monomial given by the product of the edges crossed by the arc f.*

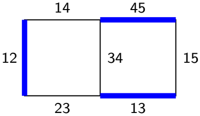
We define $\mathrm{wt} = \mathrm{wt}_x \mathrm{wt}_\theta$. The value of wt_x is the product of the weights of the edges in M with multiplicity, but the weight of each individual edge is given by a square-root.

Additionally each cycle around tiles appearing in M contributes a weight of $\theta_i \theta_j$ to wt_θ, where θ_i and θ_j label the first and last triangles of that cycle in G, respectively.

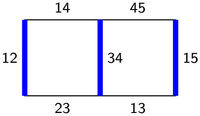
Work in Progress: A Second Combinatorial Interpretation



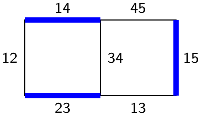
$$\textit{Recall } \lambda_{2,5} = \frac{acx_1}{x_1x_2} + \frac{abd}{x_1x_2} + \frac{bex_2}{x_1x_2} + \frac{b\sqrt{ade}}{x_1\sqrt{x_2}}\theta_1\theta_2 + \frac{a\sqrt{bcd}}{\sqrt{x_1}x_2}\theta_2\theta_3 + \frac{\sqrt{abce}}{\sqrt{x_1x_2}}\theta_1\theta_3$$



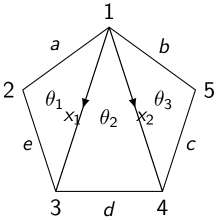
$\frac{acx_1}{x_1x_2}$



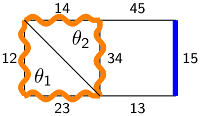
$\frac{abd}{x_1x_2}$



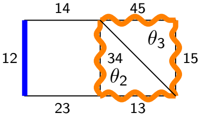
$\frac{bex_2}{x_1x_2}$



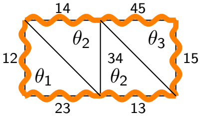
$$\textit{Recall}\ \lambda_{2,5} = \frac{acx_1}{x_1x_2} + \frac{abd}{x_1x_2} + \frac{bex_2}{x_1x_2} + \frac{b\sqrt{ade}}{x_1\sqrt{x_2}}\theta_1\theta_2 + \frac{a\sqrt{bcd}}{\sqrt{x_1}x_2}\theta_2\theta_3 + \frac{\sqrt{abce}}{\sqrt{x_1x_2}}\theta_1\theta_3$$



$\frac{b\sqrt{adex_2}}{x_1x_2}\theta_1\theta_2$



$\frac{a\sqrt{bcdx_1}}{x_1x_2}\theta_2\theta_3$



$\frac{\sqrt{abcex_1x_2}}{x_1x_2}\theta_1\theta_3$

Thank You for Listening!

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