3: Find the range of each of the following functions.

\( f: \mathbb{R} \rightarrow \mathbb{R} \).

(a) \( f(x) = x^2 + 2 \).

\( \text{range } f = [2, \infty) \), or \( \exists x \models x^2 \geq 2^3 \).

(b) \( f(x) = (x - 2)^2 + 4 \).

\( \text{has a vertex at } (2, 4) \),

\( \text{range } f = [4, \infty) \). 

(c) \( f(x) = x^2 + 6x + 4 \).

\( \text{Complete the square!} \)

\[
\begin{align*}
\quad x^2 + 6x + 9 - 9 + 4 &= (x + 3)^2 - 9 + 4 \\
&= (x + 3)^2 - 5 \\
\end{align*}
\]

So \( f(x) = (x + 3)^2 - 5 \), giving the parabola a vertex at \((-3, -5)\).

\( \text{Range } f = [-5, \infty) \).
(d) \( f(x) = 5 \cos(4x) \).

The 4 in front of the \( x \) just changes the period (and therefore the frequency). The 5 actually changes the range.

\[
\begin{align*}
\cos x : & \quad -\pi, -\frac{\pi}{2}, 0, \frac{\pi}{2}, \pi \\
\cos (4x) : & \quad -\frac{\pi}{4}, -\frac{\pi}{8}, 0, \frac{\pi}{8}, \frac{\pi}{4} \\
5 \cos (4x) : & \quad -5, -\frac{5}{2}, 0, \frac{5}{2}, 5
\end{align*}
\]

range \( f \) = \([-5, 5]\).

---

7. Classify each function as injective, surjective, bijective, or none of these:

(a) \( f : \mathbb{N} \rightarrow \mathbb{N} \); \( f(n) = n+3 \). \( \text{Injective (not surjective)} \)

(b) \( f : \mathbb{Z} \rightarrow \mathbb{Z} \); \( f(n) = n-5 \). \( \text{Bijective} \)

(c) \( f : \mathbb{R} \rightarrow \mathbb{R} \) by \( f(x) = x^3 - x \)

\[
= x(x^2-1) = x(x-1)(x+1)
\]

\( \text{Surjective (not injective)} \)

(d) \( f : [0,1) \rightarrow [0,\infty) \) by \( f(x) = x^3 - x \)

\( \text{Injective and surjective} \) \( \text{Bijective} \)

(e) \( f : \mathbb{N} \rightarrow \mathbb{Z} \) by \( f(n) = n^2 - n = n(n-1) \).

Since \( \mathbb{N} \) does not start until \( n \geq 1 \), we get rid of all "doubles," making \( f \) injective.
However, $f$ is not surjective, for $n^2 - n \geq 0$ for every $n \in \mathbb{N}$.

**Answer:** injective.

\[(f) \, f : [3, \infty) \rightarrow [5, \infty) \; ; \; f(x) = \frac{(x-3)^2 + 5}{\text{has a vertex at } (3,5)} \]

\[\text{not a part of the domain,} \]

\[\begin{array}{c}
\begin{array}{c}
5 \\
\uparrow \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\ldots \\
\downarrow \\
1, 2, 3
\end{array}
\end{array}
\]

\[f \text{ passes the horizontal line test on the domain, and for any } y \in [5, \infty), \text{ we can say } y \geq 5. \text{ Therefore, } y-5 \geq 0. \text{ So } \sqrt{y-5} \text{ exists. Let } x = \sqrt{y-5} + 3, \]

\[\text{and } f(x) = \left(\left(\sqrt{y-5} + 3\right) - 3\right)^2 + 5 \]

\[= \left(\sqrt{y-5}\right)^2 + 5 \]

\[= |y-5| + 5 \quad (\text{but } y-5 \geq 0, \text{ so } |y-5| = y-5) \]

\[= y-5 + 5 = y. \]

So $f$ is surjective.

Then $f$ is bijective.
7. (g) \( f: \mathbb{N} \rightarrow \mathbb{Q} \) by \( f(n) = \frac{1}{n} \).

