Quasiinvariants of $S_3$

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Abstract

Let $s_{ij}$ represent a transposition in $S_n$. A polynomial $P$ in $\mathbb{Q}[X_n]$ is said to be $m$-quasiinvariant with respect to $S_n$ if $(x_i - x_j)^{2m+1}$ divides $(1 - s_{ij})P$ for all $1 \leq i, j \leq n$. We call the ring of $m$-quasiinvariants, $QI_m[X_n]$. We describe a method for constructing a basis for the quotient $QI_m[X_3]/(e_1, e_2, e_3)$. This leads to the evaluation of certain binomial determinants that are interesting in their own right.

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The symmetric group $S_n$ acts on the ring of polynomials $\mathbb{Q}[X_n]$ by permuting indices. That is for any permutation $\sigma \in S_n$

$$\sigma P(x_1, \ldots, x_n) = P(x_{\sigma(1)}, \ldots, x_{\sigma(n)}).$$

A polynomial $P$ is said to be $S_n$-invariant or symmetric if and only if $\sigma(P) = P$ for all $\sigma \in S_n$. The fundamental theorem of symmetric functions [9, p. 292] states that any invariant of $S_n$ can be written as a polynomial in $\{e_1, e_2, \ldots, e_n\}$ where

$$e_k = \sum_{1 \leq i_1 < \cdots < i_k \leq n} x_{i_1}x_{i_2}\cdots x_{i_k}.$$ 

For $S_3$ we have

$$e_1 = x_1 + x_2 + x_3$$
$$e_2 = x_1x_2 + x_1x_3 + x_2x_3$$
$$e_3 = x_1x_2x_3.$$

A generalization of invariance known as “quasiinvariance” has been studied in the recent literature [1, 2, 3]. In the rest of this paper we will use the notation $s_{ij}$ to denote the transposition $(i, j)$ and will let $QI_m$ denote $\mathbb{Q}I_m[X_n]$ for convenience.

**Definition 1.** A polynomial $P$ is $m$-quasiinvariant if and only if $(1 - s_{ij})P$ is divisible by $(x_i - x_j)^{2m+1}$ for all pairs $1 \leq i < j \leq n$.

This definition is not vacuous because $(1 - s_{ij})P$ is antisymmetric with respect to the transposition $s_{ij}$ thus setting $x_i = x_j$ will yield zero. Hence $(x_i - x_j)$ divides $(1 - s_{ij})P$ and the antisymmetry forces an odd power of $(x_i - x_j)$ to divide it. We should note that an analogous condition defines $m$-quasiinvariance for any Coxeter group. In the general definition, the linear forms giving the equations of the reflecting hyperplanes play the role of the differences $x_i - x_j$.

It is easily seen that the divided difference operator $\Delta_{ij} = \frac{1 - s_{ij}}{x_i - x_j}$ is a twisted derivation [6, pp. 192-194] which means that

$$\Delta_{ij}(Q_1Q_2) = \Delta_{ij}(Q_1)Q_2 + s_{ij}(Q_1)\Delta_{ij}(Q_2).$$

Thus if $\Delta_{ij}(Q_1)$ and $\Delta_{ij}(Q_2)$ are both divisible by $(x_i - x_j)^{2m}$ then so is $\Delta_{ij}(Q_1Q_2)$. The operator $(1 - s_{ij})$ is also linear which means that each $QI_m$ is a ring. Furthermore $(1 - s_{ij})P$ will be divisible by $(x_i - x_j)^{2m+1}$ for arbitrarily large $m$ if and only if $(1 - s_{ij})P = 0$ which means that all $QI_m$ contain $\Lambda_n$ (the ring of symmetric polynomials). We thus have the inclusions

$$\mathbb{Q}[x_1, \ldots, x_n] = QI_0 \supset QI_1 \supset QI_2 \supset \cdots \supset QI_{\infty} = \Lambda_n$$

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A classic result states that \( \mathbb{Q}[x_1, \ldots, x_n] \) is a free module of rank \( n! \) over the ideal \((e_1, \ldots, e_n)\). Furthermore, the action on the quotient precisely gives the regular representation of \( S_n \) [6, p. 247].

This means that there exists a basis of \( n! \) polynomials \( \{\eta_1, \ldots, \eta_{n!}\} \) such that any \( n \)-variable polynomial can be written as a unique linear combination

\[
\sum_{i=1}^{n!} A_i \eta_i
\]

where the \( A_i \)'s are symmetric polynomials. For example, any polynomial in \( \mathbb{Q}[x_1, x_2, x_3] \) can be written uniquely as

\[
A_1 + A_2 x_2 + A_3 x_3 + A_4 x_2 x_3 + A_5 x_3^2 + A_6 x_2 x_3^2
\]

where \( A_1, \ldots, A_6 \) are symmetric polynomials. The polynomial ring can be thought of as the ring of 0-quasiinvariants and recently [3], an analogous result has been proven for the rings of \( m \)-quasiinvariants for \( m > 0 \). Namely, any element of \( QI_m \) can be written uniquely as a sum

\[
\sum_{i=1}^{n!} A_i(e_1, \ldots, e_n) \cdot \eta_i
\]

where the \( A_i \)'s are polynomials and the \( \eta_i \)'s are elements of \( QI_m \).

