Linear Systems on Tropical Curves

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Outline

1. Chip-firing, $G$-parking functions, and Riemann-Roch for graphs
2. Introduction to Tropical Arithmetic and Tropical Functions
3. Abstract Tropical Curves (Think Metric Graph)
4. Tropical Riemann-Roch and Linear Systems
5. Examples
Our story begins with the **Laplacian Matrix** and the **Matrix-Tree Theorem** for graphs.
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We let $d_{ij}$ denote the number of edges in $E$ of the form $(v_i, v_j)$ and $\text{val}(v_i) = \sum_{j=1}^{n} d_{ij}$, i.e. the number of edges incident to $v_i$. 
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We let $d_{ij}$ denote the number of edges in $E$ of the form $(v_i, v_j)$ and $\text{val}(v_i) = \sum_{j=1}^{n} d_{ij}$, i.e. the number of edges incident to $v_i$.

Define $L(G)$ to be the matrix whose diagonal entries are $\text{val}(v_i)$ and whose off-diagonal entries are $-d_{ij}$. 
Example of a Laplacian Matrix

\[
L(G) = \begin{bmatrix}
2 & -1 & -1 & 0 & 0 \\
-1 & 4 & -2 & -1 & 0 \\
-1 & -2 & 4 & -1 & 0 \\
0 & -1 & -1 & 3 & -1 \\
0 & 0 & 0 & -1 & 1
\end{bmatrix}.
\]

The Reduced Laplacian matrix \( L_0(G) \) is defined by deleting a row and column from \( L(G) \). It is a theorem (the Matrix-Tree Theorem) that \( \det L_0(G) \) does not depend on the choice of row and column deleted (as long as they are of the same index).
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For example, in the above, $\det L_0(G) = 12$. 
The Matrix-Tree Theorem

**Theorem (The Matrix-Tree Theorem or Kirchoff’s Theorem)**

The determinant of the reduced Laplacian matrix $L_0(G)$ of a graph $G$ is equal to the number of spanning trees of $G$.

For example, in the above, $\det L_0(G) = 12$. 
Sandpiles, Chip-firing, and G-parking functions

We can get other families of objects in bijection with the set of spanning trees.
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We define a chip configuration or divisor on $G$ to be an assignment of an integer to each vertex of $G$.

We say that two chip-configurations are equivalent if one can be reached from the other by a sequence of chip-firing moves.
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This means that we pick one vertex to share equally with all of its neighbors, sending one chip along each incident edge.
If a configuration has a nonnegative number of chips on each vertex and no vertex can fire, we call such a configuration stable.
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\[
\begin{array}{c}
-3 \\
1 \\
3 \\
1 \\
0
\end{array}
\]
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\[
\begin{array}{c}
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\text{April 16, 2011}
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-1 → 0 → 2

-1 0 1

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\[
\begin{align*}
\begin{array}{c}
0 \\
0 \\
0 \\
-1 \\
1
\end{array}
\rightarrow
\begin{array}{c}
2 \\
0 \\
0 \\
0 \\
0
\end{array}
\end{align*}
\]
Reduced Configurations or $G$-parking functions

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However, in the case of a stable configuration, it might be possible for multiple vertices to simultaneously fire:

We call a configuration **super-stable** (with respect to $v_0$) if no subset of vertices $S \subseteq V \setminus \{v_0\}$ can fire.

These are also known as **$G$-parking functions** or **$v_0$-reduced divisors**.
Example of Super-stables/\(G\)-Parking Functions

In this example, we have 12 super-stable configurations (with respect to vertex \(v_5\)), which are also counted by \(\det L_0(G)\).

We designate one vertex to be a sink and allow arbitrary addition or subtraction of chips to that vertex. Then up to equivalence by chip-firing moves, there is a unique super-stable configuration in each orbit.
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We define a chip configuration (equivalently a divisor $D$ on graph $G$) to be effective if the number of chips on $v$ is nonnegative for each $v \in V$.

Two divisors are linearly equivalent if their chip-configurations differ by a sequence of chip-firing moves.

