Theorem 0.1. Let $X$ be a compact Polish space and $C$ be the set of real-valued continuous functions on $X$ provided with uniform norm. Let $L$ be a linear continuous functional on $C$ such that $Lf \geq 0$ if $f \geq 0$. Then there exists a nonnegative finite measure $\mu$ on Borel subsets of $X$ such that

$$Lf = \int_X f(x) \mu(dx). \quad (0.1)$$

We need a few auxiliary results for the proof. For $F \subset X$ define

$$\nu(F) = \inf \{Lf : f \in C, f \geq I_F\}.$$ 

Obviously, for any $A, B \subset X$,

$$\nu(A \cup B) \leq \nu(A) + \nu(B),$$

and $\nu(A) \leq \nu(B)$ if $A \subset B$. We write $F \in \mathcal{B}_0$ if $\nu(\partial F) = 0$.

Lemma 0.2. The collection $\mathcal{B}_0$ is an algebra and the set-function $\nu$ is an additive function of $\mathcal{B}_0$.

Proof. Observe that $\nu(\partial F) = \nu(\partial (F^c))$ and

$$\nu(\partial (A \cup B)) \leq \nu((\partial A) \cup (\partial B)) \leq \nu(\partial A) + \nu(\partial B)$$

implying that $\mathcal{B}_0$ is an algebra.

To prove the additivity of $\nu$ on $\mathcal{B}_0$ take $A, B \in \mathcal{B}_0$ such that $AB = \emptyset$. Then

$$\bar{A}\bar{B} = (\partial A \cup A)(\partial B \cup B) = [(\partial A)\partial B] \cup [(\partial A)B] \cup [A\partial B] \subset (\partial A) \cup \partial B,$$

$$\nu(\bar{A}\bar{B}) = 0.$$ 

Hence, for any $\varepsilon > 0$ one can find $f, \phi \in C$ such that

$$f \geq I_{A \cup B}, \quad 1 \geq \phi \geq I_{\bar{A}\bar{B}}, \quad Lf \leq \nu(A \cup B) + \varepsilon, \quad L\phi \leq \varepsilon.$$ 

Define

$$f_1(x) = 1 \quad \text{for} \quad x \in \bar{A}, \quad f_1(x) = \phi(x) \quad \text{for} \quad x \in \bar{B},$$

and extend $f_1$ as a continuous function on $X$ such that

$$0 \leq f_1(x) \leq f(x) + \phi(x)$$

(take any continuous extension $g \geq 0$ and to satisfy the above inequality consider $\min(g, f + \phi)$). Also set $f_2 = f + \phi - f_1$. Then $f_2 \in C$, $f_2 \geq 0$, and $f_2 = f \geq 1$ on $B$. Hence,

$$\nu(A) + \nu(B) \leq Lf_1 + Lf_2 = Lf + L\phi \leq \nu(A \cup B) + 2\varepsilon.$$ 

This finishes the proof of the lemma.
Remark 0.1. If $\bar{A}B = \emptyset$, one can take $\phi \equiv 0$ and in that case even for $A, B \notin \mathfrak{B}_0$ we have

$$\nu(A \cup B) = \nu(A) + \nu(B).$$

Lemma 0.3. The set-function $\nu$ is regular on $\mathfrak{B}_0$, that is, for any $A \in \mathfrak{B}_0$ and $\varepsilon > 0$ there exists a closed set $\Gamma \subset A$ and an open set $G \supset A$ such that $\Gamma, G \in \mathfrak{B}_0$ and $\nu(G) - \varepsilon \leq \nu(A) \leq \nu(\Gamma) + \varepsilon$.

Proof. Since $\mathfrak{B}_0$ is an algebra, it suffices to concentrate on open $G \supset A$. Observe that for any $f \in C$ we have

$$\nu(\{x : f(x) = \lambda\}) = 0$$

for all $\lambda$ apart perhaps from countably many of them. This follows from Remark 0.1 since the sets $\{x : f(x) = \lambda\}$ are closed and disjoint for different $\lambda_i$

$$\sum_i \nu(\{x : f(x) = \lambda_i\}) = \nu(\cup_i \{x : f(x) = \lambda_i\}) \leq \nu(X) < \infty.$$

It follows that open sets $\{x : f(x) > \lambda\}$ belong to $\mathfrak{B}_0$ for all $\lambda$ apart perhaps from countably many of them.