These \( \eta_i \)'s are therefore a basis for \( QI_m/\langle (e_1, e_2, \ldots, e_n) \rangle \), a space which has been shown [2] to have the following Hilbert series:

\[
\sum_{i=1}^{n!} q^\text{degree}(\eta_i) = \sum_{T \in ST(n)} f_{\lambda(T)} q^{m((\binom{n}{2})-\text{content}(\lambda(T))) + \text{cocharge}(T)}
\]

In the case that \( n = 3 \), this gives that the Hilbert series of \( QI_m/\langle (e_1, e_2, e_3) \rangle \) (\( QI_m \) will always signify \( QI_m[X_3] \) from here on out) is

\[
q^0 + 2q^{3m+1} + 2q^{3m+2} + q^{6m+3}.
\]

(1)

Note also by the respective degrees of \( e_1, e_2, \) and \( e_3 \) that the Hilbert series of \( QI_m \) is

\[
\frac{q^0 + 2q^{3m+1} + 2q^{3m+2} + q^{6m+3}}{(1-q)(1-q^2)(1-q^3)}
\]

(2)

It is easily shown that the Vandermonde determinant \( \Delta(x) = (x_1 - x_2)(x_1 - x_3)(x_2 - x_3) \) raised to the power \( 2m+1 \) accounts for the term \( q^{6m+3} \) and clearly
the constants account for \( q^0 \). So the interesting problem arises to construct the four \( m \)-quasiinvariants that account for the terms \( 2q^{3m+1} + 2q^{3m+2} \). The explicit construction of these four \( m \)-quasiinvariants is the goal and motivating force which led to the results of this paper. It developed that this construction required the evaluation of two binomial determinants which are interesting in their own right and deserve a special mention here. The two resulting identities may be stated as follows.

**Theorem 1.**

\[
\det \left| \begin{array}{c}
(C + \alpha i) \\
(E + \beta j)
\end{array} \right| - \left| \begin{array}{c}
(D - \alpha i) \\
(E + \beta j)
\end{array} \right|_{i,j=1}^k = \frac{(C + D)(C + D + 1) \cdots (C + n\alpha)}{(C + \alpha)(C + 2\alpha) \cdots (C + n\alpha)} \cdot |F| \tag{3}
\]

where \( F \) denotes the collection of \( k \)-tuples of non-intersecting lattice paths respectively joining the points

\[
\{(D - k\alpha, D - k\alpha), (D - (k-1)\alpha, D - (k-1)\alpha), \ldots, (D - \alpha, D - \alpha)\}
\]

and throughout avoiding the line \( y = -x + C + D \).

It is also worthy of notice the fact that the entries of the determinant in (3) are differences of binomial coefficients where the tops are different and the bottoms are the same. A literature search found no determinant results covering this particular case. Nevertheless, a manipulation suggested by an argument of Gessel and Viennot in [5] enabled us to derive Theorem 1 from the following general result:

**Theorem 2.** For any integers \( a, b, c, d, e \), the determinant

\[
\det \left| \begin{array}{c}
a + bi \\
c + dj
\end{array} \right| - \left| \begin{array}{c}
a + bi \\
e - dj
\end{array} \right|_{i,j=1}^n
\]

is the number of families of non-intersecting lattice paths with NORTH and WEST steps, respectively joining the points

\[
\{(c + d, c + d), (c + 2d, c + 2d), \ldots, (c + nd, c + nd)\}
\]

and throughout avoiding the line \( y = -x + (c + e) \).

Our main result is that a basis for the quotient of the \( m \)-quasiinvariants of \( S_3 \) can be found by computing the 1-dimensional null space of particular matrices.
The non-vanishing of the determinant (3) provides the crucial step in proving the null space in question is indeed 1-dimensional.

Our presentation is divided into four parts. In the first part we show (non-constructively) that quasiinvariants of a certain nice form exist. In the second part, we find a system of equations that the coefficients of these quasiinvariants must satisfy. In the third part, we show that we can solve this system by computing a 1-dimensional null space. In the final part we complete the construction, and prove the elements we’ve constructed complete a basis for the quotient.

We should mention that Feigin and Veselov in [1] have given explicit module bases for the \( m \)-quasiinvariants of all Dihedral groups \( D_n \). But so far there are no other Coxeter groups for which explicit constructions have been given. The Feigin-Veselov construction is based on complex number techniques that are very suitable in the dihedral case. Although \( D_3 \) \( m \)-quasiinvariants can be easily converted into \( S_3 \) \( m \)-quasiinvariants, our work efforts have been guided by the need of developing methods that can be extended to the general case. Our results may be taken as an instance of such methods. Extensions of the present construction to \( S_n \) will be the topic of a forthcoming publication.

1 Quasiinvariants with a nice form

We begin by defining the following elements of the group algebra of \( S_3 \):

\[
[S_3] = \frac{1}{6} \sum_{\sigma \in S_3} \sigma,
\]

\[
[S_3]' = \frac{1}{6} \sum_{\sigma \in S_3} \text{sgn}(\sigma)\sigma
\]

\[
\pi_1 = \frac{1}{3}(1 + s_{23})(1 - s_{12}),
\]

\[
\pi_2 = \frac{1}{3}(1 + s_{12})(1 - s_{23})
\]

These defined, the following identities are easily verified:

\[
(\pi_1)^2 = \pi_1, (\pi_2)^2 = \pi_2
\]

\[
[S_3]'\pi_1 = \pi_1\pi_2 = \pi_2\pi_1 = 0
\]

\[
[S_3] + \pi_1 + \pi_2 + [S_3]' = 1
\]

\[
s_{23}\pi_1 = \pi_1
\]

\[
\pi_2 s_{12}\pi_1 = -s_{13}\pi_1
\]

We now show that there exist quasiinvariants satisfying certain symmetry and independence conditions.

