**6551 Euler Method**

Use the Euler method with the indicated step size to compute approximate values of $y$ for the solution of the following initial value problem. Compute $y_1, y_2, y_3,$ and $y_4$.

1. $y' = 2x + y \quad y(1) = 2 \quad h = 0.4$

   $X_0 = 1$
   $X_1 = 1.4$
   $X_2 = 1.8$
   $X_3 = 2.2$

   $y_n = y_n + \frac{4}{10} (2x_n + y_n)$

   $y_0 = 2$
   $y_1 = 3.6$
   $y_2 = 6.16$
   $y_3 = 10.064$
   $y_4 = 15.8496$

2. $y' = y - 2x \quad y(1) = 5 \quad h = 0.4$

   $X_0 = 1$
   $X_1 = 1.4$
   $X_2 = 1.8$
   $X_3 = 2.2$

   $y_n = y_n + \frac{4}{10} (y_n - 2x_n)$

   $y_0 = 5$
   $y_1 = 6.2$
   $y_2 = 7.56$
   $y_3 = 9.144$
   $y_4 = 11.0416$

3. On side of a tank that is filled with water is shown below. Find the hydrostatic force on this side of the tank. The lengths are given in meters. Recall that the density of water is 9800 Newtons/meter$^3$.

\[
y = -10 = -\frac{5}{4} (x-4)
\]

\[
x = -\frac{4}{5} y + 12
\]

\[
\Delta F = \rho (10-y) \cdot 2x \Delta y
\]

\[
\text{Force} = \int_0^{10} (10-y) \left(-\frac{8}{5} y + 24\right) dy
\]

\[
= \frac{2800}{3} \rho = \frac{9146.667}{3} \text{ Newtons}
\]

\[
= \frac{27440.000}{3} \text{ N}
\]
6552 Logistic Equations

1. First find the general solution of the differential equation. Second, solve the two initial value problems. Solve for \( y \).

\[
\begin{align*}
(a) \quad & \frac{dy}{dx} = (2 + y)(8 - y) \quad \text{and} \quad y(0) = 6 \\
(b) \quad & \frac{dy}{dx} = (2 + y)(8 - y) \quad \text{and} \quad y(0) = 10
\end{align*}
\]

\[
\ln |2+y| - \ln |8-y| = 10x + \ln C
\]

\[
\left| \frac{2+y}{8-y} \right| = C e^{10x}
\]

\[
\frac{2+y}{8-y} = C e^{10x} \quad y \neq 8
\]

\[
\frac{2+y}{8-y} = C e^{10x} \quad y \neq -2
\]

\[
y = \frac{8C e^{10x} - 2}{1 + C e^{10x}}
\]

\[
y(0) = 6 \quad \text{gives} \quad C = 4
\]

\[
y = \frac{32 e^{10x} - 2}{1 + 4 e^{10x}}
\]

\[
y = \frac{32 - 2 e^{10x}}{4 + e^{-10x}}
\]

\[
y(0) = 10 \quad \text{gives} \quad C = -6
\]

\[
y = \frac{2 + 48 e^{10x}}{6 e^{10x} - 1}
\]

2. Find the indefinite integral \( \int \frac{x^2}{\sqrt{4x+9}} \, dx \).

Let \( u^2 = 4x+9 \quad x = \frac{u^2-9}{4} \) \quad dx = \frac{u}{2} \, du

\[
\frac{1}{32} \int (u^4 - 18u^2 + 81) \, du
\]

\[
= \frac{1}{32} \left[ \frac{u^5}{5} - 6u^3 + 81u \right] + C
\]

\[
= \frac{(4x+9)^{\frac{5}{2}}}{160} - \frac{3}{16} (4x+9)^{\frac{3}{2}} + \frac{81}{32} (4x+9)^{\frac{1}{2}} + C
\]

The substitution \( u = 4x+9 \) gives

\[
\frac{1}{64} \int (u^{\frac{3}{2}} - 18u^{\frac{1}{2}} + 81u^{-\frac{1}{2}}) \, du
\]
8553 Review: Substitution

