MATH 3283W MidSemester Exam 1

Name: ____________________________

Explain all arguments clearly. All problems are worth 20 points. This is a closed book exam. No cheat sheets. Calculators may be used. You may use the back of each page if you need more room.

1) What is $\bigcup B_{B \in B}$ and $\bigcap B_{B \in B}$ when the collection $B$ is given by

$$B = \left\{ (1 - \frac{1}{n}, 2 + \frac{2}{n}) \mid n \in \mathbb{N} \right\}$$

\[ n=1: \quad \begin{array}{ccccccc}
0 & 1 & 2 & 3 & 4 & \circ & \\
\end{array} \]

\[ n=2: \quad \begin{array}{ccccccc}
0 & 1 & 2 & 3 & \circ & \\
\end{array} \]

\[ \vdots \]

\[ n=10: \quad \begin{array}{ccccccc}
0 & 1 & 2 & 3 & \circ & \\
\end{array} \]

We claim that $\bigcup B = (0, 4)$. Note that for $n=1$, $B_1 = (0, 4)$, and for all $B \in B$

$n > 1$, $1 - \frac{1}{n} > 0$ and $2 + \frac{2}{n} < 4$, so $B_n \subseteq B_1$. So all the other $B$ intervals $B \in B$ are contained in $(0, 4)$, and thus, $\bigcup B = (0, 4)$.

We also claim that $\bigcap B = [1, 2]$. For $x < 1$, there is an $n$ large enough that $x < 1 - \frac{1}{n}$, and so $x$ is not in the corresponding $B$. Likewise, for $x > 2$, there is an $n$ large enough so that $x > 2 + \frac{2}{n}$, meaning $x$ is not in the corresponding $B$. However, for all $x \in [1, 2]$, $x \in (1 - \frac{1}{n}, 2 + \frac{2}{n})$ for all $n \in \mathbb{N}$.

Hence, $\bigcap B = [1, 2]$. 
2) Make a truth table for $p; q; p \Rightarrow q; \sim q; \sim p$. Use it to verify

$$[(p \Rightarrow q) \land (\sim q)] \Rightarrow \sim p$$

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<th>$p \Rightarrow q$</th>
<th>$\sim p$</th>
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3) Let \( S \) be a set and \( \sim \) an equivalence relation defined on \( S \). Let \( x, y \in S \).

(a) Define the equivalence class \( E_x \) (also denoted \([x]\)) of \( x \).

(b) Prove that \( E_x = E_y \) or \( E_x \cap E_y = \emptyset \).

\[
\begin{align*}
(a) \quad E_x &= \{ y \in S \mid x \sim y \} \\
(b) \quad \text{There are two cases to consider: let } y \in S. \text{ Then either } & x \sim y \quad \text{or} \quad x \not\sim y. \\
& \text{Since these are negations of each other, one or the other must be true.} \\
& x \not\sim y: \text{ Since } y \not\sim x \text{ and } \sim \text{ the transitive property holds for } \sim,
\end{align*}
\]

\[
\begin{align*}
z \in E_x & \iff z \sim x, \quad \text{and recall } y \sim x \\
& \iff y \sim z \\
& \iff z \in E_y.
\end{align*}
\]

So every element in \( E_x \) is in \( E_y \), and vice versa. Thus, \( E_x = E_y \).

\[
x \not\sim y: \text{ Let us say that } x \not\sim y \text{ but } E_x \cap E_y \neq \emptyset. \text{ Then, there must be some element } z \in E_x \cap E_y. \text{ Because } z \in E_x \cap E_y, \\
z \sim x \text{ and } z \sim y. \text{ By the transitive and symmetric properties, this means } x \sim y, \text{ a contradiction. So } E_x \cap E_y = \emptyset.
\]
4) Let \( f : X \to Y \) be a function. Let \( C_1 \) and \( C_2 \) be subsets of \( X \).

a. Prove \( f(C_1 \cap C_2) \subseteq f(C_1) \cap f(C_2) \).

b. Prove if \( f \) is injective, \( f(C_1 \cap C_2) = f(C_1) \cap f(C_2) \).

(a) We wish to show that if \( y \in f(C_1 \cap C_2) \), then \( \exists x \in C_1 \cap C_2 \) such that \( f(x) = y \).

Start by setting \( y \in f(C_1 \cap C_2) \). That means there is some \( x \in C_1 \cap C_2 \) s.t. \( f(x) = y \). So, \( x \in C_1 \) and \( y \in f(C_1) \) and \( x \in C_2 \) and \( y \in f(C_2) \). Therefore, \( y \in f(C_1) \cap f(C_2) \).

We conclude that \( f(C_1 \cap C_2) \subseteq f(C_1) \cap f(C_2) \). \( \blacksquare \)

(b) We have already shown inclusion in one direction, so we only need to establish that \( f \) is injective implies \( f(C_1 \cap C_2) = f(C_1) \cap f(C_2) \).

Let us assume we can find some \( y \in f(C_1) \cap f(C_2) \) such that \( y \notin f(C_1 \cap C_2) \). Since \( y \in f(C_1) \cap f(C_2) \), there is some \( x_1 \in C_1 \) and some \( x_2 \in C_2 \) such that \( f(x_1) = y = f(x_2) \). If \( y \notin f(C_1 \cap C_2) \), then there is no element \( x \in C_1 \cap C_2 \) such that \( y = f(x) \). So the \( x_1 \) cannot be in \( C_2 \) (it is already in \( C_1 \)) and \( x_2 \notin C_1 \). Therefore, \( x_1 \neq x_2 \). But \( f(x_1) = f(x_2) \) and \( f \) is injective \( \neq \). So \( y \notin f(C_1) \cap f(C_2) \), giving us the inclusion \( f(C_1 \cap C_2) \subseteq f(C_1) \cap f(C_2) \).

Hence, \( f(C_1) \cap f(C_2) = f(C_1 \cap C_2) \) when \( f \) is injective \( \square \).
5) Let \( A \) and \( B \) be two sets. Let \( \mathcal{P}(A) \) and \( \mathcal{P}(B) \) be their respective power sets. Using the definition of power set, prove: if \( |A| \leq |B| \) then \( |\mathcal{P}(A)| \leq |\mathcal{P}(B)| \).

Recall that \( |A| \leq |B| \) iff there is an injection between \( A \) and \( B \). Let \( f \) be that injection, which sends elements \( a \in A \) to \( f(a) \in B \). We will use \( f \) to define an injection \( F: \mathcal{P}(A) \rightarrow \mathcal{P}(B) \). Given a subset \( S \subseteq A \), let \( F(S) = \{ f(x) | x \in S \} \). Then, if \( S \neq T \), we have \( S \neq T \) or \( T \neq S \).

Without loss of generality, assume \( S \neq T \). Then there is an \( x \in S \) s.t. \( x \notin T \). Since \( f \) is injective, this means \( f(x) \notin F(S) \) and we show that \( f(x) \notin F(T) \). If \( f(x) \in F(T) \), then there is some \( y \in T \) such that \( f(y) = f(x) \). But \( y \in T \) and \( x \notin T \), so \( y \neq x \) since \( f \) is injective. Thus, \( f(x) \notin F(T) \). Hence \( F(S) \neq F(T) \), and so \( F(S) \neq F(T) \). Thus \( F \) is an injection from \( \mathcal{P}(A) \) to \( \mathcal{P}(B) \), so \( |\mathcal{P}(A)| \leq |\mathcal{P}(B)| \).