**Lemma 1.** For all \( m \geq 0 \), there exist non-symmetric \( m \)-quasiinvariants \( A_1, A_2 \) of degrees \( 3m + 1, 3m + 2 \), respectively, such that \( s_{23}(A_1) = A_1 \) and in the quotient \( QI_m/\langle (e_1, e_2, e_3) \rangle \), the image of \( A_1 \) and \( s_{12}(A_1) \) are linearly independent. Further all four of these will be independent of \( \Delta^{2m+1}(x) \).
Proof. It is easy to see that the image of $[S_3]$ in the quotient is the constant terms. We also note that any polynomial in the image of $[S_3]$ is alternating and any alternating $m$-quasiinvariant must be divisible by $\Delta^{2m}(x)$, which has degree $6m$. Thus, from the Hilbert series (1), there must exist quasiinvariants $B_i$ of degree $3m + i$, ($i \in \{1, 2\}$) such that if we apply equation (6) to $B_i$ we have

$$\pi_1(B_i) + \pi_2(B_i) \neq 0.$$  

(9)

Assume without loss that $\pi_1(B_i) \neq 0$, and set

$$A_i = \pi_1(B_i).$$  

(10)

Equation (7) immediately gives that $s_23(A_i) = A_i$. Now suppose we had symmetric functions $S, T$ such that

$$SA_i + T(s_{12}A_i) = 0$$  

(11)

Applying $\pi_2$ to this gives (by (5) and (8)):

$$T \pi_2 s_{12} \pi_1 B_i = 0 \quad (12)$$

$$- T s_{13} \pi_1 B_i = 0 \quad (13)$$

$$- T s_{13} A_i = 0 \quad (14)$$

Since $A_i$ was assumed to be non-zero, this gives $T = 0$ and (11) gives $S = 0$. Now assume there was a nontrivial relationship between these and $\Delta^{2m+1}(x)$

$$c_1 A_1 + c_2 (s_{12} A_1) + c_3 A_2 + c_4 (s_{12} A_2) + c_5 \Delta^{2m+1}(x) = 0$$  

(15)

Applying $[S_3]'$ gives (by (5))

$$c_2 [S_3]' (s_{12} A_1) + c_4 [S_3]' (s_{12} A_2) + c_5 \Delta^{2m+1}(x) = 0$$  

(16)

But $[S_3]' s_{12} = s_{12} [S_3]'$ and $[S_3]' A_i = 0$ so (16) gives $c_5 = 0$.

\[ \square \]

Since $s_{23}(A_i) = A_i$, $A_i$ is symmetric with respect to $x_2$ and $x_3$. This means that we can write the $m$-quasiinvariants $A_1$ and $A_2$ as

$$A_1 = \sum_{0 \leq i \leq j \leq d} C_{[i,j]} x_1^{d-i-j} m_{[i,j]}(x_2, x_3).$$  

(17)

and

$$A_2 = \sum_{0 \leq i \leq j \leq d} \tilde{C}_{[i,j]} x_1^{d-i-j} m_{[i,j]}(x_2, x_3).$$  

(18)

for $d = 3m + 1$ or $3m + 2$, respectively. In fact we can make the following stronger statement about the form of the $A_i$:
Lemma 2. There exist $m$-quasiinvariants $A_1$ and $A_2$, satisfying the conditions of Lemma 1, of the form

$$A_1 = \sum_{0 \leq i \leq j \leq m} C_{[i,j]} x_1^{3m+1-i-j} m_{[i,j]}(x_2, x_3)$$

and

$$A_2 = \sum_{0 \leq i \leq j \leq m+1} \tilde{C}_{[i,j]} x_1^{3m+2-i-j} m_{[i,j]}(x_2, x_3).$$

**Proof.** We first prove this result for $A_1$. By grouping together monomials with similar exponent sequences, we can rewrite the above sum (17) as

$$\sum_{\substack{0 \leq i < j < k \leq m}} \left( C_{[i,j]}(x_1^i x_2^j x_3^k + x_1^i x_2^j x_3^k) + C_{[i,k]}(x_1^i x_2^j x_3^k + x_1^i x_2^j x_3^k) ight) + \sum_{\substack{0 \leq i < j \leq m+1}} C_{[i,j]}(x_1^i x_2^j x_3^k + x_1^i x_2^j x_3^k) + C_{[i,i]} x_1^i x_2^j x_3^k.$$

Using this decomposition, we find that $(1 - s_{13})A_1$ is the sum

$$\sum_{\substack{0 \leq i < j < k \leq m+1}} \left( (C_{[i,k]} - C_{[i,j]})(x_1^i x_2^j x_3^k - x_1^i x_2^j x_3^k) + (C_{[i,k]} - C_{[i,j]})(x_1^i x_2^j x_3^k - x_1^i x_2^j x_3^k) + (C_{[i,j]} - C_{[i,i]})(x_1^i x_2^j x_3^k - x_1^i x_2^j x_3^k) + \sum_{\substack{2i+j \leq 3m+1}} (C_{[i,j]} - C_{[i,i]})(x_1^i x_2^j x_3^k - x_1^i x_2^j x_3^k).$$

We can now discover properties of the coefficients by focusing on one summand at a time. For instance, given a specific composition $[i, j, k]$ of $3m + 1$ such that $0 \leq i < j < k$, the fact that $i + j + k > 3m$ means that the largest exponent, namely $k$, will be greater than $m$. However, $A_1$ $m$-quasiinvariant means that $(x_1 - x_3)^{2m+1}(1 - s_{13})A_1$ and thus the highest power of $x_2$ that can appear in $(1 - s_{13})A_1$ will be $(3m + 1) - (2m + 1) = m$. Thus $x_2^k$ cannot appear in a term of $(1 - s_{13})A_1$ with a nonzero coefficient, and thus we obtain $C_{[i,k]} = C_{[j,k]}$. If both the exponents $j$ and $k$ happen to be greater than $m$, then by similar logic we conclude that $C_{[i,j]} = C_{[i,k]} = C_{[j,k]}$. Finally, if we are given the composition $[i, i, j]$ with $i > m$, we see that $C_{[i,j]} = C_{[i,i]}$. We summarize these conditions
here:

\[
C_{i,k} = C_{i,i} \quad \text{when } i < j < k
\]

(19)

\[
C_{i,j} = C_{i,k} = C_{j,k} \quad \text{when } i < j < k, \quad j > m
\]