1. Find the following antiderivative using a u-substitution. Show all steps.

\[ \int (4x^2 + 16x + 11)^3/5(x + 2) \, dx. \]

Let \( u = 4x^2 + 16x + 11 \)
\[ du = (8x + 16) \, dx = 8(x + 2) \, dx \]
\[ \frac{1}{8} \int u^{3/5} \, du = \frac{1}{8} \frac{u^{8/5}}{8/5} + c = \frac{5}{64} u^{8/5} + c \]
\[ = \frac{5}{64} \left[ (4x^2 + 16x + 11)^{8/5} \right] + c \]

2. Convert the following definite integral into another definite integral with variable of integration \( \theta \) using the substitution \( 3x = 5 \sin \theta \). Show all work.

\[ \int_{5/6}^{5\sqrt{3}/6} \frac{x^2 \, dx}{(25 - 9x^2)^{3/2}} \]

When \( x = \frac{5}{6} \) \( \sin \theta = \frac{1}{2} \) \( \theta = \frac{\pi}{6} \)

When \( x = \frac{5\sqrt{3}}{6} \) \( \sin \theta = \frac{\sqrt{3}}{2} \) \( \theta = \frac{\pi}{3} \)

\[ \frac{1}{27} \int_{\pi/6}^{\pi/3} \tan^2 \theta \, d\theta \]

3. Evaluate the following integral:

\[ \int \frac{64x + 6}{(x^2 + 16)(x - 9)} \, dx. \]

\[ \int \frac{-6x + 10}{x^2 + 16} + \frac{6}{x - 9} \, dx \]
\[ = -3 \ln (x^2 + 16) + \frac{5}{2} \arctan \left( \frac{x}{4} \right) + 6 \ln |x - 9| + C \]
1. Calculate the first moments $M_x$ and $M_y$ for the lamina of density $\rho$ which covers the region bounded by the parabola $y = 5x - x^2$ and the line $y = x$. Find the mass of the lamina. Also find the first moment about the line $x = -3$.

$$\Delta M_x = \frac{\rho}{2} \int_0^4 y^2 \Delta x - \frac{\rho}{2} \int_0^4 y^2 \Delta x = \frac{\rho}{2} \left[ (5x-x^2)^2 - x^2 \right] \Delta x$$

$$M_x = \rho \int_0^4 (12x^2 - 5x^3 + \frac{1}{2}x^4) \Delta x = \frac{192}{5} \rho$$

$$\Delta M_y = \rho \int_0^4 x y_0 \Delta x - \rho \int_0^4 x y_0 \Delta x = \rho \int_4^0 (5x-x^2-x) \Delta x$$

$$M_y = \int_0^4 (5x-x^2-x) dx = \frac{64}{3} \rho$$

$$\text{Mass} = \rho \left( \text{Area} \right) = \frac{32}{3} \rho$$

$$\text{Moment} = 3 \left( \text{Mass} \right) + \frac{64}{3} \rho = \frac{160}{3} \rho$$

2. The actual error $E_T$ made when using the trapezoid rule to approximate an integral satisfies $|E_T| \leq \frac{K(b-a)^3}{12n^2}$. Find a bound on the error when approximating the integral $\int_1^5 (5x + 8x^3 - x^4) dx$ for the trapezoid rule using $n = 200$ subintervals.

$$f''(x) = 48x - 12x^2$$
$$f'''(x) = 48 - 24x$$
$$f''(1) = 24 \text{ local min}$$
$$f''(5) = -72 \text{ local min}$$
$$f''(2) = -24 \text{ local max}$$

$$f''(1) = 36, f''(3) = 36, f''(5) =$$

$$|f''(x)| \leq 60 \text{ for } 1 \leq x \leq 5$$
$$K = 60$$

$$|E_T| \leq \frac{60(5-1)^3}{12(200)^2} = \frac{1}{125} = 0.008$$

3. Suppose $f(x)$ is such that $|f''(x)| \leq 9$ for $0 \leq x \leq 10$. Find the smallest value of $n$ such that when using the trapezoid approximation to compute $\int_0^6 f(x) dx$ the actual error $E_T$ satisfies $|E_T| \leq 5(10^{-3})$.