Given a divisor $D$, the linear system of $D$, denoted as $|D|$, is the set of all effective divisors that are linearly equivalent to $D$. 
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In other words, the set $\mathbb{Z}^{|V|}_{\geq 0}$ breaks up into equivalence classes via chip-firing. The linear systems are the orbits and each orbit has a representative which is of the form $S + dv_0$ where $S$ is a super-stable configuration (with respect to sink $v_0$) and $d \in \mathbb{Z}_{\geq 0}$. 
Example of a Linear System

Let $G$ be as above and $D$ be the divisor:

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Example of a Linear System

Let $G$ be as above and $D$ be the divisor:

0 1
1 0
0 0

Then the linear system $|D|$ consists of $D$ and the following four divisors:

0 2
2 0
0 0
0 0

0 1
1 0
0 0
0 0

0 0
0 0
2 0
0 0

0 0
0 0
0 2
0 0

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Riemann-Roch Theorem for Graphs

Define the **degree** of a divisor to be the total number of chips in the configuration.

Let $K_G$ (the **canonical divisor**) be the chip-configuration such that there are $\text{val}(v) - 2$ chips on each vertex $v$.

The **genus** $g(G)$ of the graph is $|E| - |V| + 1 = b_1(G)$. 
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The **genus** \( g(G) \) of the graph is \( |E| - |V| + 1 = b_1(G) \).

We also have to define a **rank function** \( r(D) = r(|D|) \) defined as follows:

1) If \( D \) is not effective nor linearly equivalent to an effective divisor, then \( r(D) = -1 \).

2) If \( D \) is linearly equivalent to an effective divisor, i.e. \( |D| \neq \emptyset \), then \( r(D) \geq 0 \).

3) If \( |D - E| \neq \emptyset \) for any effective divisor \( E \) of degree \( k \), then \( r(D) \geq k \).
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3) If $|D - E| \neq \emptyset$ for any effective divisor $E$ of degree $k$, then $r(D) \geq k$.

**Theorem (Baker-Norine 2007)** We have the following equality for any graph $G$ and any divisor $D$.

$$r(D) - r(K_G - D) = \deg(D) - g(G) + 1.$$
Example: Let $D$ and $G$ be as follows:
Then the **canonical divisor** for this graph is

$K_G$ is $1_{-1}$, and $K_G - D$ is $\sim$. 
Example of Riemann-Roch

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Then the canonical divisor for this graph is

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Then $g(G) = 3$, $\deg(D) = 2$, $r(D) = r(K - D) = 1$, and the Riemann-Roch equality $1 - 1 = 2 - 3 + 1$ is satisfied.

(To see that $r(D) = 1$, note that we can subtract a chip from any vertex and we are still linearly equivalent to an effective divisor.

However, it is possible to subtract two chips and get a non-effective.)
And now for something completely different . . .
Tropical Arithmetic

We work over the tropical semi-ring

$$(\mathbb{R} \cup \{-\infty\}, \oplus, \odot)$$

where $a \oplus b = \max(a, b)$ and $a \odot b = a + b$. 
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where \(a \oplus b = \max(a, b)\) and \(a \odot b = a + b\).

Notice that \(a + \max(b, c) = \max(a + b, a + c)\), so we have the tropical distributive law

\[a \odot (b \oplus c) = (a \odot b) \oplus (a \odot c).\]
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We also have the tropical commutative and associative laws. Also,

$$a \oplus (-\infty) = a \quad \text{and} \quad b \odot 0 = b$$

for any $a$ and $b$, so we have additive and multiplicative identities.

Lastly, we have multiplicative inverses, but we do not have additive inverses.
We can form **Tropical Polynomials** such as

\[ P = x^\odot 3 \oplus 2 \odot x \oplus 0 = \max(3x, 2 + x, 0). \]

A tropical polynomial is a piecewise linear function with integer slopes, whose image is **convex**, and a finite number of linear pieces.
A Tropical Rational Function is also a piecewise linear function of the same form, but the requirement of convexity is dropped.

