I understand that we haven’t had a lot of time to cover triple integrals, and I feel it is a very sad thing. These were the things I remembered the most from my multivariable calculus class, and they were probably what I enjoyed the most. So, I’m writing up an example that I think is particularly cool. I’m going to calculate the volume of a sphere, using only triple integrals. (because we’re doing volume, the inside function is 1).

First, I’m going to set up the triple integral with a generic function \( f(x, y, z) \). I want you to see that it doesn’t matter what the function is; the setup for the integral over the space will be the space occupied is what counts. In the case of a sphere, this is determined by the function \( x^2 + y^2 + z^2 = C^2 \), where \( C \) is the constant radius. For sake of conventional order, I’m going to choose to solve for \( z \) first, and therefore integrate with respect to \( z \) on the inner-most integral. Note that we’ll get two explicit functions. The following graphs show what the functions look like (top and bottom).

\[
\begin{align*}
z &= \sqrt{C^2 - y^2 - x^2} \\
z &= -\sqrt{C^2 - y^2 - x^2}
\end{align*}
\]

So, the inner-most limits of integration will be \(-\sqrt{C^2 - y^2 - x^2}\) and \(\sqrt{C^2 - y^2 - x^2}\) on bottom and top, respectively. Next, we explore where \( y \) can range. We look at the place, where \( y \) can range the most: when \( z = 0 \).

\[
\begin{align*}
y &= \sqrt{C^2 - x^2} \\
y &= -\sqrt{C^2 - x^2}
\end{align*}
\]

So, the limits for the next integral (with respect to \( y \)) will be \(-\sqrt{C^2 - x^2}\) and \(\sqrt{C^2 - x^2}\) for the bottom and top, respectively. Finally, we look for the limits with respect to \( x \). Once again, we look where \( x \) has its largest range: when both \( y \) and \( z \) are 0. Here, \( x \) ranges from \(-C\) to \( C \), which make the lowr and upper limits, respectively.
We can now put all of our information together:

\[
\int_{-C}^{C} \left( \int_{-\sqrt{C^2-x^2}}^{\sqrt{C^2-x^2}} \left( \int_{-\sqrt{C^2-y^2-x^2}}^{\sqrt{C^2-y^2-x^2}} f(x, y, z) \, dz \right) \, dy \right) \, dx
\]

Now, we can consider that we only care about the volume of the sphere. So, \( f(x, y, z) = 1 \) for our example. The integral reduces to:

\[
\int_{-C}^{C} \left( \int_{-\sqrt{C^2-x^2}}^{\sqrt{C^2-x^2}} \left( \int_{-\sqrt{C^2-y^2-x^2}}^{\sqrt{C^2-y^2-x^2}} \sqrt{C^2 - y^2 - x^2} + \sqrt{C^2 - y^2 - x^2} \, dy \right) \, dz \right) \, dx
\]

\[
= \int_{-C}^{C} \left( \int_{-\sqrt{C^2-x^2}}^{\sqrt{C^2-x^2}} (2\sqrt{C^2 - y^2 - x^2}) \, dy \right) \, dz \, dx
\]

\[
= \int_{-C}^{C} \left( \int_{-\sqrt{C^2-x^2}}^{\sqrt{C^2-x^2}} (2\sqrt{C^2 - x^2 - y^2}) \, dy \right) \, dx
\]

Now, we do that dreaded thing from Calc 2: the trig substitution. We want \( y^2 \) to equal \( (C^2 - x^2) \sin^2(\theta) \), so we set

\[
y = \sqrt{C^2 - x^2} \sin(\theta)
\]

\[
\Rightarrow dy = \sqrt{C^2 - x^2} \cos(\theta) \, d\theta
\]

( limits: and when \( y = \sqrt{C^2 - x^2} \), \( \sin(\theta) = 1 \), so \( \theta = \pi/2 \)

and when \( y = -\sqrt{C^2 - x^2} \), \( \sin(\theta) = -1 \), so \( \theta = -\pi/2 \)

