Functions, composition, and inverse

1. Let $f : A \to B$ be a function. Let $C, C_1, C_2$ be subsets of $A$, and let $D, D_1, D_2$ be subsets of $B$. In the book, we learned that in general $C \subseteq f^{-1}(f(C))$ and $f(f^{-1}(D)) \subseteq D$. In this problem, we investigate how $f$ and $f^{-1}$ interact with intersection. (Notes: The statements here appear as part of Theorem 2.3.16 in the text. Remember that, here, the notation $f^{-1}$ means preimage of a set, not inverse function.)

(a) Prove that $f^{-1}(D_1 \cap D_2) = f^{-1}(D_1) \cap f^{-1}(D_2)$.

(b) Prove that $f(C_1 \cap C_2) \subseteq f(C_1) \cap f(C_2)$.

(c) Give a counterexample to the converse in (b).
2. (Exercises 2.3 #26-28) Let \( f : A \rightarrow B \) and \( g : B \rightarrow C \). Give examples (you might like to give examples of both real-valued functions and functions on small finite sets):

(a) \( f \) and \( g \circ f \) are injective, but \( g \) is not.
(b) \( g \) and \( g \circ f \) are surjective, but \( f \) is not.
(c) \( g \circ f \) is bijective, but neither \( f \) nor \( g \) is.

3. Consider the bijections (injective and surjective) \( f_1(x) = x + 1 \), \( f_2(x) = x/2 \), and \( f_3(x) = x^3 \). Form the six possible compositions of the three functions, writing the six compositions both as functions of \( x \) and using composition notation, and then write the inverses of the six, both as functions (of \( y \), let’s say) and as compositions of the inverses of the original functions.