Now take a nonnegative $f \in C$ such that $f \geq I_A$ and

$$Lf \leq \nu(A) + \varepsilon/2.$$ 

Take a $\lambda \in (0, 1)$ such that the open set $G := \{x : f(x) > \lambda\} \in \mathfrak{B}_0$. Observe that $G \supset A$. Also

$$\nu(G) \leq L(f/\lambda) \leq \nu(A)/\lambda + \varepsilon/(2\lambda) = \nu(A) + \frac{1 - \lambda}{\lambda} \nu(A) + \varepsilon/(2\lambda),$$

and it only remains to take $\lambda$ so close to 1 that what we add to $\nu(A)$ above is less than $\varepsilon$. The lemma is proved.

Lemma 0.4. The additive set-function $\nu$ on $\mathfrak{B}_0$ has a $\sigma$-additive extension as a measure on $\sigma(\mathfrak{B}_0)$.

Proof. It suffices to show that for any decreasing sequence of sets $A_n \in \mathfrak{B}_0$ such that $\cap_n A_n = \emptyset$, we have $\nu(A_n) \to 0$ as $n \to \infty$.

We argue by contradiction and assume that there exists an $\varepsilon > 0$ and a decreasing sequence of sets $A_n \in \mathfrak{B}_0$ such that $\cap_n A_n = \emptyset$ but $\nu(A_n) \geq \varepsilon$. Then we take closed $\Gamma_n \in \mathfrak{B}_0$ such that $\Gamma_n \subset A_n$ and $\nu(\Gamma_n) \geq \nu(A_n) - \varepsilon/2^{n+1}$.

The closed sets $B_n = \cap_{k \leq n} \Gamma_k$ are nested and their intersection is empty. Then $B_n$ is empty for an $n$ ($X$ is a compact set). However,

$$A_n \setminus B_n = \left( \bigcap_{k \leq n} A_k \right) \setminus \bigcap_{k \leq n} \Gamma_k \subset \bigcup_{k \leq n} (A_k \setminus \Gamma_k),$$

$$\nu(A_n \setminus B_n) \leq \varepsilon \sum_k 2^{-k-1} = \varepsilon/2, \quad \mu(B_n) \geq \nu(A_n) - \varepsilon/2 \geq \varepsilon/2,$$

and this is the desired contradiction.

We call $\mu$ the extended $\nu$. 
The end of proof of the theorem. The only things which remain to be proved are that $\sigma(B_0)$ contains the Borel $\sigma$-field and that (0.1) holds. Observe that $\rho(x, x_0)$ is a continuous function of $x$ for any $x_0$. By what was said above, $B_r(x_0) \in B_0 \subset \sigma(B_0)$ for almost all $r > 0$. Since $\sigma(B_0)$ is a $\sigma$-field we conclude that $B_r(x_0) \in \sigma(B_0)$ for all $r > 0$ and all Borel sets are also in $\sigma(B_0)$.

To prove (0.1) take $f \in C$ such that $0 \leq f \leq 1$. Let $c_0 < 0 < c_1 < ... < c_{n-1} < 1 < c_n$ be such that $\{x : f(x) < c_i\} \in B_0$. Then for any $\varepsilon > 0$ one can find $\phi_i \in C$ such that

$$\phi_i \geq I_{E_i}, \quad L\phi_i \leq \nu(E_i) + \varepsilon/n = \mu(E_i) + \varepsilon/n,$$

where $E_i = \{x : c_i \leq f(x) < c_{i+1}\}$. Then

$$f \leq \sum_{i=0}^{n-1} c_{k+1}\phi_k, \quad Lf \leq \sum_{i=0}^{n-1} c_{k+1}L\phi_k \leq \sum_{i=0}^{n-1} c_{k+1}\mu(E_i) + \varepsilon,$$

$$\leq \int_X f(x) \mu(dx) + \varepsilon + \max(c_{k+1} - c_k).$$

Since what we add can be made arbitrarily small

$$Lf \leq \int_X f(x) \mu(dx), \quad L(1 - f) \leq \int_X (1 - f(x)) \mu(dx)$$

and the theorem is proved because $L1 = \nu(X) = \mu(X)$.