(20)

\[
C_{i,j} = C_{i,i} \quad \text{when } m < i < j.
\]

(21)

The idea now will be to subtract certain symmetric functions from \(A_1\) in order to get rid of exponents of \(x_2\) and \(x_3\) greater than \(m\), without changing the equivalence class of \(A_1\) in the quotient. For every triplet \{\(i, j, k\)\} of exponents with \(i < j < k, j \leq m\), we see that \(A_1 - C_{i,k} m_{i,j,k}\) has

\[
(C_{i,j} - C_{i,k}) (x_1^{k_i} x_2^{k_j} x_3^{k_k} + x_1^{k_i} x_2^{k_j} x_3^{k_k})
\]

(22)

as the only monomials with exponent sequence a permutation of \((i, j, k)\), by (19). (Here \(m_{i,j,k}\) is the monomial symmetric function with exponents \(i, j, k\)).

For every triplet \{\(i, j, k\)\} of exponents with \(i < j < k, j > m\), we have that \(A_1 - C_{i,k} m_{i,j,k}\) has no monomials with exponent sequence a permutation of \((i, j, k)\), by (20).

For every remaining triplet \{\(i, i, j\)\} of exponents we see that \(A_1 - C_{i,j} m_{i,i,j}\) has

\[
(C_{i,i} - C_{i,j}) x_1^{k_i} x_2^{k_j} x_3^{k_j}
\]

(23)

as the only monomial with exponent sequence a permutation of \((i, i, j)\), which by (21) is only nonzero when \(i \leq m\). Thus, after subtracting appropriate symmetric functions we are left with a sum containing only monomials such that the exponents of \(x_2\) and \(x_3\) are less than or equal to \(m\). This gives the stated result for \(A_1\).

Since \(A_2\) has degree \(3m+2\), the highest power of \(x_2\) that can appear in \((1 - s_{13}) A_2\) is \(m + 1\). Thus any composition \([i, j, k]\) such that \(0 \leq i < j < k\) and \(i + j + k = 3m + 2\) will have to satisfy \(k > m + 1\), which will allow us to equate certain coefficients as above. Any composition where \(0 \leq i, j\) and \(2i + j = 3m + 2\) will only yield three terms, two of which have the same coefficient. Either way, we will analogously be able to use appropriate symmetric functions to subtract from \(A_2\) so that monomials with powers of \(x_2\) or \(x_3\) exceeding \(m+1\) will disappear. \(\square\)

Later on, we will demonstrate that, for \(A_2\), we can strengthen the result of Lemma 2. Namely we will prove that there exists a quasiiinvariant \(A_2\) of degree \(3m + 2\) that satisfies the properties of Lemma 1 and is of the form

\[
A_2 = \sum_{0 \leq i, j \leq m} \tilde{C}_{i,j} x_1^{3m+2-i-j} m_{i,j}(x_2, x_3).
\]

(24)

Note that the indices of the sum are now less than \(m+1\). The proof of this will require the explicit construction of \(A_1\), and will be necessary to explicitly construct \(A_2\).
2 Relations satisfied by the coefficients \( C_{[i,j]} \)

In this section we show the \( C_{[i,j]} \) satisfy certain relations. We begin by setting \( d = 3m + 1 \), and

\[
A_{i,j,k,l} = \begin{cases} 
\binom{i}{k} \binom{d-i-k}{l} - \binom{i}{k} \binom{2i-k}{l} & \text{if } i = j, \\
\binom{i}{k} \binom{d-j-k}{l} + \binom{j}{l} \binom{d-i-k}{l} - \left( \binom{i}{l} + \binom{j}{l} \right) \binom{i+j-k}{l} & \text{otherwise.}
\end{cases}
\]

We can now state the main result of this section.

**Lemma 3.** The coefficients \( C_{[i,j]} \) satisfy the linear equations

\[
\sum_{0 \leq j \leq l \leq m} A_{i,j,k,l} C_{[i,j]} = 0 \tag{25}
\]

for \( k \in \{0, \ldots, m\} \) and \( l \in \{1, 3, 5, \ldots, 2m - 1\} \).

**Proof.** By definition, if \( i > j \), then \( C_{[i,j]} \) is the coefficient of

\[
x_1^{d-i-j} m_{[i,j]}(x_2, x_3) = x_1^{d-i-j} \left( x_2^i x_3^j + x_2^j x_3^i \right).
\]

If instead \( i = j \), then \( C_{[i,j]} \) is the coefficient of \( x_1^{d-2i} x_3^i \). Consequently, inside of \( (1 - s_{13})A_1 \), \( C_{[i,j]} \) is the coefficient of the polynomial

\[
x_1^{d-i-j} x_2^i x_3^j + x_1^{d-i-j} x_2^j x_3^i - x_1^{d-i-j} x_2^i x_3^j - x_1^{d-i-j} x_2^j x_3^i
\]

if \( i > j \) and

\[
x_1^{d-i-j} x_2^i x_3^j - x_1^{d-i-j} x_2^j x_3^i
\]

if \( i = j \). Using the substitutions \( y_1 = x_2 - x_1 \) and \( y_2 = x_1 - x_3 \), we rewrite these polynomials. For the case \( i = j \) we have

\[
(1 - s_{13})A_1 \big|_{C_{[i,i]}} = x_1^{d-2i} (y_1 + x_1)^i x_3^i - x_1^{d-2i} (y_1 + x_1)^i x_3^i
\]

\[
= \sum_{k=0}^{i} \binom{i}{k} x_1^{d-i-k} y_1^k x_3^i - \binom{i}{k} x_1^{2i-k} y_1^k x_3^i
\]

\[
= \sum_{k=0}^{i} \binom{i}{k} (y_2 + x_3)^{d-i-k} y_1^k x_3^i - \binom{i}{k} (y_2 + x_3)^{2i-k} y_1^k x_3^i
\]

