The final examination is scheduled on Monday, Dec. 11, 2017, same room, 10:00-11:55 am. Here are 14 problems, 7 of which will be given on final. If you solve all of them, you can just hand in your solutions of the seven chosen or all solutions before 11:55 am on Monday, Dec. 11, 2017 to me personally or put under the door of my office. In the latter case I want to have an email notification that you did that.

1. Let \( \nu \) be an outer measure on \( \Omega \). prove that if \( A, B \in \Sigma \) and \( AB = \emptyset \), then \( \nu(X(A \cup B)) = \nu(XA) + \nu(XB) \). In particular, when \( X = \Omega \), \( \nu(A \cup B) = \nu(A) + \nu(B) \).

2. Let \( F \) be a nondecreasing finite left-continuous function on \( \mathbb{R} \). Define \( E \) as the collection of finite unions of disjoint intervals of the type \((a, b]\) with \(-\infty \leq a < b \leq \infty \) as in the lecture notes. If \( A \in E \) is given by \( A = \bigcup_{i=1}^{n} (a_i, b_i] \) with disjoint \((a_i, b_i]\) then set

\[
R(A) = \sum_{i=1}^{n} R((a_i, b_i]),
\]

where \( R((a, b]) = F(b) - F(a) \) if \( a \leq b \), \( F(\infty) = \lim_{x \to \infty} F(x) \), \( F(-\infty) = \lim_{x \to -\infty} F(x) \).

Show that \( R \) is an additive but not a \( \sigma \)-additive function on \( E \) if \( F \) has at least one point of discontinuity.

3. We know that \( \beta(t) := (1 - |t|)_+ \) is the characteristic function of a distribution on \( \mathbb{R} \). One can scale \( \beta \) and this will preserve the property. Prove that, if \( a_1, ..., a_n \) and \( c_1, ..., c_n \) are positive numbers with

\[
\sum_{k=1}^{n} a_k = 1,
\]

then

\[
\sum_{k=1}^{n} a_k \beta(c_k t)
\]

is a characteristic function. By using this prove the following result of Polya: If \( \gamma(t) \geq 0 \) is an even, continuous function on \( \mathbb{R} \) such that \( \gamma(t) \) is convex and decreasing on \([0, \infty)\) and \( \gamma(0) = 1 \), then \( \gamma \) is the characteristic function of a probability distribution. (Hint: Approximate \( \gamma \) with broken lines and use the first part of the problem.)

4. Let numbers \( b_k, a_k^n \) and \( a_k \) be given for \( n, k = 1, 2, ... \). Assume that \( |a_k^n| \leq b_k \) and \( a_k^n \to a_k \) as \( n \to \infty \) for any \( k \). Also assume \( \sum_k b_k < \infty \). Prove
then that
\[ \lim_{n \to \infty} \sum_{k} a_n^k = \sum_{k} a_k. \]

5. Let \((\Omega, \mathcal{F}, \mu)\) be a measure space and let \(f \geq 0\) defined on \(\Omega\) be measurable. Assume that \(\int_{\Omega} f(x) \mu(dx) < \infty\) and prove that if \(A_n \in \mathcal{F}\) are such that \(\mu(A_n) \to 0\), then \(\int_{\Omega} f(x) I_{A_n}(x) \mu(dx) \to 0\).

6. By using the central limit theorem applied to \(X_1 + \ldots + X_n\), where \(X_k\) are iid having the Poisson distribution with parameter 1, prove that
\[ \lim_{n \to \infty} e^{-n} \sum_{k=0}^{n} \frac{n^k}{k!} = \frac{1}{2}. \]

7. Let \(X_n, n = 1, 2, \ldots\), be pairwise independent random variables such that
\[ P(X_n \in (a,b)) = (1 - 2^{-n})(b - a) \quad \text{for} \quad 0 \leq a \leq b \leq 1 \quad \text{and} \quad P(X_n = 2^n) = 2^{-n}. \]
Show that there exists a constant \(c\) such that
\[ \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} X_k = c \quad \text{(a.s.)} \]
and find this constant.

8. If \(g\) is any function on a Polish space \(X\) denote by \(\Delta_g\) the set of points of discontinuity of \(g\) and prove that \(\Delta_g\) is a Borel set. (Hint: Introduce
\[ M_n(x) = \sup\{|f(y) - f(z)|: \rho(y, x), \rho(z, x) < 1/n\}, \]
\(\Delta_{n,m} = \{x : M_n(x) > 1/m\}\) and prove that the sets \(\Delta_{n,m}\) are open and the set of discontinuity of \(f\) is
\[ \bigcup_{m=1}^{\infty} \bigcap_{n=1}^{\infty} \Delta_{n,m}. \]

9. We know that if \(\Gamma_n\) is a decreasing sequence of closed sets in a Polish space such that \(\text{diam} \Gamma_n \to 0\) as \(n \to \infty\), then \(\cap_n \Gamma_n\) is nonempty and consists of only one point.
Let \(f(x) = \sin(1/x)\) and in \(C([0, 1])\) consider the sets
\[ \Gamma_n = \{x(\cdot) \in C([0, 1]) : \sup_{1 \geq t \geq 1/n} |x(t) - f(t)| \leq 1/2, \sup_{0 \leq t \leq 1} |x(t)| \leq 2\} \]
for \(n \geq 1\). Show that the sets \(\Gamma_n\) are bounded, closed, nested, and \(\cap_n \Gamma_n = \emptyset\).

10. (Problem 14.31) Let probability distributions \(Q\) and \(Q_n\) on a Polish space have densities \(f\) and \(f_n\) with respect to a common \(\sigma\)-finite measure \(\mu\). Assume that \(f_n \to f\) \(\mu\)-a.e.. Prove that \(Q_n \Longrightarrow Q\).
11. (Problem 18.9) We know that the space $C[0,1]$ of real-valued bounded continuous functions on $[0,1]$ provided with the metric
\[ \rho(f, g) = \max_{t \in [0,1]} |f(t) - g(t)| \]
is a Polish space. Let $X_n$ be $C[0,1]$-valued random variable converging to $X$ in distribution. Prove that $\max_{[0,1]} X_n(t)$ converge to $\max_{[0,1]} X(t)$ in distribution.

12. (∼ Problem 13.13) By using characteristic functions prove that, if $\varepsilon_n, n = 1, 2, \ldots$, are iid with $P(\varepsilon_1 = 1) = P(\varepsilon_1 = -1) = 1/2$, then
\[ \xi := \sum_{n=1}^{\infty} \frac{\varepsilon_n}{2^n} \]
is uniformly distributed on $[-1, 1]$. (Hint: Use that $2 \cos x \sin x = \sin(2x).$

13. Prove that if $u_n(t), n = 1, 2, \ldots$, are equicontinuous on $[a,b]$ and converge to $u(t)$ for each $t \in [a,b]$, then $u$ is continuous on $[a,b]$ and the convergence is uniform.

14. Prove that if $Q_n, n \geq 1$, is a sequence of distributions on $\mathbb{R}^d$ such that the sequence of the corresponding characteristic functions converges pointwise to a function, say $f$, which is continuous at zero, then the convergence of the characteristic functions to $f$ is uniform on any bounded subset of $\mathbb{R}^d$. 