8.2 3b) \[ \sum \frac{(-2)^n}{n^2}. \] Since \[ \frac{2^n}{n^2} \] is increasing (for \( n > 1 \)), \( \frac{2^n}{n^2} \to 0 \), so \( \sum \frac{(-2)^n}{n^2} \) diverges.

d) \[ \sum \frac{-5^n}{2^n} = \sum \left( \frac{-5}{2} \right)^n \] is a geometric series. Since \( \frac{-5}{2} > 1 \), it diverges.

f) \[ \sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n} : \text{ Check for absolute convergence first: } \int_2^{\infty} \frac{1}{x \ln x} \, dx = \int_2^{\infty} \frac{1}{u} \, du = \ln u \bigg|_2^{\infty} \] diverges.

Since \( \frac{1}{n \ln n} \) is decreasing and approaches 0, \( \sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n} \) converges by the alt. series test, so converges conditionally.

g) \[ \sum \frac{(-1)^n}{n^{1/3}} : \text{ check for abs. conv. } \lim_{n \to \infty} \frac{\frac{1}{n^{1/3}}}{\frac{1}{n^{2/3}}} = \lim_{n \to \infty} \frac{1}{n} = 0 \text{ which is finite and non-zero. So by the limit comparison test (proved later in this text—we could had done integral instead)} \sum \frac{1}{n^{2/3}} \text{ diverges.} \]

\( \frac{1}{n^{1/3}} \) is decreasing and approaches 0, so \( \sum \frac{(-1)^n}{n^{1/3}} \) converges by the alt. series test, so conditionally convergent.

i. \[ \sum \left( \frac{1}{n} - \frac{1}{n^2} \right). \]

\[ = \sum \frac{n^2 - 1}{n}. \] Since \( \frac{n^2 - 1}{n} > \frac{1}{n} \) for \( n > 1 \) and \( \sum \frac{1}{n} \) diverges, we have \( \sum \frac{1}{n^2} - \frac{1}{n} \) diverges as well.

8. (a) Let \( \sum a_n \) and \( \sum b_n \) be two series of positive numbers with \( \left( \frac{a_n}{b_n} \right) \to c \) \( 0 < c < \infty \). Suppose \( \sum a_n \) converges. Since \( \left( \frac{a_n}{b_n} \right) \to c \), \( \frac{b_n}{a_n} \to \frac{1}{c} \), so \( \frac{b_n}{a_n} \) is bounded, say by \( M \). Then \( b_n = \frac{b_n}{a_n} \cdot a_n < M a_n \), so since \( \sum a_n \) converges, \( \sum b_n \) converges as well by the comparison test.
8.3 a) False, \( R = \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right| \), if this limit exists. Not the same for, say, \( \sum 2^n x^n \)

b) False \( \sum \frac{x^n}{n} \) has \( R = 1 \), but interval \([-1, 1]) \), which is neither open nor closed

c) False \( \sum \frac{x^n}{n} \) has \( a_n > 0 \) \( \forall n \), but converges conditionally for \( x = -1 \).

5. a) \( \sum \frac{(2n)!}{(n!)^2} x^n \), 
\( \lim_{n \to \infty} \frac{(2n+2)!}{(n+1)!} x^n = \lim \frac{(2n+2)(2n+1)}{(n+1)(n+1)} x^n = 4x \)

so \( R = 1/4 \)

c) \( \sum \frac{n!}{n^n} x^n \), 
\( \lim \frac{((n+1)!)x^n}{(n+1)!} \frac{n^n}{n!} = \lim \frac{n^n}{n!} x^n = \frac{1}{e} x \)

so \( R = e \)

7. Suppose \( (a_n) \) is bounded but \( \sum a_n \) diverges. Show the radius of convergence of \( \sum a_n x^n \) is equal to 1.

\( \sum a_n x^n \) is centred at 0, so since it diverges for \( x = 1 \), \( R \leq 1 \).

Since \( a_n \) is bounded, \( |a_n| < M \) for some \( M \). So for \( |x| < 1 \) we may compare \( \sum |a_n x^n| \) with the convergent geometric series \( \sum M|x|^n \). Thus for \( |x| < 1 \), \( \sum a_n x^n \) converges absolutely, so \( R = 1 \).