- \( f \) is injective: \( \frac{1}{n} = \frac{1}{m} \iff n = m \), for \( n, m \in \mathbb{N} \).
- \( f \) is not surjective: \( \frac{2}{3} \neq \frac{1}{n} \) for any \( n \in \mathbb{N} \), so \( \frac{2}{3} \notin \mathbb{Q} \) but \( \frac{2}{3} \notin \text{range } f \).

So \( f \) is injective (not surjective).

10. In each part, find a function \( f: \mathbb{N} \rightarrow \mathbb{N} \) that has the desired properties.

(a) Surjective, not injective.

\[ f(n) = |n - 3| + 1. \]

\[ \text{has its vertex at (3,1).} \]

Surjective: let \( n \in \mathbb{N} \) be given. Try to find an \( m \) such that \( f(m) = n \). If \( m \geq 3 \), then \( |m-3| = m-3 \).

This would make \( |m-3| + 1 = m-3 + 1 = m-2 \).

So let us set \( m = n+2 \). Then, since \( n \geq 1 \), \( m \geq 3 \), and so we get

\[ f(n+2) = |n+2-3| + 1 = n-1 + 1 = n. \]

Not injective: \( f(2) = f(4) = 2 \).
10 (b) Injective but not surjective:

\[ f(n) = n^2. \]

Not surjective: There is no \( m \in \mathbb{N} \) such that \( m^2 = 3 \in \mathbb{N} \).

So \( 3 \notin \text{codom}(f) \) but \( 3 \notin \text{range}(f) \).

Injective: Since \( n > 0 \) for all values in the domain,

\[
\text{There is only one solution if } n^2 = m^2, \text{ then } \sqrt{n^2} = |n| = n = m = |m| = \sqrt{m^2}.
\]

because \( n \) is positive and hence \( m \) is positive.

(c) Neither surjective nor injective.

Take the example from (a) and raise it up by 1.

\[ f(n) = |n-3| + 2. \]

There is no \( n \) such that \( f(n) = 1 \),

and \( f(2) = f(4) = 3. \)

(d) Let \( f(n) = n \). Clearly bijective.
32. Suppose that \( f : A \rightarrow B \) is any function. Then a function \( g : B \rightarrow A \) is called a

\[
\text{left inverse for } f \text{ if } g(f(x)) = x \text{ for all } x \in A, \\
\text{right inverse for } f \text{ if } f(g(y)) = y \text{ for all } y \in B.
\]

(a) Prove that \( f \) has a left inverse \( \iff \) \( f \) is injective.

(b) Prove that \( f \) has a right inverse \( \iff \) \( f \) is surjective.

(a) Assume \( f \) has a left inverse. That is, there is some \( g : B \rightarrow A \) such that

\[ g(f(x)) = x \text{ for all } x \in A. \]

Then assume \( f \) is not injective. Then \( \exists x, z \in A \) such that \( x \neq z \) and \( f(x) = g(f(z)) \). Then

\[ x = g(f(x)) = g(f(z)) = z, \]

but \( x \neq z \). \( \ast \) Therefore, \( f \) must have been injective.

Now assume \( f \) is injective. We always have a preimage for any \( f(x) \in B \), but it may not, in general, be a single element. Since \( f \) is injective, there is only one element \( x \) with the image \( f(x) \). Let \( g(y) = f^{-1}(y) \), \( \forall y \in f(A) \subseteq B \).

For the other elements in \( B \), it doesn't matter how \( g \) maps. Then for any \( x \in A \), \( g(f(x)) = f^{-1}(f(x)) = x \). The function is well defined \( \Box \)
(b) Assume \( f \) is surjective, or that \( f(A) = B \). That is, for all \( y \in B \) there is some \( x \in A \) such that \( f(x) = y \). There may be more than one such \( x \). Let \( g(y) \) be one, and only one, of those. Then

\[
    f(g(y)) = y \quad \text{for all } y \in B,
\]

and \( g \) is a right inverse.

Now, assume there is some \( g : B \to A \) such that

\[
    f(g(y)) = y \quad \text{for all } y \in B,
\]

and assume that \( f \) is not surjective. Then there is some \( y' \in B \) such that \( y' \notin f(A) \). That is, there is no \( x \in A \) such that \( f(x) = y' \). Then, no matter which element \( g(y') \) is assigned, \( f(g(y')) \neq y' \). *

Thus, \( f \) must be surjective.