1. (a) To show that \((a_n)\) is decreasing: \(a_1 = 4\) and \(a_2 = 14/5 < a_1\). Suppose that \(a_{k+1} < a_k\). Then
\[
a_{k+2} = \frac{1}{5}(3a_{k+1} + 2) < \frac{1}{5}(3a_k + 2) = a_{k+1}.
\]
Hence, by induction, \(a_{n+1} < a_n\) for all \(n\). To show that \(a_n > 0\), for all \(n\): \(a_1 > 0\). Suppose that \(a_k > 0\). Then \(a_{k+1} = \frac{1}{5}(3a_k + 2) > \frac{2}{5} > 0\). Hence, by induction, \(a_n > 0\) for all \(n\). (We could have, in fact, shown that \(a_n > 1\) for all \(n\), but it is not necessary.) Thus, the sequence converges, by the Monotone Convergence Theorem. The limit \(a\) must satisfy the equation \(a = \frac{1}{5}(3a + 2)\). Hence \(a = 1\).

(b) To show that \((a_n)\) is increasing: \(a_1 = 3\) and \(a_2 = \sqrt{11} > 3\). Suppose that \(a_k < a_{k+1}\). Then
\[
a_{k+2} = \sqrt{4a_{k+1} - 1} > \sqrt{4a_k - 1} = a_{k+1}.
\]
Hence, by induction, \((a_n)\) is increasing. To show that \(a_n < 4\), for all \(n\): \(a_1 = 3 < 4\). Suppose that \(a_k < 4\). Then \(a_{k+1} = \sqrt{4a_k - 1} < \sqrt{15} < 4\). Hence, by induction, \((a_n)\) is bounded. MCT \(\rightarrow\) convergence. The limit \(a\) must satisfy the equation \(a = \sqrt{4a - 1}\). Hence, \(a = 2 + \sqrt{3}\). Since \(a_1 > 2 - \sqrt{3}\) and \((a_n)\) is an increasing sequence, we must have \(a = 2 + \sqrt{3} \approx 3.7\).

2. To show that \((s_n)\) is increasing, by induction: \(s_1 = \sqrt{6}\) and \(s_2 = \sqrt{6 + \sqrt{6}} > \sqrt{6 + 0} = s_1\). Suppose that \(s_{k+1} > s_k\). \(s_{k+2} = \sqrt{6 + s_{k+1}} > \sqrt{6 + s_k} = s_{k+1}\). Thus \((s_n)\) is increasing. To show \(s_n < 3\) for all \(n\): \(s_1 = \sqrt{6} < \sqrt{9} = 3\). Suppose that \(s_k < 3\). Then \(s_{k+1} = \sqrt{6 + s_k} < \sqrt{6 + 3} = 3\), so \(s_n < 3\) for all \(n\). Thus the sequence converges by MCT, and the limit \(s\) satisfies the equation \(s = \sqrt{6 + s}\), which implies that \(s = -2\) or \(s = 3\). Since \(s_n\) is a positive increasing sequence, the limit must be 3.

3. Let \(k > 0\) and \(x > 0\). To show that the sequence is positive, first note \(s_1 = k > 0\), and now suppose that \(s_i > 0\). Then \(s_{n+1} = (s_n^2 + x)/(2s_n) > 0\) as well. Thus \(s_n > 0\) for all \(n\).

To show that \((s_n)\) is (eventually) decreasing: first of all, notice that for certain choices of \(k\) and \(x\), we have \(s_1 < s_2\): if \(k = 1\), and \(x = 2\), then \(s_1 = 1\) and \(s_2 = 3/2 > 1\). Nonetheless, we can show that \(s_n \geq s_{n+1}\) for all \(n \geq 2\).

For all \(n\),
\[
\frac{s_n^2}{2s_n} - x = \left(\frac{s_n^2 + x}{2s_n}\right)^2 - x = s_n^4 + 2xs_n^2 + x^2 - 4xs_n^2 = 4k^2 = x^2 \geq 0.
\]

Then for all \(n \geq 2\),
\[
s_n - s_{n+1} = s_n - \frac{s_n^2 + x}{2s_n} = \frac{2s_n^2 - s_n^2 - x}{2s_n} = \frac{s_n^2 - x}{2s_n} \geq 0.
\]

Since \(2s_{n+1}s_n = s_n^2 + x\), the limit \(s\) must satisfy the equation \(2s^2 = s^2 + x\). Since \((s_n)\) is a positive sequence, we have \(s = \sqrt{x}\).