Something like Section 19.3

**Theorem 0.5** (Bachelier). For every $t \in (0, 1]$ we have $\max_{s \leq t} w_s \sim |w_t|$, which is to say that for every $x \geq 0$

$$P\{\max_{s \leq t} w_s \leq x\} = \frac{2}{\sqrt{2\pi t}} \int_0^x e^{-y^2/2t} \, dy.$$

Proof. Take independent identically distributed random variables $\eta_k$ so that $P(\eta_k = 1) = P(\eta_k = -1) = 1/2$, and define $\xi^n_t$ by

$$\xi^n_t := S_{[nt]}/\sqrt{n} + (nt - [nt])\eta_{[nt]+1}/\sqrt{n},$$

where $S_k := \eta_1 + ... + \eta_k$. First we want to find the distribution of

$$\zeta^n = \max_{[0,1]} \xi^n_t = n^{-1/2} \max_{k \leq n} S_k.$$

Observe that, for each $n$, the sequence $(S_1, ..., S_n)$ takes its every particular value with the same probability $2^{-n}$. In addition, for each integer $i > 0$, the number of sequences favorable for the events

$$\{\max_{k \leq n} S_k \geq i, S_n < i\} \quad \text{and} \quad \{\max_{k \leq n} S_k \geq i, S_n > i\} \quad (0.2)$$
is the same. One proves this by using the reflection principle; that is, one takes each sequence favorable for the first event, keeps it until the moment when it reaches the level $i$ for the last time before $n$ and then reflects its remaining part about this level. This implies equality of the probabilities of the events in (0.2). Furthermore, due to the fact that $i$ is an integer, we have

$$\{\zeta^n \geq in^{-1/2}, \xi^n_1 < in^{-1/2}\} = \{\max_{k\leq n} S_k \geq i, S_n < i\}$$

and

$$\{\zeta^n \geq in^{-1/2}, \xi^n_1 > in^{-1/2}\} = \{\max_{k\leq n} S_k \geq i, S_n > i\}.$$  

Hence,

$$P\{\zeta^n \geq in^{-1/2}, \xi^n_1 < in^{-1/2}\} = P\{\zeta^n \geq in^{-1/2}, \xi^n_1 > in^{-1/2}\}.$$  

Moreover, obviously,

$$P\{\zeta^n \geq in^{-1/2}, \xi^n_1 > in^{-1/2}\} = P\{\xi^n_1 > in^{-1/2}\},$$

$$P\{\zeta^n \geq in^{-1/2}\} = P\{\zeta^n \geq in^{-1/2}, \xi^n_1 > in^{-1/2}\} + P\{\zeta^n \geq in^{-1/2}, \xi^n_1 < in^{-1/2}\} + P\{\xi^n_1 = in^{-1/2}\}.$$  

It follows that

$$P\{\zeta^n \geq in^{-1/2}\} = 2P\{\xi^n_1 > in^{-1/2}\} + P\{\xi^n_1 = in^{-1/2}\}$$  

for every integer $i > 0$. The last equality also obviously holds for $i = 0$. We see that for numbers $a$ of type $in^{-1/2}$, where $i$ is a nonnegative integer, we have

$$P\{\zeta^n \geq a\} = 2P\{\xi^n_1 > a\} + P\{\xi^n_1 = a\}.$$  

Certainly, the last probability goes to zero as $n \to \infty$ since $\xi^n_1$ is asymptotically normal with parameters $(0,1)$. Also, keeping in mind Donsker’s theorem, it is natural to think that

$$P\{\max_{s \leq 1} \xi^n_s \geq a\} \to P\{\max w_s \geq a\}, \quad 2P\{\xi^n_1 > a\} \to 2P\{w_1 > a\}.$$  

Therefore, (0.4) naturally leads to the conclusion that

$$P\{\max w_s \geq a\} = 2P\{w_1 > a\} = P\{|w_1| > a\} \quad \forall a \geq 0,$$

and this is our statement for $t = 1$.

To justify the above argument, notice that (0.3) implies that
\[ P\{\zeta^n = in^{-1/2}\} = P\{\zeta^n \geq in^{-1/2}\} - P\{\zeta^n \geq (i+1)n^{-1/2}\} \]
\[ = 2P\{\xi^n = (i+1)n^{-1/2}\} + P\{\xi^n = in^{-1/2}\} - P\{\xi^n = (i+1)n^{-1/2}\} \]
\[ = P\{\xi^n = (i+1)n^{-1/2}\} + P\{\xi^n = in^{-1/2}\}, \quad i \geq 0. \]

Now for every bounded continuous function \( f(x) \) which vanishes for \( x < 0 \) we get

\[Ef(\zeta^n) = \sum_{i=0}^{\infty} f(in^{-1/2})P\{\zeta^n = in^{-1/2}\} = Ef(\xi^n - n^{-1/2}) + Ef(\xi^n).\]

By Donsker’s theorem and by the continuity of the function \( x \to \max_{[0,1]} xt \) we have

\[Ef(\max_{[0,1]} w_t) = 2Ef(w_1) = Ef(|w_1|).\]

We have proved our statement for \( t = 1 \). For other values of \( t \) one uses that \( cw_{s/t} \) is a Wiener process if \( c > 0 \). The theorem is proved.

