Practice problems for Midterm 2. Do not assume the midterm will look just like these problems. Also, these are not meant to replace the usual homework problems on my website.

1. Find the arc length of the curve \( y = \frac{1}{2}x^2 \) from \( x = 0 \) to \( x = 1 \).

   \[
   \text{Arc length} = \sqrt{2 + \ln(1 + \sqrt{2})}
   \]

2. The curve \( x = \sqrt{a^2 - y^2}, 0 \leq y \leq a/2 \), is rotated around the \( y \)-axis. Find the resulting surface area.

   Surface area = \( \pi a^2 \) (this was also a homework problem)

3. Verify that \( y = -t \cos t - t \) solves the differential equation

   \[
   t \frac{dy}{dt} = y + t^2 \sin t
   \]

   with the initial condition \( y(\pi) = 0 \).

   To check that \( y \) solves the differential equation, we simply plug it in. Since \( y = -t \cos t - t \) we have \( \frac{dy}{dt} = -\cos t + t \sin t - 1 \) and so

   \[
   t \frac{dy}{dt} = -t \cos t + t^2 \sin t - t = (-t \cos t - t) + t^2 \sin t = y + t^2 \sin t.
   \]

   Furthermore

   \[
   y(\pi) = -\pi \cos \pi - \pi = -\pi - \pi = 0.
   \]

4. i) Sketch a direction field for \( y' = x - y + 1 \), and, without solving the differential equation, use the direction field to sketch the solution curve with initial condition \( y(0) = 1 \).

   ii) Solve the differential equation by writing it at as a first order linear equation and using an integrating factor.

   Solving for \( y \) we get \( y = x + Ce^{-x} \). Using the initial condition \( y(0) = 1 \) we get \( C = 1 \) and so \( y = x + e^{-x} \). (See plot below.)

5. Solve the initial value problem

   \[
   \frac{dy}{dx} = \frac{\ln x}{xy}, \quad y(1) = 2.
   \]

   \[
   y = \sqrt{(\ln x)^2 + 4}
   \]
6. i) Solve the separable differential equation \( y'(x) = x - xy(x). \)

\[ y(x) = 1 + Ce^{-x^2/2} \]

ii) Using your answer to part i), solve the integral equation

\[ y(x) = 2 + \int_2^x (t - ty(t)) \, dt. \]

By the fundamental theorem of calculus, the solution \( y(x) \) to the integral equation also satisfies the differential equation in i), so

\[ y = 1 + Ce^{-x^2/2}. \]

Notice also that \( y(2) = 2 \). Therefore \( 2 = 1 + Ce^{-2} \), so \( C = e^2 \) and

\[ y = 1 + e^{2-x^2/2}. \]

7. Suppose a population \( P(t) \) satisfies

\[ \frac{dP}{dt} = \frac{1}{10} P - \frac{1}{100,000} P^2, \quad P(0) = 10. \]

i) What is the population carrying capacity, \( M \)?

\( M = 10,000 \)

iii) What is \( P'(0) \), and what does it represent?
\[ P'(0) = \frac{999}{1000}. \] It represents the instantaneous population growth rate at time 0.

ii) Write down a simpler differential equation which is approximately satisfied by \( P \) when \( P \) is small compared to \( M \).

\[ \frac{dP}{dt} = \frac{1}{10}P \]

8. Sketch the curve with parametric equations \( x = \sqrt{t - 2}, \ y = t - 1 \) by eliminating the parameter to find a corresponding Cartesian equation. Be careful to locate any endpoints of the parametric curve. Indicate with an arrow the direction in which the curve is traced out.

The curve is \( y = x^2 + 1, \ x \geq 0 \). (It’s the right half of an upward-opening parabola with vertex at \((0, 1)\).) The curve begins at \((0, 1)\) and is traced out to the right.

9. Find the area enclosed by the curve \( x = \sqrt{t}, \ y = t^2 - 2t \), and the \( x \)-axis.

\[ \text{Area} = \frac{8\sqrt{2}}{15} \]

10. i) Find a polar equation for the curve represented by the Cartesian equation \( 4y^2 = x \).

\[ r = \frac{1}{4} \cot \theta \csc \theta \]

ii) Write down a polar equation which represents a spiral.

\[ r = \theta^2 \] would work. (So would \( r = |\theta|, \ r = \sqrt{\theta}, \ r = |\theta|e^0 \), etc.)
Parametric equations to know:
Circles, ellipses, lines, and segments thereof

Integrals to know:
1. \[ \int x^a \, dx = \frac{x^{a+1}}{a+1} + C \text{ if } a \neq -1 \]
2. \[ \int x^a \, dx = \ln |x| + C \]
3. \[ \int a^x \, dx = \frac{a^x}{\ln a} + C \text{ if } a > 0 \]
4. \[ \int e^x \, dx = e^x + C \]
5. \[ \int \cos x \, dx = \sin x + C \]
6. \[ \int \sin x \, dx = -\cos x + C \]
7. \[ \int \sec^2 x \, dx = \tan x + C \]
8. \[ \int \sec x \tan x \, dx = \sec x + C \]
9. \[ \int \csc^2 x \, dx = -\cot x + C \]
10. \[ \int \csc x \cot x \, dx = -\csc x + C \]
11. \[ \int \sec x \, dx = \ln |\sec x + \tan x| + C \]
12. \[ \int \csc x \, dx = \ln |\csc x - \cot x| + C \]
13. \[ \int \frac{1}{x^2 + a^2} \, dx = \frac{1}{a} \tan^{-1} \left( \frac{x}{a} \right) + C \]
14. \[ \int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \left( \frac{x}{a} \right) + C \text{ if } a > 0 \]

*Memorization of these can be avoided by noticing that \( \sin \left( \frac{\pi}{2} - \theta \right) = \cos \theta \), \( \cos \left( \frac{\pi}{2} - \theta \right) = \sin \theta \), and therefore \( \tan \left( \frac{\pi}{2} - \theta \right) = \cot \theta \); one can then make the substitution \( x = \frac{\pi}{2} - \theta \) and use (7)-(8).

**Memorization of these can be avoided by remembering the tricks for integrating them (e.g. multiplying the integrand in (11) by \( \frac{\sec x + \tan x}{\sec x + \tan x} \)).

Identities to know:
15. \[ \sin^2 x + \cos^2 x = 1 \]
16. \[ \sec^2 x = \tan^2 x + 1 \]
17. \[ \csc^2 x = \cot^2 x + 1 \]
18. \[ \sin 2x = 2 \sin x \cos x \]
19. \[ \sin^2 x = \frac{1}{2} (1 - \cos 2x) \]
20. \[ \cos^2 x = \frac{1}{2} (1 + \cos 2x) \]

***These follow easily from (15).

During exams you can use the formulas (1)-(20) without justification. For integrals not appearing in the above list, you’ll have to justify your calculations.