Sample Lab Write-Up for MATH 2374

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Exercise 1. The goal for this problem is to find the tangent lines to the circle \( x^2 + y^2 = 4 \) at all points for which \( y = \frac{1}{2} \).

To begin, we shall first find all points for which \( y = \frac{1}{2} \). Plugging \( y = \frac{1}{2} \) into the circle equation yields \( x^2 + \left( \frac{1}{2} \right)^2 = 4 \). We use Mathematica to solve this for \( x \):

\[
\text{Solve}[x^2 + (1/2)^2 == 4, x]
= \{\{x \to -\sqrt{15/2}\}, \{x \to \sqrt{15/2}\}\}
\]

Thus, our points are \((-\sqrt{15/2}, 1/2)\) and \((\sqrt{15/2}, 1/2)\).

Of course, in order to describe the tangent lines, we will need to know the slope. To do this, we first implicitly differentiate the circle equation:

\[
2x + 2yy' = 0.
\]

Solving this for \( y' \) we get

\[
y' = -\frac{x}{y}.
\]

By plugging the points \((-\sqrt{15/2}, 1/2)\) and \((\sqrt{15/2}, 1/2)\), we find \( y' = \sqrt{15} \) at \((-\sqrt{15/2}, 1/2)\) and \( y' = -\sqrt{15} \) at \((\sqrt{15/2}, 1/2)\). These values of \( y' \) are the slopes of the respective tangent lines.

Using point-slope form, we can plug in the points and the slopes to determine the tangent line. At \((-\sqrt{15/2}, 1/2)\) we get

\[
y - \frac{1}{2} = \sqrt{15}(x + \frac{\sqrt{15}}{2})
\]
as the tangent line, while at \( \left( \frac{\sqrt{15}}{2}, \frac{1}{2} \right) \) we have

\[
y - \frac{1}{2} = -\sqrt{15}\left(x - \frac{\sqrt{15}}{2}\right)
\]

for the tangent line.

We can confirm that these are the correct lines by graphing the circle simultaneously with the two lines. A look at the picture convinces us that we have indeed found the tangent lines.

ContourPlot\[
\{x^2 + y^2 == 4, y - 1/2 == Sqrt[15]*(x + Sqrt[15]/2), y - 1/2 == -Sqrt[15]*(x - Sqrt[15]/2)\}, \{x, -3, 3\}, \{y, -3, 3\}\]
**Exercise 2** I shall attempt to find the area enclosed between the graphs of the functions $f(x) = 4 + \frac{\sin(\pi x)}{2}$ and $g(x) = (x - 2)^2$.

To begin, I made a plot with which to visualize the area in question.

\[
\text{Plot}\left[\left\{4 + \frac{\sin(\pi x)}{2}, (x - 2)^2\right\}, \{x, -2, 5\}\right]
\]

Recall from calculus 1 that this area can be found by integrating the upper function minus the lower function. Thus, the integrand will be $4 + \frac{\sin(\pi x)}{2} - (x - 2)^2$. The question remains: over what bounds?

It sure looks like the graphs intersect at $x = 0$ and $x = 4$. This is easily confirmed by plugging these values in:

\[f(0) = g(0) = f(4) = g(4) = 4.\]

So the area in question will equal

\[
\int_0^4 4 + \frac{\sin(\pi x)}{2} - (x - 2)^2 dx.
\]

Mathematica evaluates this to $\frac{32}{3}$, as seen below.

\[
\text{Integrate}\left[4 + \frac{\sin(\pi x)}{2} - (x - 2)^2, \{x, 0, 4\}\right]
\]