10. Prove or give a counterexample:

(a) \( A \times B = B \times A \).

False. Let \( A = \{0, 1\} \), \( B = \{2, 3\} \).

Then \( A \times B = \{(0,2), (0,3), (1,2), (1,3)\} \) and \( B \times A = \{(2,0), (2,1), (3,0), (3,1)\} \).

Note that \( (0,2) \in A \times B \) but \( (0,2) \notin B \times A \).

The sets are not equal.

Note: They are isomorphic, but that is not what is being asked out of the material up to this point.

(b) \((A \cup B) \times C = (A \times C) \cup (B \times C)\).

True: let us prove this in two parts.

\((A \cup B) \times C \subseteq (A \times C) \cup (B \times C)\): let \((x, y) \in (A \cup B) \times C\).

Then, \( x \in A \cup B \) and \( y \in C \). That is, \( x \in A \) or \( x \in B \), and \( y \in C \).

This implies \( x \in A \) and \( y \in C \) or \( x \in B \) and \( y \in C \).

(see the truth table on the next page)

So \((x, y) \in A \times C \) or \((x, y) \in B \times C\). This is equivalent to \((x, y) \in (A \times C) \cup (B \times C)\) and we conclude

\((A \cup B) \times C \subseteq (A \times C) \cup (B \times C)\).
For the truth table, let $p \iff "x \in A," \quad q \iff "x \in B," \quad \text{and} \quad r \iff "y \in C." \quad \text{Are} \quad (p \lor q) \land r \quad \text{and} \quad (p \land q) \lor (q \lor r)\quad \text{equivalent?}

\[
\begin{array}{c|c|c|c|c|c}
 p & q & r & (p \lor q) \land r & (p \land q) \lor (q \lor r) \\
 T & T & T & T & T \\
 T & T & F & T & T \\
 T & F & T & F & T \\
 T & F & F & F & T \\
 F & T & T & F & T \\
 F & T & F & F & T \\
 F & F & T & T & T \\
 F & F & F & T & T \\
\end{array}
\]

Yes, they are equivalent. We can also use this in the other direction.

\[
(A \times C) \cup (B \times C) \subseteq (A \cup B) \times C.\quad \text{Let} \quad (x, y) \in (A \times C) \cup (B \times C). \quad \text{Then}
\]

both $x \in A$ and $y \in C$ or both $x \in B$ and $y \in C$.

We have already shown this is actually equivalent to, and therefore implies, $x \in A$ or $x \in B$ and also $y \in C$.

So $(x, y) \in (A \cup B) \times C$.

With containment both directions ($\subseteq$ and $\supseteq$) we have shown $(A \cup B) \times C = (A \times C) \cup (B \times C)$. 

10. (c) \((A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)\).

**True:** Let’s show containment in both directions at once. Let \((x, y) \in (A \times B) \cap (C \times D)\). That is, \(x \in A\) and \(y \in B\) and \(x \in C\) and \(y \in D\).

This is true exactly when (or if and only if) all four basic statements are true, regardless of the order. Thus, this is equivalent to \(x \in A\) and \(x \in C\) and \(y \in B\) and \(y \in D\), which is another way of saying \((x, y) \in (A \cap C) \times (B \cap D)\). 

**Note:** Two sets \(S, T\) are equal if and only if \(S \subseteq T\) and \(T \subseteq S\), which is the same as \(x \in S\) if and only if \(x \in T\).

Thus, \((A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)\). \(\square\)

10. (d) \((A \times B) \cup (C \times D) = (A \cup C) \times (B \cup D)\).

**False:** For an idea of why, let \(A\) and \(C\) not be equal, and let \(B\) and \(D\) not be equal.
(d) (cont'd):

As a counter example, let \( A = [0,1] \), \( C = [2,3] \),
\( B = [-2,-1] \), \( D = [3,4] \).

Let \( (x,y) = (2.5, -1.5) \).

Then, \( (x,y) \in C \times B \) (because \( 2.5 \in [2,3] \) and \(-1.5 \in [-2,-1]) \).

This element is in \( (A \cup C) \times (B \cup D) \) because
\[ x \in C \leq A \cup C \quad \text{and} \quad y \in B \leq B \cup D. \]

However, \( (x,y) \notin (A \times B) \cup (C \times D) \) on the set of ordered
\[ (a,b) \quad \text{where} \quad (a \in A \land b \in B) \lor (a \in C \land b \in D), \]

since neither \( p \lor q \) holds for \((2.5, -1.5)\).

because we "proved" it false by providing a counter-exampe.
Let \( S = \{a, b, c, d, e\} \) and define the equivalence relation
\[ R = \{(a, a), (b, b), (c, c), (d, d), (e, e), (a, b), (b, a), (a, d), (d, a), (b, d), (d, b)\} \]

Describe the partition \( P \) determined by \( R \) by listing the pieces in \( P \).

