Solutions for Homework and Tests

Assignment 1 due January 29 from Section 1.13:
1.1 (explain your answer), 1.2, 1.6, 1.7 (c) (in 1.7 interpret the phrase “did not rain on Sunday or Monday” as “both Sunday and Monday were sunny days”)

1.1. We have
\[ P(X_3 = 2, X_2 = 1, X_1 = 2) = P(Y_0 = 1, Y_1 = 1, Y_2 = 0, Y_3 = 2) = 0, \]
\[ P(X_3 = 2, X_2 = 1) = P(Y_1 = 0, Y_2 = 1, Y_3 = 1) = 1/8. \]

Hence
\[ P(X_3 = 2 | X_2 = 1, X_1 = 2) \neq \frac{P(X_3 = 2 | X_2 = 1)}{8P(X_2 = 1)} \]
and \(X_n\) is not a Markov chain.

1.2. All possible values of \(X_n\), that is states of the chain, are 0, 1, 2, 3, 4, 5. Notice that always \(|X_{n+1} - X_n| = 0 \) or \(1\), so that
\[ P(X_{n+1} = j | X_n = i) = 0 \text{ if } |j - i| > 1, \]
and we only have to consider the cases \(j = i, j = i + 1, \) and \(j = i - 1\).

Given that \(X_n = i\), we have \(i\) white and \(5 - i\) black balls in the left urn and \(5 - i\) white and \(i\) black balls in the right urn. The event \(X_{n+1} = i\) only happens if either we take for exchanging a black ball from the left urn and a black ball from the right urn (which happens with probability \(\frac{5-i}{5} \cdot \frac{i}{5}\)) or we take a white ball from the left and a white ball from the right (which happens with probability \(\frac{i}{5} \cdot \frac{5-i}{5}\)). Hence
\[ P(X_{n+1} = i | X_n = i) = \frac{5-i}{5} \cdot \frac{i}{5} + \frac{i}{5} \cdot \frac{5-i}{5} = 2i(5-i) \]
\[ = \frac{2i(5-i)}{25}. \]

Under the same condition that \(X_n = i\), the event \(X_{n+1} = i + 1\) only happens if \(i < 5\) and we take a black ball from the left and a white ball from the right, that is
\[ P(X_{n+1} = i + 1 | X_n = i) = \frac{5-i}{5} \cdot \frac{5-i}{5} = \frac{(5-i)^2}{25}. \]
(Observe that the formula is valid even if \(i = 5\).)

Finally, given \(X_n = i\), the event \(X_{n+1} = i - 1\) means that \(i \geq 1\), we took a white ball from the left and a black one from the right. Hence,
\[ P(X_{n+1} = i - 1 | X_n = i) = \frac{i}{5} \cdot \frac{i}{5} = \frac{i^2}{25}. \]
1.6. Let $X_n$ be the chain taking three values $A, B, C$. We are given that $X_{n+1} \neq X_n$, $P(X_{n+1} = B|X_n = A) = P(X_{n+1} = C|X_n = A) = 1/2$, $P(X_{n+1} = A|X_n = B) = P(X_{n+1} = A|X_n = C) = 3/4$. This yields the transition matrix given in the Answers: If we assign number 1 to $A$, 2 to $B$ and 3 to $C$

$$p = (p(ij))_{i,j=1}^3 = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 3/4 & 0 & 1/4 \\ 3/4 & 1/4 & 0 \end{pmatrix}.$$  

Then

$$p^2 = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 3/4 & 0 & 1/4 \\ 3/4 & 1/4 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1/2 & 1/2 \\ 3/4 & 0 & 1/4 \\ 3/4 & 1/4 & 0 \end{pmatrix} = \begin{pmatrix} 3/16 & 7/16 & 3/8 \\ 3/16 & 3/8 & 7/16 \end{pmatrix},$$

$$p^3 = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 3/4 & 0 & 1/4 \\ 3/4 & 1/4 & 0 \end{pmatrix} \cdot \begin{pmatrix} 3/16 & 7/16 & 3/8 \\ 3/16 & 3/8 & 7/16 \end{pmatrix} = \begin{pmatrix} 39/64 & 13/32 & 13/32 \\ 39/64 & 13/32 & 13/32 \end{pmatrix}.$$