So, the integral becomes:

\[
\int_{-C}^{C} \left( \int_{-\pi/2}^{\pi/2} \left( 2\sqrt{(C^2 - x^2) - (C^2 - x^2) \sin^2(\theta) \sqrt{(C^2 - x^2) \cos(\theta)}} \right) \, d\theta \right) \, dx
\]

\[
\int_{-C}^{C} \left( \int_{-\pi/2}^{\pi/2} \left( 2\sqrt{(C^2 - x^2)(1 - \sin^2(\theta)) \sqrt{(C^2 - x^2) \cos(\theta)}} \right) \, d\theta \right) \, dx
\]

\[
\int_{-C}^{C} \left( \int_{-\pi/2}^{\pi/2} \left( 2\sqrt{(C^2 - x^2) \cos^2(\theta) \sqrt{(C^2 - x^2) \cos(\theta)}} \right) \, d\theta \right) \, dx
\]

\[
\int_{-C}^{C} \left( \int_{-\pi/2}^{\pi/2} \left( 2\sqrt{(C^2 - x^2)^2 \cos^2(\theta) \sqrt{(C^2 - x^2) \cos(\theta)}} \right) \, d\theta \right) \, dx
\]

\[
\int_{-C}^{C} \left( \int_{-\pi/2}^{\pi/2} \left( 2(C^2 - x^2)^2 \cos^3(\theta)) \, d\theta \right) \, dx \text{ (but x, C, and 2 have nothing to do with } \theta)
\]

\[
\int_{-C}^{C} \left( 2(C^2 - x^2) \int_{-\pi/2}^{\pi/2} (\cos^2(\theta)) \, d\theta \right) \, dx
\]
We now focus on the inside integral.

\[ A = \int_{-\pi/2}^{\pi/2} \left( \cos^2(\theta) \right) d\theta \]

We then assign: \( u = \cos(\theta) \), so \( du = -\sin(\theta) d\theta \) and \( dv = \cos(\theta) d\theta \), so \( v = \sin(\theta) \).

\[ A = \cos(\theta) \sin(\theta) \bigg|_{-\pi/2}^{\pi/2} - \left( - \int_{-\pi/2}^{\pi/2} \sin^2(\theta) d\theta \right) \]

\[ = \cos(\pi/2) \sin(\pi/2) - \cos(-\pi/2) \sin(-\pi/2) + \int_{-\pi/2}^{\pi/2} (1 - \cos^2(\theta)) d\theta \]

\[ = 0 - 0 + \int_{-\pi/2}^{\pi/2} 1 d\theta - \int_{-\pi/2}^{\pi/2} \cos^2(\theta) d\theta \]

\[ = \theta \bigg|_{-\pi/2}^{\pi/2} - A \]

\[ = \pi/2 - (-\pi/2) - A \]

\[ = \pi - A \]

We take the very top, and it is equal to the very bottom, so \( A = \pi - A \), or \( A = \pi/2 \). Our overall integral is now:

\[ \int_{-C}^{C} \left( 2(C^2 - x^3) \right) \left( \int_{-\pi/2}^{\pi/2} \left( \cos^2(\theta) \right) d\theta \right) dx \]

\[ = \int_{-C}^{C} 2(C^2 - x^3) Adx \]

\[ = \int_{-C}^{C} 2(C^2 - x^2)(\pi/2)dx \]

\[ = \pi \int_{-C}^{C} (C^2 - x^2)dx \]

\[ = \pi \left( C^2x - \frac{x^3}{3} \right) \bigg|_{-C}^{C} \]

\[ = \pi \left( C^3 - \frac{C^3}{3} - \left( (-C)^3 - \frac{(-C)^3}{3} \right) \right) \]

\[ = \pi \left( 2C^3 - 2 \frac{C^3}{3} \right) \]

\[ = \frac{4\pi C^3}{3} \]