\[
= \sum_{k=0}^{i} \binom{i}{k} \sum_{l=0}^{d-i-k} \binom{d-i-k}{l} y_1^k y_2^l x_3^{d-k-l} - \sum_{l=0}^{2i-k} \binom{2i-k}{l} y_1^k y_2^l x_3^{d-k-l}
\]

\[
= \sum_{k=0}^{i} \sum_{l=0}^{\max(d-i-k, 2i-k)} \binom{i}{k} \binom{d-i-k}{l} - \binom{2i-k}{l} y_1^k y_2^l x_3^{d-k-l}
\]

\[
= \sum_{k=0}^{i} \sum_{l=0}^{\max(d-i-k, 2i-k)} A_{i,i,k,l} y_1^k y_2^l x_3^{d-k-l}
\]
For $i > j$ we have

$$(1 - s_{13}) A_1 \bigg|_{C_{i,j}} = x_1^{d-i+j} (y_1 + x_1)^j x_3^j + x_1^{d-i+j} (y_1 + x_1)^j x_3^j - x_1^j (y_1 + x_1)^j x_3^{d-i-j}$$

$$= \sum_{k=0}^{i} \binom{i}{k} x_1^{d-j-k} y_1^k x_3^j + \binom{j}{k} x_1^{d-i-k} y_1^k x_3^j$$

$$- \binom{i}{k} x_1^{i+j-k} y_1^k x_3^{d-i-j} - \binom{j}{k} x_1^{i+j-k} y_1^k x_3^{d-i-j}$$

$$= \sum_{k=0}^{i} \binom{i}{k} (y_2 + x_3)^{d-j-k} y_1^k x_3^j + \binom{j}{k} (y_2 + x_3)^{d-i-k} y_1^k x_3^j$$

$$= \sum_{k=0}^{i} \max\{d-j-k, i+j-k\} \sum_{l=0}^{\max\{d-j-k, i+j-k\}} \binom{i}{k} \binom{d-j-k}{l} + \binom{j}{k} \binom{d-i-k}{l} y_1^k y_2^l x_3^{d-k-l}$$

$$= \sum_{k=0}^{i} \max\{d-j-k, i+j-k\} \sum_{l=0}^{\max\{d-j-k, i+j-k\}} A_{i,j,k,l} y_1^k y_2^l x_3^{d-k-l}$$

By definition, $A_1$ is $m$-quasiinvariant if and only if $(1 - s_{13}) A_1$ is divisible by $y_2^{2m+1}$. Solving the equations implies that $(1 - s_{13}) A_1 \bigg|_{C_{i,j}}$ has even order or order greater than $2m - 1$ with respect to $y_2$. Since $(1 - s_{13}) A_1$ is divisible by an odd power of $(x_1 - x_3)$, we make the following statement: for fixed $k \in \{0, \ldots, m\}$ and fixed odd $l < 2m + 1$, we must have

$$\sum_{0 \leq j \leq i \leq m} A_{i,j,k,l} C_{i,j} y_1^k y_2^l x_3^{d-k-l} = 0.$$

The lemma is an immediate consequence.

### 3 The coefficients have a one-dimensional solution space

Once we verify that the relations in (25) have a one-dimensional solution space, it is a straightforward (although time-intensive) process to find a representative solution. This will allow us to explicitly construct $A_1$, for which we currently have only an existence proof. We begin by computing the determinants of certain matrices, beginning with Theorem 2, stated in the introduction.
Proof of Theorem 2. We first show that the number of lattice paths from \((c + jd, c + jd)\) to \((0, a + ib)\) which avoid the line \(y = -x + (c + e)\) is \(\binom{a+bi}{c+dj} - \binom{a+bi}{c-e-dj}\).

Consider the following two diagrams:

**Figure 1:** Counting paths from \((c + jd, c + jd)\) to \((0, a + ib)\).

**Figure 2:** ‘Bad’ paths from \((c + jd, c + jd)\) to \((0, a + ib)\).

The number of bad paths in rectangle \(A\), namely the ones that go through the forbidden line, is in bijection with the number of total paths in rectangle \(B\); we replace WEST steps with NORTH steps and NORTH steps with WEST steps following the first touch of the forbidden line. This is known as Andrée’s
Reflection Principle [4]. Thus the number of good paths in rectangle $A$ is exactly the correct difference of binomials.

This shown, a classical involution of Lindström [8] and Gessel-Viennot [5] shows that when the entries of a matrix count paths, the determinant counts families of non-intersecting paths. This completes the proof.

We now are in a position to prove Theorem 1, as stated in the introduction.

Proof of Theorem 1. We begin by considering a more general form of this matrix and factoring it. This factorization was suggested by an argument of Gessel and Viennot [5]:

$$
\det \begin{pmatrix} \frac{c}{c} & \cdots & \frac{c}{c} \\ \vdots & \ddots & \vdots \\ \frac{c}{c} & \cdots & \frac{c}{c} \end{pmatrix} \cdot \det \begin{pmatrix} \frac{c}{c} & \cdots & \frac{c}{c} \\ \vdots & \ddots & \vdots \\ \frac{c}{c} & \cdots & \frac{c}{c} \end{pmatrix} = \det \begin{pmatrix} \frac{c}{c} & \cdots & \frac{c}{c} \\ \vdots & \ddots & \vdots \\ \frac{c}{c} & \cdots & \frac{c}{c} \end{pmatrix} 
$$