The image of a Tropical Rational Function:

A zero of the Tropical Rational Function is a point where the slope increases, and a pole is a point where the slope decreases.

Notice that the image is convex at zeros, but is concave at poles.
**Tropical Curves**

The **Corner Locus** of a Tropical Function is the set of all points where the slope changes (i.e. the maximum is achieved twice.)

1 – $D$: the corner locus would be the set of zeros and poles.

2 – $D$: The corner locus looks like a Metric Graph (plus unbounded rays).

**Tropical Line**: $a \odot x \oplus b \odot y \oplus c$ and **Tropical Cubic**: \( \bigoplus_{i+j \leq 3} x^i y^j \).

The **Degree** of the polynomial equals the \# of rays in each direction.
Tropical Riemann-Roch

An **Abstract Tropical Curve** $\Gamma$ is simply a Metric Graph, where we allow leaf edges to be of infinite length. The **genus** of $\Gamma$ is $g(\Gamma) = |E| - |V| + 1$.

**Examples** (Finite portions of Genus 2):
Tropical Riemann-Roch

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Examples (Finite portions of Genus 2):

A Chip Configuration $C$ of $\Gamma$ is a formal linear combination of points of $\Gamma$:

$$C = \sum_P c_P P$$

(only finitely many $c_P$'s are nonzero).

The Canonical Chip Configuration $K = K(\Gamma) = \sum_{V \in \Gamma} (\text{val}(V) - 2)V$.

(Gathmann-Kerber, Mikhalkin-Zharkov): The Baker-Norine rank function $r(C)$ satisfies Riemann-Roch for Tropical Curves

$$r(C) - r(K - C) = \deg C + 1 - g(\Gamma).$$
Given a tropical rational function $f$, we let $\text{ord}_P(f)$ denote the sum of the outgoing slopes locally at point $P$ with respect to the function $f$.

The Chip Configuration of $f$ is defined as $(f) = \sum_{P \in \Gamma} \text{ord}_P(f)P$.

Examples: $g_1 = -P_1 + P_2 + P_3 - P_4$, and $g_2 = -2Q_1 + Q_2 + Q_3$. 
Tropical Linear Systems

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Then $(g_1) = -P_1 + P_2 + P_3 - P_4$. and $(g_2) = -2Q_1 + Q_2 + Q_3$.

Can also think of these transformations as weighted chip-firing moves. (We can fire a subgraph of $\Gamma$ in place of a subset of vertices.)

The Tropical Linear System of $C$ (following Gathmann-Kerber):

$|C| = \{ C' \geq 0 : C' = C + (f) \text{ for some tropical rational function } f \}$. 
For $\Gamma = $ with $C$ as specified, we have $|C|$ is

The Linear System $|C|$ contains six 0-cells, seven 1-cells and two 2-cells.
$|C|$ and $R(C)$ as polyhedral cell complexes

Recall $|C| = \{ C' \geq 0 : C' = C + (f) \text{ for some tropical rational function } f \}$. Let $R(C) = \{ f : C + (f) \geq 0 \}$. This is a tropical semi-module of functions.
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**First observation:** $R(C)$ is naturally embedded in $\mathbb{R}^\Gamma$ and $|C|$ is a subset of the $d$th symmetric product of $\Gamma$, where $d = \deg C$. 

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**First observation:** $R(C)$ is naturally embedded in $\mathbb{R}^\Gamma$ and $|C|$ is a subset of the $d$th symmetric product of $\Gamma$, where $d = \deg C$.

Let $\mathbb{1}$ denote the set of constant functions on $\Gamma$. (Note that if $f$ is constant, then the chip configuration $(f) = 0$.)

