10. (a) \( a_1 = 3.1 \quad a_2 = 3.14 \quad a_3 = 3.141 \quad a_9 = \text{approx. of } \pi \text{ to } n \text{ decimal places.} \) 
\( a_n \) is rational, since the decimal representations are finite in length.
Show \( (a_n) \to \pi \); let \( \varepsilon > 0 \). Let \( N \) be the natural number corresponding to the first non-zero decimal place in a decimal approximation of \( \varepsilon \). Then for \( n > N \), 
\[ |a_n - \pi| < \varepsilon. \] So \( (a_n) \to \pi \).

(b) Let \( b_n = \pi - a_n \), where \( a_n \) is defined as above. Since \( a_n \) is rational, \( \pi - a_n \) is irrational. 
Show \( (b_n) \to 0 \); let \( \varepsilon > 0 \) and choose \( N \) as above. Then 
\[ |b_n - 0| = |a_n - \pi| < \varepsilon. \] So \( (b_n) \to 0 \).

12. (a) Suppose \( \lim s_n = 0 \). If \( (t_n) \) is a bounded sequence, show \( \lim (s_n t_n) = 0 \).

Proof. Suppose \( \lim s_n = 0 \), suppose \( (t_n) \) is a bounded sequence. Then \( \exists M \in \mathbb{R} \) so \( |t_n| < M \) for all \( n \in \mathbb{N} \). Let \( \varepsilon > 0 \). Since \( (s_n) \to 0 \), \( \exists N_1 \) st. for \( n > N_1 \), \( |s_n| < \varepsilon/4M \).
Then for \( n > N_1 \), 
\[ |s_n t_n - 0| < |s_n||t_n - b| < |s_n||t_n - b| < |s_n||t_n - b| < M \frac{|t_n - b|}{|t_n - b|} = \frac{\varepsilon}{4M} < \varepsilon. \] So \( (s_n t_n) \to 0 \).

(b) Let \( s_n = \frac{1}{n} \), let \( t_n = n^2 \). Then \( (s_n t_n) = (n) \), which does not converge to 0.

13. Suppose \( (a_n), (b_n), (c_n) \) are sequences st. \( a_n \leq b_n \leq c_n \) for all \( n \in \mathbb{N} \), and such that
\( \lim a_n = \lim c_n = b. \) Show that \( (b_n) \to b. \)

Let \( \varepsilon > 0 \). Since \( (a_n) \to b \), \( \exists N_1 \in \mathbb{N} \) st. for \( n > N_1 \), \( |a_n - b| < \varepsilon \).
In particular \( b - \varepsilon < a_n \). Also, since \( (c_n) \to b \), \( \exists N_2 \in \mathbb{N} \) st. for \( n > N_2 \), \( |c_n - b| < \varepsilon \).
In particular, \( c_n < b \). Let \( N = \max \{N_1, N_2\} \). For \( n > N \), we have
\[ b - \varepsilon < a_n < c_n < b + \varepsilon, \] so \( b - \varepsilon < b_n < b + \varepsilon, \) so \( -\varepsilon < b_n - b < \varepsilon, \)
and thus \( |b_n - b| < \varepsilon. \) So \( (b_n) \to b. \).
15 a) Let \( x \) be an accumulation point of \( S \). For each \( n \in \mathbb{N} \), choose \( s_n \in N^k(x, \frac{1}{n}) \cap S \). Since \( x \) is an accumulation point, we may always make such a choice of \( s_n \).

Claim: \( (s_n) \rightarrow x \). Let \( \varepsilon > 0 \). By the Archimedean property, there exists \( N \in \mathbb{N} \) so that \( \frac{1}{N} < \varepsilon \).

Then for \( n > N \), \( \left| s_n - x \right| < \frac{1}{n} < \frac{1}{N} < \varepsilon \). So \( (s_n) \rightarrow x \). Also, since \( s_n \in N^k(x, \frac{1}{n}) \cap S \), we have \( s_n \neq x \) \( \forall n \in \mathbb{N} \), and \( s_n \in S \), so \( (s_n) \subseteq S \setminus \{x\} \).

Now suppose there is such a sequence \( (s_n) \). Let \( \varepsilon > 0 \). Since \( (s_n) \rightarrow x \), \( \exists N \in \mathbb{N} \) so \( \left| s_n - x \right| < \varepsilon \) for \( n > N \). Also, since \( (s_n) \subseteq S \setminus \{x\} \), we have \( s_n \neq x \) \( \forall n \in \mathbb{N} \). Then \( s_{n+1} \in N(x, \varepsilon) \cap S \), so \( x \) is an accumulation point.

b) Suppose \( S \) is closed. Then \( S' \subseteq S \) (by 3.4.17). Let \( (s_n) \rightarrow x \) be a convergent sequence of points in \( S \). If \( s_n = x \) for some \( k \in \mathbb{N} \), then since \( (s_n) \subseteq S \), \( x \in S \). Otherwise \( (s_n) \subseteq S \setminus \{x\} \), so by part (a) \( x \in S' \), so \( x \in S \).

Now suppose every convergent sequence in \( S \) converges to a limit in \( S \). To show \( S \) is closed, it is sufficient by 3.4.17 to show \( S' \subseteq S \).

So suppose \( x \in S' \). By (a) \( \exists (s_n) \rightarrow x \) of points in \( S \). Then by hypothesis \( x \in S \).

4.2.3 (a) \( \lim_{n \to \infty} \frac{5n^2 + 4n}{7n^2 - 3n} = \lim_{n \to \infty} \frac{\frac{5n^2 + 4n}{n^2}}{\frac{7n^2 - 3n}{n^2}} = \lim_{n \to \infty} \frac{5 + \frac{4}{n}}{7 - \frac{3}{n}} \)

\[(4.2.1.d)\]

\[= \frac{\lim_{n \to \infty} 5n^2}{\lim_{n \to \infty} 7n^2} - \frac{\lim_{n \to \infty} 4n}{\lim_{n \to \infty} 3n} = \frac{5}{7} - \frac{4}{3} \lim_{n \to \infty} \frac{1}{n} = \frac{5}{7} - \frac{4}{3} \cdot \frac{1}{n} \]

(b) \( \lim_{n \to \infty} \frac{2n^4 + 7}{n^5 - 3n} = \lim_{n \to \infty} \frac{\frac{2n^4 + 7}{n^4}}{\frac{n^5 - 3n}{n^4}} = \lim_{n \to \infty} \frac{2 + \frac{7}{n^4}}{1 - \frac{3}{n^4}} \)

\[(4.2.1.d)\]

\[= \lim_{n \to \infty} \frac{2 + \lim_{n \to \infty} \frac{7}{n^4}}{1 - \frac{3}{n^4}} = \frac{2 + 0}{1 - 0} = 2 \]

This level of detail not required (esp. on a test) but it's worth seeing you can show this using only \( \frac{1}{n} \to 0 \) and from 4.2.1.