Proposition 14 of [5] used an analogous factorization for the determinant of a matrix of single binomial coefficients. Our factorization also works by the symmetry $\binom{c}{a_i} = \binom{c}{a_j}$. This implies that the same quotient of binomials can be factored out of both terms that appear as a difference in our entries. Returning to the proof of Theorem 1, we let $a_i = C + \alpha \beta$, $b_j = E + \beta j$, and $c = C + D$ and find

$$
\det \begin{pmatrix} C + \alpha \beta j \\ E + \beta j \end{pmatrix} - \begin{pmatrix} D - \alpha \beta j \\ E + \beta j \end{pmatrix} = \begin{pmatrix} C + D \alpha \beta j + \alpha \beta j \\ C + D \alpha \beta j + \alpha \beta j \end{pmatrix}. 
$$

$$
\det \begin{pmatrix} C + D - E - (k - i + 1) \beta \\ D - (k - j + 1) \alpha \end{pmatrix} - \begin{pmatrix} C + D - E - (k - i + 1) \beta \\ C + (k - j + 1) \alpha \end{pmatrix} = \begin{pmatrix} C + D - E - (k - i + 1) \beta \\ C + (k - j + 1) \alpha \end{pmatrix}. 
$$

Notice that now the tops of the binomial coefficients are the same and the bottoms are different. This allows us to apply Theorem 2 to obtain the result.

We now see how these results can help us with our system of equations. Notice that in (25) there are $\binom{m+2}{2}$ coefficients $C_{i,j}$ and $m(m+1)$ equations. We define $B_m$ as the restriction of the matrix given by (25) to the $\binom{m+2}{2}$ sub-matrix where $[i,j] \neq [m,m], 0 \leq k \leq m-1$ and $l \in \{2m-2k-1, \ldots, 2m-3, 2m-1\}$ or $k = m$ and $l \in \{1, 3, 5, \ldots, 2m-1\}$. 

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Lemma 4. The matrix $B_m$ is nonsingular

Proof. By using an ordering for the pairs $(k, l)$ where the $k$’s increase and the $l$’s decrease while lexicographically ordering the $[i, j]$s, the matrix $B_m$ becomes block triangular. Furthermore, there is one block of size 1, one block of size 2, . . . , one block of size $m - 1$, and two blocks of size $m$. This block triangularity follows from the fact that for $i, j$ such that $0 \leq j \leq i < k$, then $\binom{j}{i} = \binom{i}{j} = 0$ and thus the $A_{i, j, k, l}$’s of equation (25) are all zero.

Furthermore, the entries of $B_m$ inside these blocks, where $j$ runs over the interval $0 \leq j \leq i = k$, are much simpler than the general case. For such $i, j$’s, the $A_{i, j, k, l}$’s of equation (25) simplify to

$$A_{k, j, k, l} = \begin{cases} \binom{k}{l} \left( d - 2k \right) - \binom{k}{j} \binom{k}{l} - \binom{k}{j} \binom{k}{l} - \binom{k}{j} \binom{k}{l} & \text{if } j = k, \\ \binom{k}{l} \left( d - j - k \right) - \binom{k}{j} \binom{k}{l} & \text{otherwise}. \end{cases}$$

But since $\binom{k}{l} = 1$ and $\binom{k}{l} = 0$ if $j < k$ we obtain

$$A_{k, j, k, l} = \binom{d - j - k}{l} - \binom{k}{l}. \quad (26)$$

For $f \in \{1, 2, \ldots, m\}$, we let $B_{f, m}^{f, m}$ denote the $f$th block matrix on the diagonal of $B_m$, which forces $f = k + 1$, and set $B_m$ to be the final block matrix. Setting $d = 3m + 1$ and utilizing (26) allows us to describe the entries of these blocks as follows:

For $j \in \{0, \ldots, f - 1\}$ and $l \in \{2m - 2f + 1, 2m - 2f - 1, \ldots, 2m - 1\}$,

$$B_{f, j}^{f, m} = \binom{3m + 1 - j - (f - 1)}{l} - \binom{j}{l}$$

and for $j \in \{0, \ldots, m - 1\}$ and $l \in \{1, 3, \ldots, 2m - 1\}$,

$$B_{l, j}^{m} = \binom{2m + 1 - j}{l} - \binom{j}{l}.$$

At this point, we re-index the matrix $B_{f, m}^{f, m}$, replacing the current indices of $j$ and $l$ with the standard indices $i, j \in \{1, \ldots, f\}$. This gives

$$B_{f, m} = \begin{pmatrix} \binom{3m + 1 - (j - 1) - (f - 1)}{2m + 1 - 2i} - \binom{j - 1}{2m + 1 - 2i} \end{pmatrix} \bigg|_{i, j=1}^{f} \quad (27)$$

and
\[
B^m = \left| \begin{array}{c}
(2m + 1 - (j - 1)) \\
2m + 1 - 2i
\end{array} \right|_{i,j=1}^m \cdot (28)
\]

Applying Theorem 1 to the transpose of this matrix, we find the determinant of (27) is

\[
\left( \frac{3m+2-f}{2m-1} \right) \left( \frac{3m+2-f}{2m-3} \right) \cdots \left( \frac{3m+2-f}{3m-2f+1} \right) \cdot |F|
\]

where \( F \) is the set of families of non-intersecting lattice paths from \( \{(0,0), (1,1), \ldots, (f-1, f-1)\} \) to \( \{(0, m-f+3), (0, m-f+5), \ldots, (0, m+f+1)\} \) which stay below the line \( y = -x + 3m + 2 - f \). Since this family of paths is non-empty, we conclude that the matrices \( B_f^m \) are non-singular for \( f \in \{1, 2, \ldots, m\} \). Similarly we find that the determinant of (28) is positive and thus \( B^m \) is also non-singular. Since the diagonal blocks of \( B_m \) are non-singular, the matrix \( B_m \) must also be.  