In fact, there is the natural homeomorphism:

$$R(C)/\mathbb{1} \longrightarrow |C|$$

$$f \mapsto C + (f).$$

So a linear system can be described also by tropical rational functions modulo tropical multiplication (i.e. translation by adding a constant function). Only local slope changes matter, not the function values.
Back To Barbell Example

In terms of tropical rational functions, we obtain the following labeling of the polyhedral complex's vertices instead:

Each of the 1-cells and 2-cells are tropically convex.
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\[ h = f_0 \oplus \left( \frac{1}{4} \odot f_1 \right) \oplus \left( \frac{1}{3} \odot f_4 \right) = \]
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In particular, every tropical rational function on $\Gamma$ is the tropical convex hull of the 0-cells $\{f_0, f_1, \ldots, f_5\}$. 
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More strongly, every tropical rational function on $\Gamma$ is tropical convex hull of $\{f_0, f_2, f_3\}$. Generators of this minimal set are called extremals.
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For example, $g = f_1 \oplus (+1/4 \odot f_5) = f_2 \oplus (+1/4 \odot f_3) =$
Main Results

**Theorem (HMY 2009)** \( R(C) \) is a finitely generated tropical semimodule.

If \( C' \in |C| \), with \( C' = C + (f) \), is in the cell with vertices \( C_1, C_2, \ldots, C_k \) (with corresponding \( f_1, f_2, \ldots, f_k \)), then

\[
f = (c_1 \circ f_1) \oplus (c_2 \circ f_2) \oplus \cdots \oplus (c_k \circ f_k),
\]

i.e. the cells of \( |C| \) are *tropically convex*. 
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If $C' \in |C|$, with $C' = C + (f)$, is in the cell with vertices $C_1, C_2, \ldots, C_k$ (with corresponding $f_1, f_2, \ldots, f_k$), then

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i.e. the cells of $|C|$ are tropically convex.

In particular, $R(C)/\mathbb{1} \cong |C|$ is finitely generated by the 0-cells of $|C|$.

Theorem (HMY 2009) The 0-cells of $|C|$, as well as all other $d$-cells, can be described explicitly.
**Definition.** A point $P \in \Gamma$ is **smooth** if it has valence two and is not a vertex (i.e. the interior of an edge).

**Definition.** The **support** of a chip configuration $C$ is the set of points of $\Gamma$ with nonzero coefficients in $C$.

Let $I(\Gamma, C') = \Gamma \setminus (\text{Supp } C' \cap \{\text{Smooth points}\})$.

**Theorem (HMY 2009)** The cell containing chip configuration $C'$ is of Dimension $= \# (\text{Connected components of } I(\Gamma, C')) - 1$. 
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Theorem (HMY 2009) The cell containing chip configuration $C'$ is of Dimension $= \# (\text{Connected components of } I(\Gamma, C')) - 1$.

Corollary (HMY 2009) The 0-cells, i.e. a set of generators for $R(C)/\mathbb{1}$, correspond to the $C'$’s whose smooth support does not disconnect $\Gamma$.

The extremals lie inside this set: They are the functions $f$ precisely such that no two proper subgraphs $\Gamma_1$ and $\Gamma_2$ of $\Gamma$ covering $\Gamma$ (i.e. $\Gamma_1 \cup \Gamma_2 = \Gamma$) can both fire on the chip configuration $C + (f)$. 
Another return to the barbell

For $\Gamma = \begin{array}{c} \text{1} \\ \text{1} \end{array}$ with $C$ as specified, we have $|C|$ is

Notice that removal of the smooth support of $C'$ (for $C'$ a 0-cell) does not disconnect the graph $\Gamma$. 
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For $\Gamma = \begin{array}{c}
\begin{array}{c}
\bullet \\
\bullet \\
\bullet
\end{array}
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Chip configurations corresponding to tropical rational functions $g$ and $h$ correspond to the interiors of 1-cells and 2-cells.

Removal of their breakpoints disconnects the graph into 2 and 3 pieces.
Other Results

**Theorem (HMY)** If $R(D) = \text{tconv}(f_0, f_1, \ldots, f_r)$, then

$$\phi : \Gamma \rightarrow TP^r$$

$$x \mapsto (f_0(x), \ldots, f_r(x))$$

satisfies $|D| \cong \text{tconv}(\phi(\Gamma))$.