If the driver starts at the airport, his distribution

at time 1 = $(p(1,1), p(1,2), p(1,3)) = (0, 1/2, 1/2)$,

at time 2 = $(p^2(1,1), p^2(1,2), p^2(1,3)) = (3/4, 1/8, 1/8),$

at time 3 = $(p^3(1,1), p^3(1,2), p^3(1,3)) = (3/16, 13/32, 13/32).$

1.7 (c). State space $(1,2,3,4) = (RR, RS, SR, SS)$. The question is to find $P(X_2 \in \{1,3\} \mid X_0 = 4) = p^2(4, 1) + p^2(4, 3)$. The transition matrix

$$p = \begin{pmatrix} 0.6 & 0.4 & 0 & 0 \\ 0 & 0 & 0.6 & 0.4 \\ 0.6 & 0.4 & 0 & 0 \\ 0 & 0 & 0.3 & 0.7 \end{pmatrix}$$

$$p^2(4, 1) = (0, 0, 0.3, 0.7) \cdot (0.6, 0, 0.6, 0) = 0.18,$$

$$p^2(4, 3) = (0, 0, 0.3, 0.7) \cdot (0, 0.6, 0, 0.3) = 0.21.$$  

The answer is $0.18+ = 0.21 = 0.39$.

An alternative solution. For $SS$ to get transformed to $SR$ in two steps, we need the first step to lead to $SS$ and the next step to yield $SR$. The probability of this is $P(SR \mid SS)P(SS \mid SS) = 0.3 \times 0.7 = 0.21$. There is another (only) way to get rain on Wednesday: to transform $SS$ to $RR$. For that $SS$ should go first to $SR$ and then $SR$ should go to $RR$. The probability that both steps happen is $0.3 \times 0.6 = 0.18$. 

Grade distribution for Assignment 1:
Assignment 2 due Feb 12 from Section 1.13: Exercise 1.38 and do what is required in Exercise 1.13 but with $p$ given by

$$p = \begin{pmatrix} 0 & 0 & \frac{3}{5} & \frac{2}{5} \\ 0 & 0 & \frac{1}{5} & \frac{4}{5} \\ \frac{1}{4} & \frac{3}{4} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \end{pmatrix}.$$  

[Hint: Stationary distributions for $p^2$ form a one parameter family]

1.13. (a) We have

$$p^2 = \begin{pmatrix} 0 & 0 & \frac{3}{5} & \frac{2}{5} \\ 0 & 0 & \frac{1}{5} & \frac{4}{5} \\ \frac{1}{4} & \frac{3}{4} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & \frac{3}{5} & \frac{2}{5} \\ 0 & 0 & \frac{1}{5} & \frac{4}{5} \\ \frac{1}{4} & \frac{3}{4} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \end{pmatrix} = \begin{pmatrix} \frac{7}{20} & \frac{13}{20} & 0 & 0 \\ \frac{9}{20} & \frac{11}{20} & 0 & 0 \\ 0 & 0 & \frac{3}{10} & \frac{7}{10} \\ 0 & 0 & \frac{2}{5} & \frac{3}{5} \end{pmatrix}.$$  

(b) Stationary distribution for $p$. Any stationary distribution $\pi = (\pi(1), \pi(2), \pi(3), \pi(4))$ for $p$ satisfies $\pi = \pi p$ by definition. In the coordinate form

$$\begin{align*}
\pi(1) &= \pi(3)\frac{1}{2} + \pi(4)\frac{1}{2}, \\
\pi(2) &= \pi(3)\frac{3}{4} + \pi(4)\frac{1}{2}, \\
\pi(3) &= \pi(1)\frac{3}{5} + \pi(2)\frac{1}{5} \\
\pi(4) &= \pi(1)\frac{2}{5} + \pi(2)\frac{4}{5}
\end{align*}.$$  

We rewrite the system as

$$\begin{align*}
0 &= -\pi(1) + \pi(3)\frac{1}{2} + \pi(4)\frac{1}{2}, \\
0 &= -\pi(2) + \pi(3)\frac{3}{4} + \pi(4)\frac{1}{2}, \\
0 &= \pi(1)\frac{3}{5} + \pi(2)\frac{1}{5} - \pi(3), \\
0 &= \pi(1)\frac{2}{5} + \pi(2)\frac{4}{5} - \pi(4).
\end{align*}$$  

