Accumulation points, isolated points and closure

1. Recall that, given \( x \in \mathbb{R} \) and \( \varepsilon > 0 \), we define \( N^*(x, \varepsilon) = \{ y : 0 < |y - x| < \varepsilon \} \). Write

\[
N^*(2, 2) \cap N^*(3, 2) = (2-2, 2+2) \times (3-2, 3+2) = (-2, 2) \times (1/2, 3/2)
\]

and

\[
N^*(2, 2) \cup N^*(3, 2) = (0, 2) \cup (2, 4) \cup (1, 3) \cup (3, 5)
\]

as intervals or unions of intervals.

\[
N^*(2, 2) \cup N^*(3, 2) = (0, 2) \cup (2, 4) \cup (1, 3) \cup (3, 5)
\]

\[
\boxed{(0, 5)}
\]

2. For the following sets find \( S' \) (the set of accumulation points), the set of isolated points, and the closure \( \overline{S} \).

(a) \( S = [0, 1] \)
\( S' = (0, 1] \)
isolated pts: \( \emptyset \)
\( \overline{S} = [0, 1] \)

(b) \( S = (0, 1) \)
\( S' = [0, 1] \)
isolated pts: \( \emptyset \)
\( \overline{S} = [0, 1] \)

(c) \( S = \mathbb{N} \)
\( S' = \emptyset \)
isolated pts: \( \mathbb{N} \)
\( \overline{S} = \mathbb{N} \)

(d) \( S = \mathbb{Q} \)
\( S' = \emptyset \)
isolated pts: \( \emptyset \)
\( \overline{S} = \mathbb{Q} \)

(e) \( S = \{ \frac{1}{n} : n \in \mathbb{N} \} \)
\( S' = \emptyset \)
isolated pts: \( \mathbb{N} \)
\( \overline{S} = \mathbb{R} \)

(f) \( S = \{ s \in \mathbb{Q} : 0 < s < \sqrt{2} \} \)
\( S' = [0, \sqrt{2}] \)
isolated pts: \( \emptyset \)
\( \overline{S} = [0, \sqrt{2}] \)

3. In lecture you defined \( \overline{S} = S \cup \partial S \), and the book gives you the equivalent definition \( \overline{S} = S \cup S' \). It is not the case, however, that \( S' = \partial S \) in general.

(a) Find examples of (i) an accumulation point that is not a boundary point and (ii) a boundary point that is not an accumulation point.

(i) \( S = (0, 2) \). Then \( 1 \in S' \), but \( 1 \not\in \partial S \).

(ii) \( S = \emptyset \). Then \( S' = \emptyset \), but \( 0 \in \partial (S = \emptyset) \).

(b) Prove that if \( x \) is an accumulation point of \( S \), then either \( x \) is an interior point of \( S \) or a boundary point of \( S \).

Let \( x \in S' \) and assume \( x \not\in S \). Then, for all \( \varepsilon > 0 \), \( N^*(x, \varepsilon) \cap S \neq \emptyset \) because \( x \in S' \). Then,

\[
\emptyset = (N^*(x, \varepsilon) \cap S) \subseteq (N^*(x, \varepsilon) \cup \partial S) \subseteq (N^*(x, \varepsilon) \cup N^*(x, \varepsilon) \cap S) = (N^*(x, \varepsilon) \cap S)
\]

which is therefore nonempty. So \( x \in \partial S \).

(c) Prove that if \( x \) is a boundary point of \( S \), then either \( x \) is an accumulation point of \( S \) or an isolated point of \( S \).

Let \( x \in \partial (S) \), so for all \( \varepsilon > 0 \), \( N(x, \varepsilon) \cap S \neq \emptyset \) and \( N(x, \varepsilon) \cap S^c \neq \emptyset \). Now assume \( x \) is not an accumulation point. So, there is an \( \varepsilon_0 \) s.t. \( N^*(x, \varepsilon_0) \cap S = \emptyset \). In fact, for all \( \varepsilon > 0 \) but \( \varepsilon < \varepsilon_0 \),

\[
\left[ N^*(x, \varepsilon) \cap S \right] = [N^*(x, \varepsilon) \cap S] = \emptyset.
\]

Yet, \( N(x, \varepsilon) = N^*(x, \varepsilon) \cup \emptyset \), so \( \emptyset \neq N(x, \varepsilon) \cap S = (N^*(x, \varepsilon) \cup \emptyset) \cap S = \emptyset \).

(on back)
\[(N^*(x, \varepsilon) \cap S) \cup (\varepsilon x \cap S) = \emptyset \cup (\varepsilon x \cap S) = \varepsilon x \cap S.\]

Because \(\varepsilon x \cap S \neq \emptyset\), \(x \in S\). So \(x \in S\) but \(x \notin S'\). Then,
\(x\) is an isolated point.

Thus, if \(x \in \partial S\), \(x\) is either an accumulation point or an isolated pt. \(\Box\)