Recall that the tropical convex hull of two points is the tropical line segment between them.
Letting $\Gamma = \begin{array}{c}
\circ \\
1 \\
- \\
1 \\
\end{array}$ with $D$ as specified, we note that the extremals of $|D|$ are $f_0$, $f_2$, and $f_3$ in the picture.
Letting $\Gamma = \begin{array}{c} 1 \\ \hline \\ 1 \end{array}$ with $D$ as specified, we note that the extremals of $|D|$ are $f_0, f_2,$ and $f_3$ in the picture.

Letting $P$ be the leftmost point of $\Gamma$, up to vertical translation (i.e. tropical projective scaling), we can assume that $f_0(P) = f_2(P) = f_3(P) = 0$. 
Embedding the Barbell

Letting $\Gamma = \bullet \quad 1 \quad 1 \quad \bullet$ with $D$ as specified, we note that the extremals of $|D|$ are $f_0$, $f_2$, and $f_3$ in the picture.

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Graphing $f_0$, $f_2$, and $f_3$ along $\Gamma$, we get an infinite matrix with three rows and columns indexed by points of $\Gamma$.

$$
\begin{bmatrix}
0 & \ldots & 0 & \ldots & 0 & \ldots & 0 & \ldots & 0 \\
0 & \ldots & 1 & \ldots & 3/2 & \ldots & 2 & \ldots & 2 \\
0 & \ldots & 0 & \ldots & -1/2 & \ldots & -1 & \ldots & -2
\end{bmatrix}
$$
Embedding the Barbell

We then plot the columns as projective points (ignoring the zeroes in the first row)

\[
\begin{bmatrix}
0 & \ldots & 0 & \ldots & 0 & \ldots & 0 \\
0 & \ldots & 1 & \ldots & 3/2 & \ldots & 2 \\
0 & \ldots & 0 & \ldots & -1/2 & \ldots & -1 \\
\end{bmatrix}
\]

\[(0,0) \quad (1,0) \quad (2,-1) \quad (2,-2)\]
Embedding the Barbell

We then plot the columns as projective points (ignoring the zeroes in the first row)

\[
\begin{bmatrix}
0 & \ldots & 0 & \ldots & 0 & \ldots & 0 \\
0 & \ldots & 1 & \ldots & 3/2 & \ldots & 2 \\
0 & \ldots & 0 & \ldots & -1/2 & \ldots & -1 \\
\end{bmatrix}
\]

The second plot is the tropical convex hull of the points in the first. Observe that \( tconv(f_0, f_2, f_3) \) in \( \mathbb{T}\mathbb{P}^2 \cong \) the linear system \( |D| \).
Final Examples: Genus One Circle Graph

Take the circle $\Gamma = S^1$ on one vertex and a chip configuration of degree $d$. E.g. $d = 3$ or 4:

Black Vertices correspond to Extremals. $|C|$ is a subdivision of a $(d - 1)$-simplex.

In the case of $d = 4$, $|C|$ is a cone over the triangle that is shown. The cone point is the constant function, and is another extremal.
Final Examples: Complete Graph on 4 Vertices

For $\Gamma = K_4$ with edges of equal length and $K$ the canonical chip configuration with 1 at all four vertices: $|K|$ is a cone over the Petersen graph from point $K$.

Theorem (HMY) For any $\Gamma$, the fine subdivision of $\text{link}(K, |K|)$ contains the fine subdivision of the Bergman complex $B(M^*(\Gamma))$ as a subcomplex.
Final Examples: Complete Graph on 4 Vertices (Continued)

Fourteen 0-cells, seven (black vertices) of which (not $K$) are extremal.

This is a 2-dimensional cell complex: including $K$ (at the bottom), here is a close-up of a quadrilateral cell. In particular, $|K|$ is not simplicial.
Open Questions

**Question:** Is there a relationship between geometric properties of the polyhedral cell complex $|C|$ and the Baker-Norine rank function satisfying Tropical Riemann-Roch?
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Thanks for Listening!