An example may help to clarify things at this point. When \( m = 3 \) and \( d = 10 \), we have the matrix

\[
\begin{bmatrix}
252 & 378 & 126 & 308 & 182 & 56 & 273 & 147 & 75 \\
0 & 126 & 56 & 252 & 133 & 42 & 378 & 174 & 75 \\
0 & 84 & 56 & 168 & 147 & 68 & 252 & 184 & 125 \\
0 & 0 & 0 & 56 & 21 & 6 & 168 & 63 & 19 \\
0 & 0 & 0 & 56 & 35 & 20 & 168 & 105 & 66 \\
0 & 0 & 0 & 8 & 6 & 4 & 21 & 15 & 11 \\
0 & 0 & 0 & 0 & 0 & 0 & 21 & 6 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 35 & 20 & 10 \\
0 & 0 & 0 & 0 & 0 & 0 & 7 & 5 & 3
\end{bmatrix}
\]

This matrix is the matrix of coefficients \( A_{i,j,k,l} \) where the columns are indexed by the \( [i, j] \)'s and the rows are indexed by the pairs \( (k, l) \). In this example, the columns have the order

\[
[i, j] = [0, 0], [1, 0], [1, 1], [2, 0], [2, 1], [2, 2], [3, 0], [3, 1], [3, 2]
\]

and the rows have the order:

\[
(k, l) = (0, 5), (1, 5), (1, 3), (2, 5), (2, 3), (2, 1), (3, 5), (3, 3), (3, 1).
\]

We also have the following block sub-matrices:

\[
B_{1,3} = \begin{bmatrix}
252
\end{bmatrix}
\]

\[
B_{2,3} = \begin{bmatrix}
126 & 56 \\
84 & 56
\end{bmatrix}
\]
\[ B^{3,3} = \begin{bmatrix} 56 & 21 & 6 \\ 56 & 35 & 20 \\ 8 & 6 & 4 \end{bmatrix} \]

\[ B^3 = \begin{bmatrix} 21 & 6 & 1 \\ 35 & 20 & 10 \\ 7 & 5 & 3 \end{bmatrix} \]

Lemma 5. The equations given in (25) have a solution that is unique up to scalar multiples.

Proof. Since the system in (25) has an \( \binom{m+2}{2} - 1 \) \( \times \) \( \binom{m+2}{2} - 1 \) nonsingular sub-matrix, it must be true that the rank of the system in (25) is \( \geq \binom{m+2}{2} - 1 \). Thus the null space has dimension \( \leq 1 \). However, since we know by Lemma 2 that \( A_1 \) is a solution, the dimension of the null space must be exactly one. \( \square \)

4 Constructing \( A_2 \) and a basis for the quotient

In the first section, we showed the existence of nonzero (in the quotient) \( m \)-quasiinvariants \( A_1, A_2 \) of degrees \( 3m+1 \) and \( 3m+2 \), respectively, that are both symmetric with respect to \( s_{23} \). In the previous two sections we illustrated an explicit construction of the element \( A_1 \). We now give an explicit construction of the element \( A_2 \), which will be linearly independent of \( A_1 \). We have deferred this construction until now since this argument is dependent on the explicit form of \( A_1 \). We begin by strengthening Lemma 2.

Lemma 6. There exists an \( m \)-quasiinvariant of degree \( 3m+2 \), satisfying the conditions of Lemma 1, which has the form given in equation (24).

Proof. First, we observe that the Hilbert series (1) and Lemma 2 tell us there is a nonzero \( m \)-quasiinvariant of degree \( 3m+1 \),

\[ A_1 = \sum_{0 \leq i \leq j \leq m} C_{[i,j]} x_1^{3m+1-i-j} m_{[i,j]}(x_2, x_3), \]

as well as a nonzero \( m \)-quasiinvariant of degree \( 3m+2 \),

\[ A_2 = \sum_{0 \leq i \leq j \leq m+1} \tilde{C}_{[i,j]} x_1^{3m+2-i-j} m_{[i,j]}(x_2, x_3). \]

We proved in the last section that the set of possible coefficient vectors \( \langle C_{[i,j]} \rangle \) comprises a 1-dimensional space. Eliminating the last column of the matrix of entries \( A_{i,i,k,l} \)'s is like setting the coefficient \( C_{[m,m]} = 0 \). Since the sub-matrix \( B_m \) also lacks that column and is nonsingular we conclude that the nonzero \( m \)-quasiinvariant \( A_1 \) satisfies \( C_{[m,m]} \neq 0 \). Consequently, \( e_1 A_1 \) has a nonzero multiple of \( x_1^{m+1} x_2^{m+1} x_3^m \) as one of its terms while at the same time the term \( x_1^m x_2^{m+1} x_3^{m+1} \) will not appear. With no cancellation therefore possible, the
quantity $(1 - s_{13})e_1 A_1$ will contain the term $C(x_1 - x_3)^{2m+1}x_2^{m+1}$ for some nonzero $C$.

Since $(1 - s_{13})A_2 = (x_1 - x_3)^{2m+1}(C' x_2^{m+1} + \text{terms of lower order})$, we find that $(1 - s_{13})(C' e_1 A_1 - CA_2)$ contains no term with $x_2^{m+1}$. We thus re-define $A_2$ as the quantity $C'(A_1 - CA_2)$ (which still meets the conditions of Lemma 1). Recall that in the proof of Lemma 2, the crucial step that proved the result for $A_1$ was the fact that we could eliminate every term in $(1 - s_{13})A_1$ containing a power of $x_2$ exceeding $m$. Now we can utilize this fact for $A_2$ also. The rest of the proof goes through as before and we conclude that $A_2$ can be written as

$$\sum_{0 \leq i \leq j \leq m} \tilde{C}_{[i,j]} x_1^{3m+2-i-j} m_{[i,j]}(x_2, x_3).$$ (29)

We now examine how the construction of $A_1$ can be applied to construct $A_2$. In section 2, we used the fact that $A_1$ had the form

$$\sum_{0 \leq i \leq j \leq m} C_{[i,j]} x_1^{3m+1-i-j} m_{[i,j]}(x_2, x_3).$$

to obtain a linear system of relations that the $C_{[i,j]}$’s satisfy. Since we now know that $A_2$ has an analogous form, namely (29), we can apply the same proof (setting $d = 3m + 2$) to obtain an analogous system for the $\tilde{C}_{[i,j]}$’s.