Let us start with the elements that are never paired with a different element. These are \( c \) and \( e \).

We list the \( x \in S \) such that \((c, x) \in R\). Since \((c, c)\)
is the only one of these, \( c \) is the only element in \( S \) such that \( c \) relates to it. Another way of saying this is
\[
(c, c) \in R \text{ or } \quad cRc.
\]

Since it's the only one, \( E_c = \{c\} \).

Similarly, \( E_e = \{e\} \) because \((e, e) \in R \) but \((e, x) \notin R \) for any \( x \neq e \).

Now, let us determine \( E_a \), the equivalence class of \( a \).
We know \((a, a) \in R\), so \( a \in E_a \). Also, \((a, b), (b, a) \in R\), so \( b \in E_a \). Finally, \((d, a), (a, d) \in R \) so \( d \in E_a \). Thus,
\[
E_a = \{a, b, d\}.
\]

Our partition of \( S \), determined by \( R \) is
\[
P = \{\{c\}, \{e\}, \{a, b, d\}\}.
\]
28. Let \( A = \mathbb{N} \times \mathbb{N} = \{ (x, y) \mid x, y \in \mathbb{N} \} \). Define \( R \) by 
\[
(a, b)R(c, d) \iff a^b = c^d.
\]
(a) Show that \( R \) is an equivalence relation. (Read "prove" when you see "show").

Elements of \( A \) are ordered pairs, so for reflexivity, we want \( (a, b)R(a, b) \).

1. **Reflex:** Let \( a, b \in \mathbb{N} \). Then \( a^b = b^a \iff (a, b)R(a, b) \). The former was certainly true, so \( (a, b)R(a, b) \). \( \checkmark \)

2. **Sym:** Let \( a, b, c, d \in \mathbb{N} \). Assume \( (a, b)R(c, d) \). That is,
\[
a^b = c^d \iff c^d = a^b, \text{ so } (c, d)R(a, b),
\]
as desired. \( \checkmark \)

3. **Trans:** Let \( a, b, c, d, f, g \in \mathbb{N} \). Assume \( (a, b)R(c, d) \) and \( (c, d)R(f, g) \), or 
\[
a^b = c^d = f^g.
\]
Since equality is transitive, \( a^b = f^g \), or \( (a, b) = (f, g) \), as desired. \( \checkmark \)

We have shown all three requirements, so \( R \) is an equivalence relation. \( \checkmark \)

(b) List the elements in the equivalence class \( E(9, 2) \).

We wish to find \( \{ (a, b) \mid a, b \in \mathbb{N} \text{ and } a^b = 9^2 = 81 \} \).

Let's break 81 down to its prime factorization:
\[
81 = 3^4.
\]
So any \( a \) such that \( a^b = 81 \) must be a power of 3 and it must be less than or equal to 81. Our choices are 3, 9, 27, and 81. There is no integer, let alone natural number, such that \( 27^b = 81 \). So we cannot use 27. This leaves 3, 9, and 81.
Notice that \(3^4 = 81\)
\(9^2 = 81\)
\(81^1 = 81\).

These must be our exponents. So \(E_{(9, 2)} = \{(3, 4), (9, 2), (81, 1)\}\).

(c) Find an equivalence class with exactly 82 elements.

Let \((a, b) = (2, 2)\). Then
\[
E_{(2, 2)} = \{ (a, b) \mid a^b = 2^2 = 4 \}
= \{ (2, 2), (4, 1) \}.
\]

(d) Find an equivalence class with exactly four elements.

Since \(E_{(9, 2)}\) had exactly 3, let’s analyse 81 to see why.

\[
81 = 3^4 = 3^{2^2}\quad \text{this exponent gets done first, or last.}
\]

Let’s set \(n = 3^{2^2}\) and see what happens.

\[
3^{2^2} = 3^8 = (3^2)^4 = 9^4
= (3^{2^2})^2 = 81^2\quad \text{because } x^2 = x \text{ for any number.}
= (3^{2^2})^1 = 6561^1.
\]

\[
E_{(6561, 1)} = \{(3, 8), (9, 4), (81, 2), (6561, 1)\}.
\]

A smaller acceptable answer is (256, 1)
\[
E_{(256, 1)} = \{ (2, 8), (4, 4), (16, 2), (256, 1) \}.
\]

Note:
\[
8 \sim 4 \sim 2 \sim 1,
\]
is \(3^3 \sim 2^2 \sim 1^2\) and
\[
3 \sim 9 \sim 81 \sim 6561.
\]

\[
3 \sim 3^2 \sim 9 \sim 81 \sim 6561.
\]