Supremum, infimum, and the completeness axiom

The following two theorems will be useful in these problems.

**Theorem 1**

These are equivalent (and are all considered forms of the Archimedean property):

1. The Archimedean property: \( \mathbb{N} \) is unbounded above in \( \mathbb{R} \).
2. \( \forall z \in \mathbb{R}, \exists n \in \mathbb{N} \ni n > z \).
3. \( \forall x > 0, \forall y \in \mathbb{R}, \exists n \in \mathbb{N} \ni nx > y \).
4. \( \forall x > 0, \exists n \in \mathbb{N} \ni 0 < 1/n < x \).

**Theorem 2**

\( \mathbb{Q} \) is dense in \( \mathbb{R} \); that is, if \( \forall x, y \in \mathbb{R} \) with \( x < y \), \( \exists r \in \mathbb{Q} \ni x < r < y \).

1. For each subset of the real numbers below, give its supremum, maximum, infimum, and minimum, if they exist. Justify your claims, and note especially when you use a form of the Archimedean property in your justification.
   
   (a) \[ \left\{ \frac{n}{n+1} : n \in \mathbb{N} \right\} \]
   
   (b) \[ \left\{ \frac{n + (-1)^n}{n} : n \in \mathbb{N} \right\} \]
2. Prove that if $x$ and $y$ are real numbers with $x < y$, then there exist infinitely many rational numbers in the interval $[x, y]$. 

3. Prove that what we have called a closed interval, $[x, y]$ where $x < y$ in $\mathbb{R}$, is in fact close, using the definition involving boundary points. We break this proof up into a few steps:

(a) Write the definition of a closed set.

(b) What are the boundary points of $[x, y]$? Justify. Use the definition of a boundary point.

(c) Prove that $[x, y]$ fits the definition for a closed set.