These coefficients can be explicitly computed by finding the null space of the matrix given by the linear system

$$\sum_{0 \leq j \leq i \leq m} A_{i,j,k,l} \tilde{C}_{[i,j]} = 0$$ (30)

for $k \in \{0, \ldots, m\}$ and $l \in \{1, 3, 5, \ldots, 2m - 1\}$. As in the $A_1$ case, this null space is 1-dimensional and we prove this by showing that the matrix $\tilde{B}_m$ is nonsingular, where $\tilde{B}_m$ is the restriction of the matrix given by (30) to the $\left(\begin{pmatrix} m+2 \cr 2 \end{pmatrix} - 1 \right) \times \left(\begin{pmatrix} m+2 \cr 2 \end{pmatrix} - 1 \right)$ sub-matrix where $[i,j] \neq [m,m]$, $0 \leq k \leq m - 1$ and $l \in \{2m - 2k - 1, \ldots, 2m - 3, 2m - 1\}$ or $k = m$ and $l \in \{1, 3, 5, \ldots, 2m - 1\}$.

The matrix $\tilde{B}_m$ is block triangular and thus we prove that it is nonsingular by proving that its blocks

$$\tilde{B}^{i,m} = \left(\begin{pmatrix} 3m + 2 - (j - 1) - (f - 1) \cr 2m + 1 - 2i \end{pmatrix} - \begin{pmatrix} j - 1 \cr 2m + 1 - 2i \end{pmatrix}_{i,j=1} \right)_{i,j=1}^{f}$$ (31)
for \( f \in \{1, 2, \ldots, m\} \) as well the additional block

\[
\tilde{B}^m = \left( \begin{array}{c}
2m + 2 - (j - 1) \\
2m + 1 - 2i
\end{array} \right)_{i,j=1}^m
\]  

(32)

are nonsingular. We proceed identically to our computation of the determinant of (27). We find that the determinant of (31) is a positive scalar multiplied by the number of families of non-intersecting lattice paths from \( \{(0, 0), (1, 1), \ldots, (f - 1, f - 1)\} \) to \( \{(0, m - f + 4), (0, m - f + 6), \ldots, (0, m + f + 2)\} \) which stay below the line \( y = -x + 3m + 3 - f \). Since such paths exist, this determinant is positive. Similarly we find that the determinant of (32) is positive and thus our construction of \( A_2 \) is valid.

**Theorem 3.** The set \( \{1, A_1, s_{12}(A_1), A_2, s_{12}(A_2), \Delta_{2m+1}(x)\} \) is a basis for the quotient \( QI_m/(\langle e_1, e_2, e_3 \rangle) \).

**Proof.** It remains only to prove the independence of \( \{A_1, s_{12}(A_1), A_2, s_{12}(A_2)\} \) in the quotient. By examining the Hilbert series of \( QI_m(2) \), we find that the subspace of \( Q[x_1, x_2, x_3] \) consisting of \( 3m + 2 \)-dimensional \( m \)-quasiinvariants which are not symmetric is \( 4 \)-dimensional. Thus it is spanned by \( e_1A_1, e_1(s_{12})A_1 \) and two other elements. Since we have shown that \( A_i \) and \( s_{12}A_i \) are linearly independent for \( i \in \{1, 2\} \), it remains to show that there is no nontrivial collection of constants \( c_1, c_2, c_3, c_4 \) such that

\[
c_1e_1A_1 + c_2e_1s_{12}A_1 + c_3A_2 + c_4s_{12}A_2 = 0.
\]

(33)

We first note that

\[
A_2 \neq ce_1A_1 \text{ for any } c.
\]

(34)

This is seen by examining the terms containing \( x_2^{m+1} \) in each, as was done in the proof of Lemma 6. Now assume that (33) held. Applying \( s_{13}\pi_2 \) gives

\[
c_2e_1A_1 + c_4A_2 = 0
\]

(35)

which is in immediate contradiction of (34), unless \( c_2 = c_4 = 0 \). Returning to (33) gives

\[
c_1e_1A_1 + c_3A_2 = 0
\]

(36)

Again (34) forces \( c_1 = c_3 = 0 \). This completes the proof.

We have thus reduced the problem of finding a basis for the quasiinvariants of \( S_3 \) to finding the 1-dimensional nullspace of particular matrices, a computation easily carried out by computer. We have used this technique to explicitly compute the basis for several small values of \( m \). We conclude with the following examples:
For $m = 1$,

\[
A_1 = x_1^4 - 2x_1^3(x_2 + x_3) + 6x_1^2(x_2x_3) \\
A_2 = x_1^5 - \frac{5}{3}x_1^4(x_2 + x_3) + \frac{10}{3}x_1^3(x_2x_3).
\]

For $m = 2$,

\[
A_1 = x_1^7 - \frac{7}{2}x_1^6(x_2 + x_3) + 14x_1^5(x_2x_3) + 72x_1^4(x_2^2 + x_3^2) \\
- \frac{35}{2}x_1^4(x_2^2x_3 + x_2x_3^2) + 35x_1^3x_2^2x_3^2 \\
A_2 = x_1^8 - \frac{16}{5}x_1^7(x_2 + x_3) + \frac{56}{5}x_1^6(x_2x_3) + \frac{14}{5}x_1^5(x_2^2 + x_3^2) \\
- \frac{56}{5}x_1^5(x_2^2x_3 + x_2x_3^2) + 14x_1^4x_2^2x_3^2.
\]
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References