$$\frac{9(6-1)^3}{12n^2} \leq 5(10^{-3})$$

$$\frac{4n^2}{3(125)} \geq \frac{5}{10^3}$$

$$n^2 \geq 18750$$

$$n \geq 136.9$$

$n$ must be an integer

$$n = 137$$
6555 Review Improper Integrals

1. Explain why the following integral is improper. If the following integral is convergent, then find its value. If it is divergent, explain why it is divergent.

\[
\int_2^t \frac{x}{(x^2+5)^{3/2}} \, dx = \frac{1}{3} - \frac{1}{\sqrt{t^2+5}} \int_2^\infty \frac{x \, dx}{(x^2+5)^{3/2}}
\]

the integral is improper because it has \(\infty\) as a limit of integration.

\[
\lim_{t \to +\infty} \left[ \frac{1}{3} - \frac{1}{\sqrt{t^2+5}} \right] = \frac{1}{3}
\]

2. Explain why the following integral is an improper integral. Evaluate it.

\[
\int_4^{10} \frac{dx}{(x-4)^{3/2}} = \frac{2}{\sqrt{t-4}} - \frac{2}{\sqrt{6}} \]

the integral is improper because the integrand \((x-4)^{-3/2}\) is not defined at the lower limit of integration namely 4.

\[
\lim_{t \to 4^+} \frac{2}{\sqrt{t-4}} - \frac{2}{\sqrt{6}} = +\infty
\]

the limit does not exist, the integral is divergent.

3. Evaluate the following indefinite integral showing all steps.

\[
\int \sin^2(5x) \cos^2(5x) \, dx
\]

\[
\frac{1}{4} \int \left[ 1 - \cos(10x) \right] \left[ 1 + \cos(10x) \right] \, dx
\]

\[
= \frac{1}{8} \int \left[ 1 - \cos(20x) \right] \, dx = \frac{x}{8} - \frac{1}{160} \sin(20x) + C
\]

4. Set up the integral which when evaluated will give the surface area of the surface obtained when the curve \(y = (3x+7)^{1/2}\) from point where \(x = 1\) to the point where \(x = 7\) is revolved about the \(x\) axis.

\[
2\pi \int_1^7 \sqrt{3x+7} \left( \frac{1}{2} \right) \sqrt{\frac{12x+37}{3x+7}} \, dx = \pi \int_1^7 \sqrt{12x+37} \, dx
\]
1. Solve the initial value problem \( \frac{dy}{dx} = \frac{2y + 3}{x + 5} \) and \( y(0) = 1 \).

\[
\frac{2\frac{dy}{dx}}{2y + 3} = \frac{dx}{x + 5} \\
\ln |2y + 3| = 2 \ln |x + 5| + \ln c \quad c > 0 \\
2y + 3 = c(x + 5)^2 \quad c > 0 \\
y = -\frac{3}{2} + \frac{1}{10} (x + 5)^2
\]

2. Solve the initial value problem \( \frac{dy}{dt} = (2 - y)(10 - y) \) and \( y(0) = 4 \).

\[
\int \frac{1}{2-y} - \frac{1}{10-y} dy = dt \\
\ln \left| \frac{10-y}{2-y} \right| = \ln c e^{\delta t} \\
\frac{10-y}{2-y} = c e^{\delta t} \\
\]

3. A glass plate in the side of a water tank has the shape shown below. The top of the plate is 4 feet below the surface of the water. Find the hydrostatic force exerted by the water on this glass plate. The length units are in feet and water density is 62.5 lb/ft³.

Hint: Begin by finding the equation of the sloping line.

\[
\Delta F = \rho (\text{depth}) (x_{\text{right}} - x_{\text{left}}) \Delta y \\
\Delta F = \rho (9-y) [20-2y-4+15] \Delta y \\
\]

\[
\text{Force} = \rho \int_{0}^{5} (9-y) (35-3y) dy \\
= 925 \rho \\
= 57,812.5 \text{ lbs}
\]

\[
y = \frac{1}{3} x + 11 \\
y = x+15 \\
x = y - 15 \\
x = 20 - 2y
\]