**Theorem 0.6.** Let \( u \) be a bounded continuous and continuously differentiable function on \( \mathbb{R} \) such that \( u' \) is piece-wise differentiable and its derivative is bounded. Let \( c \) be a bounded Borel function on \( \mathbb{R} \) such that \( c > \delta \), where \( \delta > 0 \) is a constant. Denote

\[ f = cu - (1/2)u''. \]

Then

\[ u(0) = E \int_0^\infty e^{-\phi t} f(w_t) \, dt. \]

where

\[ \phi_t = \int_0^t c(w_s) \, ds. \]

Comment on \( u(x) \) for \( x \neq 0 \).

**Example 0.1.** Denote \( m_t = \max_{s \leq t} w_s \).

Then for \( \lambda, \mu > 0 \)

\[ E \int_0^\infty e^{-\mu m_t - \lambda t} \, dt = E \int_0^\infty e^{-\mu |w_t| - \lambda t} \, dt \]

is the value at 0 of the solution of the following equation

\[(1/2)u'' - \lambda u = -e^{-\mu|w|}. \]

Assuming that \( \mu^2 \neq 2\lambda \) on finds that

\[ u(x) = \frac{2\mu}{\sqrt{2\lambda(\mu^2 - 2\lambda)}} e^{-|x|\sqrt{2\lambda}} - \frac{2}{\mu^2 - 2\lambda} e^{-\mu|w|}. \]
\[ u(0) = \frac{2}{\mu \sqrt{2\lambda} + 2\lambda}. \]

**Example 0.2.** Let \( a > 0 \). Introduce \( T_a(s) \) as the time spent by \( w_t \) inside \((-a, a)\) during the time period \([0, s]\):

\[ T_a(s) = \int_0^s I_{(-a, a)}(w_s) \, ds. \]

Then for \( \lambda, \mu > 0 \)

\[ I_a(\mu, \lambda) := E \int_0^\infty e^{-\mu T_a(t) - \lambda t} \, dt \]

is the value at 0 of the solution of

\[ \frac{1}{2} u'' - (\lambda + \mu I_{(-a, a)}) u = -1. \]

We find that \( u \) is an even function and

\[ u(x) = \frac{1}{\lambda + \mu} + c_1 e^{-|x|\sqrt{2\lambda}} \quad \text{for} \quad |x| \geq a, \]

\[ u(x) = \frac{1}{\lambda + \mu} + c_2 \cosh(x\sqrt{2(\lambda + \mu)}) \quad \text{for} \quad |x| \leq a, \]

where \( c_1 \) and \( c_2 \) are found from the conditions that \( u(a+) = u(a-) \) and \( u'(a+) = u'(a-) \):

\[ c_1 = -c_2 e^{a \sqrt{2\lambda}} \sqrt{(\lambda + \mu) \lambda^{-1} \sinh(a \sqrt{2(\lambda + \mu)})}, \]

\[ c_2 = \frac{\mu}{\lambda(\lambda + \mu)} \left[ \cosh(a \sqrt{2(\lambda + \mu)}) + \sqrt{(\lambda + \mu) \lambda^{-1} \sinh(a \sqrt{2(\lambda + \mu)})} \right]^{-1}, \]

so that

\[ I_a(\mu, \lambda) = \frac{1}{\lambda + \mu} + c_2. \]

Take \( \mu = (2a)^{-1} \nu \) and let \( a \downarrow 0 \). We will find the distribution of the so-called local time of \( w_t \) at the origine.

