There will be three problems selected out of six below.

The problems below are taken from the book. While doing them you are allowed to use the results of previous problems without proving them. However, do explain in detail why those results are applicable.

1. (Problem 2.2.4) On the basis of Problem 2.2.3 prove that for integers $m > n \geq 0$ and $z_0, z_1, \ldots \in \mathbb{S}$

$$P(Z_{[n+1,m]} = z_{[n+1,m]} \mid Z_{[0,n]} = z_{[0,n]})$$

$$= p_m(z_m \mid z_{m-1}) \cdot \ldots \cdot p_{n+1}(z_{n+1} \mid z_n)$$

if $P(Z_{[0,n]} = z_{[0,n]}) > 0$. Then by using Problem 1.3.5 conclude that

$$P(Z_{[n+1,m]} = z_{[n+1,m]} \mid Z_n = z_n) = p_m(z_m \mid z_{m-1}) \cdot \ldots \cdot p_{n+1}(z_{n+1} \mid z_n)$$

if $P(Z_n = z_n) > 0$.

2. (Problem 2.2.36) Fix $d \geq 2$ and $\mathbb{S} = \{1, \ldots, d\}$. For $1 < i < d$, let $p(i, i \pm 1) = 1/2$ and let $p(1, 1) = p(1, 2) = p(d, d) = 1/2$. Prove that all entries of the matrix $p^{d-1}$ are strictly positive. In case $d \geq 3$, find $p^{d-2}_1$.

3. (Problem 3.1.6) Prove that (a.s.)

$$E(\xi_{n+1}^{-1}I_{Y_{n+1}=b} \mid Y_{[0,n]}) = \xi_{n+1}^{-1}(Y_{[0,n]}, b)E(I_{Y_{n+1}=b} \mid Y_{[0,n]})$$

$$= \begin{cases} 
1 & \text{if } P(Y_{n+1} = b \mid Y_{[0,n]}) > 0, \\
0 & \text{if } P(Y_{n+1} = b \mid Y_{[0,n]}) = 0.
\end{cases}$$

4. (Problem 3.1.7) Let $Z_n = (X_n, Y_n)$, $n = 0, 1, 2, \ldots$, be a simple symmetric random walk on $\mathbb{Z}^2$. Prove that (a.s.)

$$\xi_{n+1} = (1/2)I_{Y_n = Y_{n+1}} + (1/4)I_{Y_n \neq Y_{n+1}}$$

and then that the filtering equations are

$$\pi_{n+1}^a = (1/2)(\pi_{n}^{a-1} + \pi_{n}^{a+1})I_{Y_n = Y_{n+1}} + \pi_{n}^{a}I_{Y_n \neq Y_{n+1}}, \quad a \in \mathbb{Z}.$$
(ii) Find the least mean squared error of estimating \( X_n^2 \) in terms of \( Y_{[0,2]} \) if \( Z_n \) starts at the origin. (Hint: (i) Derive that \( m_n := E(X_n \mid Y_{[0,n]}) \) and \( d_n := E(X_n^2 \mid Y_{[0,n]}) \) satisfy
\[
m_{n+1} = (1/2) \sum_{a \in \mathbb{Z}} ((a - 1 + 1) \pi_n^{a-1} + (a + 1 - 1) \pi_n^{a+1}) I_{Y_n = Y_{n+1}} + m_n I_{Y_n \neq Y_{n+1}} = m_n, \quad d_{n+1} = d_n + I_{Y_n = Y_{n+1}}.
\]
Conclude \( m_n = E(X_0 \mid Y_0) \) and, for \( n \geq 1 \), \( d_n \) is the sum of \( E(X_0^2 \mid Y_0) \) and the number of \( i = 0, 1, \ldots, n-1 \) such that \( Y_{i+1} = Y_i \). (ii) First prove that
\[
I := E|X_2^2 - E(X_2^2 \mid Y_{[0,2]})|^2 = EX_2^4 - E|I_{Y_0 = Y_1} + I_{Y_1 = Y_2}|^2.
\]
Upon finding the distribution of \( X_2 \), one finds that the first term on the right is 2.5. In the second term the indicators are independent (why?) and \( Y_0 = 0 \), so that
\[
E|I_{Y_0 = Y_1} + I_{Y_1 = Y_2}|^2 = P(Y_1 = 0) + 2P(Y_1 = 0)P(Y_1 = Y_2) + P(Y_1 = Y_2) = 1/2 + 2/4 + 1/2 = 3/2, \quad I = 1. \)

6. (Problem 3.1.14) In the setting of Problem 3.1.7 prove that, as \( n \to \infty \),
\[
\frac{1}{n} \ln \sum_{a \in \mathbb{Z}} \rho_n^a \to -\frac{1}{2} \ln 8,
\]
so that \((\rho_n^a/\pi_n^a)^{1/n}\) tends to \(8^{-1/2}\) and \(\rho_n^a\) behaves roughly speaking as \(8^{-n/2}\pi_n^a\). In this case to find \(\pi_n^a\) from \(\rho_n^a\) for large \(n\) as in Problem 3.1.12 we have to find the ratio of two extremely small numbers. (Hint: Use the law of large numbers and the fact that the increments of \(Y_n\) are i.i.d.)