In discussion today, a student asked about the geometric interpretation of adding or subtracting two equations.

I thought about it, and I have a nice way to demonstrate what happens, but using two lines.

Ex: \( x + y = 3, \quad x - y = 5 \)

Now, the solution to this is \( x=4, \ y=-1 \). But in between, we're adding two equations. So let's do the very obvious thing and add the two equations, as given.

\[
\begin{align*}
(x + y) & = 3 \\
+ (x - y) & = 5 \\
\hline
2x + 0 & = 8,
\end{align*}
\]

which simplifies to \( x=4 \). The thing is, \( x = 4 \) is a line, a vertical line that passes through the point \( x=4 \) on the x-axis. It's still a line! It's only when we also solve for \( y \) that we get a point: \( (x,y) = (4, -1) \).

(side note: technically, we're not really adding equations; we're adding each side of the equations, but the notation works well enough not to split hairs).

Now, if that didn't convince you we just get a line, let's make this more obvious. We'll also explore what's so special about these lines. This time, we're going to do something that doesn't lead us to the solution. I'm going to multiply the first equation by 2, and add it to the second equation:

\[
\begin{align*}
2x + 2y & = 6 \\
x - y & = 5 \\
\hline
3x + y & = 11
\end{align*}
\]

If you graph all the equations: \( x+y=3, \ x-y=5, \ x=4, \) and \( 3x+y =11 \), they all have exactly one point in common: \( (4, -1) \).

When we add the equations for two lines together, we get another line that passes through the point(s) the originals had in common.

So, when you add two equations with 3 variables (so you're adding two planes), you end up with another plane, but it's a plane that passes through all the common points of the other two planes, if they have any in common. It get a little trickier with parallel planes and lines, but we don't need to worry about that at the moment.