In[2]:= Plot[{x, Sin[x]}, {x, -10, 10}]

Out[2]=

The above graph shows that x really only fits sin(x) at x=0, and it only stays close for a very small radius around x=0. Then it strays. This is the T_1(x) polynomial. Below are T_3(x), T_5(x), and T_11(x). Notice how they're staying closer to sin(x) for larger positive and negative values of x, but they still deviate at some point.

In[3]:= Plot[{x - x^3 / 3!, Sin[x]}, {x, -10, 10}]

Out[3]=

In[4]:= Plot[{x - x^3 / 3! + x^5 / 5!, Sin[x]}, {x, -10, 10}]

Out[4]=
At some point, it's harder to write every single term of the polynomial than it is to just switch into summation notation. These look a little "funny" because only the odd terms of the Taylor polynomial are not 0, when our function is sin(x). Please note that if we did this for cos(x), only the even ones would show up. For functions like e^x, every term is non-zero.

Because we only want every other term, our exponent on x is (2i-1). For i=1, this gives us 1, and for i=2, we get 3, etc. Now, we can change the number of terms in the sum just by changing the highest number i hits. We start with i going up to 10 (so \(T_{10}(x)\)) and end with 12 (\(T_{23}(x)\)).
This looks pretty good, but zooming out, we see that our function still doesn't remain close to \( \sin(x) \) forever.

The bigger our biggest exponent in the Taylor polynomial gets - that is, the bigger the \( n \) in \( T_n(x) \) - the better our approximation gets, until as \( n \to \infty \), we can't tell the difference between our "polynomial" and the original function. Of course, now that we have infinitely many terms, we really have an infinite series.
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I just don't recommend running it with very large numbers, unless you have a more powerful computer than I do. The one below will already take quite a while, which is why I don't have a plot for it. It goes up to \( i=5000 \), so we have the \( T_{2^{5000}-1}(x) = T_{9999}(x) \) polynomial.

\[
\text{In[1]} = \text{Plot}[\{\text{Sum}[-(-1)^{i + 1} x^{(2 i - 1)}/(2 i - 1)!, \{i, 5000\}], \sin[x]\}, \{x, -25, 25\}]
\]