There will be four problems selected out of eight below.

The problems below are taken from the book. While doing them you are allowed to use the results of previous problems without proving them. However, do explain in detail why those results are applicable.

1. (Problem 4.3.18) By using (4.3.8) prove that if $g$ is such that $Eg^2(X,Y) < \infty$, then

$$E\left| \int_{\mathbb{R}^d} g(x,Y)p(x \mid Y) \, dx \right|^2 \leq E\left| \int_{\mathbb{R}^d} |g(x,Y)|p(x \mid Y) \, dx \right|^2 \leq E \int_{\mathbb{R}^d} g^2(x,Y)p(x \mid Y) \, dx = E g^2(X,Y) < \infty.$$  

Conclude again that $\int_{\mathbb{R}^d} g(x,Y)p(x \mid Y) \, dx$ is finite with probability one.

2. (Problem 5.2.16) Instead of (5.2.6) consider the equation

$X_{n+1} = AX_n + B + PW_n, \quad Y_n = HX_n + QB_n, \quad n \geq 0$

with the same assumptions on $A,P,H,Q$, where $B$ is a constant vector. Derive filtering equations similar to (5.2.12) and (5.2.26). (Hint: Replace $X_n$ with $U_n = X_n - B - AB - \ldots - A^n B$. Be careful and replace $Y_n$ too.)

3. (Problem 5.2.19 corrected) Consider the following version of equation (5.2.6):

$X_{n+1} = AX_n + PW_{n+1}, \quad Y_n = HX_n + QB_n, \quad n \geq 0$

under the same assumptions as in Problem 5.2.17. Denote $\bar{X}_{n \mid n} = E(X_n \mid Y_{[0,n]})$.

(a) Prove that

$$\bar{X}_{n+1 \mid n+1} = A\bar{X}_{n \mid n} + (A\bar{\Sigma}_{n \mid n}A^* + PP^*)H^*K_n(Y_{n+1} - HA\bar{X}_{n \mid n}),$$

$$\bar{\Sigma}_{n+1 \mid n+1} = PP^* + A\bar{\Sigma}_{n \mid n}A^* - (A\bar{\Sigma}_{n \mid n}A^* + PP^*)H^*K_nH(A\bar{\Sigma}_{n \mid n}A^* + PP^*).$$
where $\Sigma_{n}|n := E(X_n - X_n|n)(X_n - X_n|n)^*$ and

$$K_n := (H\bar{\Sigma}_n|nA^*H^* + HP^*H^* + QQ^*)^{-1}.$$ 

(b) Derive a different formula for $\Sigma_{n+1}|n+1$:

$$\Sigma_{n+1}|n+1 = (A - R_nHA)\Sigma_n|n(A - R_nHA)^* + (P - R_nHP)(P - R_nHP)^* + R_nQQ^*R_n,$$

where

$$R_n = (A\bar{\Sigma}_nA^* + PP^*)H^*K_n.$$

(Hint: Use Theorem 5.2.8 after noting that $\hat{Y}_n = Y_{n+1}$, $\hat{H} = HA$, $\hat{P} = (P,0)$, $\hat{Q} = (HP,Q)$, $\hat{W}_n = \begin{pmatrix} W_{n+1} \\ B_n \end{pmatrix}$ satisfy $X_{n+1} = AX_n + \hat{P}\hat{W}_n$, $\hat{Y}_n = \hat{H}X_n + \hat{Q}\hat{W}_n$. You will see the usefulness of having the same noise term $W_n$ in both equations in (5.2.6)).

4. (Problem 6.5.9 with better hints) In the situation of Remark 6.5.8 define

$$U_t := g^{-1}(Y_t - \int_0^t \bar{X}_s ds).$$

Prove that

(i) $U_t \in \sigma(Y_{[0,t]})$,

(ii) $U_t$, $t \geq 0$, is a Wiener process (different from $B_t$) and, for any $s \leq t$ and $h > 0$ the random variables $U_{t+h} - U_t$ and $Y_s$ are independent. This will show that the increments $U_{t+h} - U_t$ of $U_t$, driving equation (6.5.30), contain only the information about $Y$ which is independent of the previous information. By this reason $U_t$ is called the innovation process. (Hint: In (ii) use that $EX_rY_s = EX_rY_s$ and $EX_r\bar{X}_u = EX_rX_u$ if $u \geq r \geq s$ while treating $E|U_t - U_s|^2$. Also derive and use

$$E\left(\int_s^t \bar{X}_r dr\right)^2 = 2E \int_s^t \left( \int_r^t \bar{X}_u \bar{X}_u du \right) dr = 2E \int_s^t \bar{X}_r(Y_t - Y_r) dr.$$ 

5. (Problem 6.5.11) In equation (6.5.10) assume $EX_0 = 0$ and $c^2 = -2aEX_0^2$ and prove that $EX_tX_s$ depends only on $|t - s|$ so that $X_t$ is a stationary process.

6. (Problem 7.2.14) Let $(X_n)$ be a mean-square stationary sequence with spectral density $f$. Take a complex number $a$ such that $|a| < 1$ and take the operator $L_a$ introduced in Problem 7.2.12. Prove that $L_a^{-1}X_n$ is a mean-square stationary sequence with spectral density

$$f(x)|e^{ia} - a|^{-2}.$$ 

(Hint: Show that what is asserted is a property of the correlation function of $X_n$.)
7. (Problem 8.1.4) Find $P$ and $Q$ if the correlation function is given as in Problem 7.2.5. (Hint: $Q \equiv 1$.)

8. (Problem 8.2.1) Take the operators $L_a$ from Problem 7.2.12 and prove that

(i) the sequence

$$L_{s_1} \cdot ... \cdot L_{s_D} X_n$$

is mean-square stationary having spectral density $|A(e^{ix})|^2$;

(ii) the sequence

$$L_{r_1}^{-1} \cdot ... \cdot L_{r_N}^{-1} L_{s_1} \cdot ... \cdot L_{s_D} X_n$$

is mean-square stationary having constant spectral density $p_0^2$ and variance $2\pi p_0^2$.

(iii) Conclude that there exists a sequence of uncorrelated random variables $(W_n, n \in \mathbb{Z})$ with variance $2\pi$ and zero mean such that, for any $n$,

$$q_0 X_n + q_1 X_{n-1} + ... + q_D X_{n-D} = p_0 W_n + p_1 W_{n-1} + ... + p_N W_{n-N}. \quad (1)$$

(Hint: Recall Problems 7.2.14, 7.2.15 and 7.2.9.)