One finds that

\[ \frac{1}{\lambda + \mu} + c_2 \to \frac{2}{\nu \sqrt{2\lambda + 2\lambda}}, \]

which as we know is the Laplace transform of

\[ E e^{-\nu m_t}. \]

Hence, the Laplace transforms of

\[ E e^{-\nu (2a)^{-1} T_a(t)} \]

converge and by the general theory for \( t > \varepsilon > 0 \)

\[ \lim_{a \downarrow 0} E e^{-\nu (2a)^{-1} T_a(t)} \leq \varepsilon^{-1} \lim_{a \downarrow 0} \int_{t-\varepsilon}^t E e^{-\nu (2a)^{-1} T_a(s)} \, ds = \varepsilon^{-1} \int_{t-\varepsilon}^t E e^{-\nu m_s} \, ds, \]

which for \( \varepsilon \downarrow 0 \) along with a similar estimate from below yields that

\[ \lim_{a \downarrow 0} E e^{-\nu (2a)^{-1} T_a(t)} = E e^{-\nu m_t}. \]
This by the general theory implies that the distributions of $(2a)^{-1}T_a(t)$ converge weakly to that of $m_t$ or $|w_t|$. It turns out that the distribution of the limit of $(2a)^{-1}T_a(t)$ in $C$ is the same as that of $m_t$.

**Example 0.3.** We are going to find the distribution of the time spent by $w_t$ on the positive half-line on the time interval $[0, s]$:

$$T_+(s) = \int_0^s I_{(0,\infty)}(w_t) \, dt.$$  

We know that for $\lambda, \mu > 0$

$$E \int_0^\infty e^{-\mu T_+(t) - \lambda t} \, dt$$

is the value at 0 of the solution of

$$\left(\frac{1}{2}\right) u'' - (\lambda + \mu I_{(0,\infty)}) u = -1.$$  

We have

$$u(x) = \frac{1}{\lambda + \mu} + c_1 e^{-x \sqrt{2(\lambda + \mu)}} \quad \text{for} \quad x \geq 0,$$

$$u(x) = \frac{1}{\lambda} + c_2 e^{x \sqrt{2\lambda}} \quad \text{for} \quad x \leq 0,$$

where $c_1$ and $c_2$ are found from the conditions that $u(0+) = u(0-)$ and $u'(0+) = u'(0-)$. 

$$c_1 = c_2 + 1 - \frac{1}{\lambda + \mu} = c_2 + \frac{\mu}{\lambda + \mu},$$

$$c_2 \sqrt{2(\lambda + \mu)} + \frac{\mu \sqrt{2(\lambda + \mu)}}{\lambda (\lambda + \mu)} = -c_2 \sqrt{2\lambda},$$

$$c_2 \sqrt{\lambda + \mu} + \frac{\mu}{\lambda \sqrt{\lambda + \mu}} = -c_2 \sqrt{\lambda},$$

$$c_2 = -\frac{\mu}{\lambda \sqrt{\lambda + \mu} (\sqrt{\lambda + \mu} + \sqrt{\lambda})} = -\frac{\sqrt{\lambda + \mu} - \sqrt{\lambda}}{\lambda \sqrt{\lambda + \mu}},$$

$$u(0) = \frac{1}{\lambda} + c_2 = \frac{1}{\sqrt{\lambda + \mu}}.$$  

Observe that $1/\sqrt{\lambda}$ is the Laplace transform of a constant times $t^{-1/2}$ and $1/\sqrt{\lambda + \mu}$, as a function of $\lambda$, is the Laplace transform of a constant (independent of $\mu$) times $t^{-1/2} e^{-\mu t}$. Hence, $u(0)$, as a function of $\lambda$ is the Laplace transform of the convolutions of the above two functions, that is equal to a constant (independent of $\mu$) times

$$\int_0^t \frac{1}{\sqrt{t-s}} \frac{1}{\sqrt{s}} e^{-\mu s} \, ds.$$  

It follows that

$$E e^{-\mu T_+(t)} = \alpha \int_0^t \frac{1}{\sqrt{t-s}} \frac{1}{\sqrt{s}} e^{-\mu s} \, ds,$$  

where $\alpha$ is a constant, the distribution of $T_+(t)$ has a density

$$\alpha \frac{1}{\sqrt{t-s}} \frac{1}{\sqrt{s}}$$

and for $r \in (0, t)$

$$P(T_+(t) \leq r) = \alpha \int_0^r \frac{1}{\sqrt{t-s}} \frac{1}{\sqrt{s}} \, ds = 2\alpha \arcsin\sqrt{\frac{r}{t}},$$

implying that $2\alpha = 2/\pi$. 