and start Gauss’s elimination by excluding $\pi(1)$ from the third equation by multiplying the first equation by $3/5$ and then adding the result to
the third equation. In the same way we eliminate \( \pi(1) \) from the fourth equation. We find
\[
0 = -\pi(1) + \pi(3)1/4 + \pi(4)1/2, \\
0 = -\pi(2) + \pi(3)3/4 + \pi(4)1/2, \\
0 = +\pi(2)1/5 - \pi(3)17/20 + \pi(4)3/10, \\
0 = +\pi(2)4/5 + \pi(3)1/10 - \pi(4)4/5.
\]
Then we eliminate \( \pi(2) \) from the third and fourth equations by using the second equation and get
\[
0 = -\pi(1) + \pi(3)1/4 + \pi(4)1/2, \\
0 = -\pi(2) + \pi(3)3/4 + \pi(4)1/2, \\
0 = -\pi(3)7/10 + \pi(4)2/5, \\
0 = +\pi(3)7/10 - \pi(4)2/5.
\]
Here the fourth equation is the third times \(-1\) and can be thrown away. After that for brievity we denote \( c = \pi(4) \) and then from the third equation we have \( \pi(3) = c4/7 \). Then the second and the first equations yield
\[
\pi(2) = c3/7 + c/2 = c13/14, \quad \pi(1) = c/7 + c/2 = c9/14.
\]
Finally from \( \pi(1) + \ldots + \pi(4) = 1 \) we infer \( 1 = c(9/14 + 13/14 + 4/7 + 1) = c44/14, \ c = 7/22, \) and
\[
\pi(1) = 9/44, \quad \pi(2) = 13/44, \quad \pi(3) = 2/11, \quad \pi(4) = 7/22.
\]
**Stationary distributions for \( p^2 \).** Here we have to solve the equation \( \pi = \pi p^2 \) which is
\[
\begin{align*}
\pi(1) &= \pi(1)7/20 + \pi(2)9/20, \\
\pi(2) &= \pi(1)13/20 + \pi(2)11/20, \\
\pi(3) &= \pi(3)3/10 + \pi(4)2/5, \\
\pi(4) &= \pi(3)7/10 + \pi(4)3/5.
\end{align*}
\] (1)
This system splits into two
\[
\begin{align*}
\pi(1) &= \pi(1)7/20 + \pi(2)9/20, \\
\pi(2) &= \pi(1)13/20 + \pi(2)11/20, \\
\pi(3) &= \pi(3)3/10 + \pi(4)2/5, \\
\pi(4) &= \pi(3)7/10 + \pi(4)3/5.
\end{align*}
\] (2)
System (2) transforms into
\[
\begin{align*}
0 &= -\pi(1)13/20 + \pi(2)9/20, \\
0 &= \pi(1)13/20 - \pi(2)9/20,
\end{align*}
\] which after denoting \( \pi(1) = c_1 \) is equivalent to
\[
(\pi(1), \pi(2)) = c_1(1, 13/9).
\] (4)
System (3) transforms into

\[ 0 = -\pi(3)7/10 + \pi(4)2/5, \]
\[ 0 = \pi(3)7/10 - \pi(4)2/5, \]

which after denoting \( \pi(3) = c_2 \) is equivalent to

\[ (\pi(3), \pi(4)) = c_2(1, 7/4). \]  (5)

We conclude from (4) and (4) that any solution of system (1) is written as

\[ (\pi(1), \pi(2), \pi(3), \pi(4)) = c_1(1, 13/9, 0, 0) + c_2(0, 0, 1, 7/4). \]

Since we are only looking for solutions which are distributions we have to take into account two more conditions: \( \pi(i) \geq 0 \) and \( \sum \pi(i) = 1 \). The latter is

\[ c_1 \frac{22}{9} + c_2 \frac{11}{4} = 1, \quad c_2 = 4/11 - c_1 8/9. \]

Thus, there are many stationary distributions and they all are given by

\[ (\pi(1), \pi(2), \pi(3), \pi(4)) = c_1(1, 13/9, 0, 0) + (4/11 - c_1 8/9)(0, 0, 1, 7/4) \]

\[ = (c_1, c_1 13/9, 4/11 - c_1 8/9, 4/11 - c_1 8/9) \frac{7}{4}. \]

where \( c_1 \) is any constant satisfying \( 0 \leq c_1 \leq 9/22 \).

(c) The Markov chain corresponding to \( p^2 \) has two closed irreducible sets \( A = \{1, 2\} \) and \( B = \{3, 4\} \). Since the diagonal terms are positive all states have period one. Therefore, for \( x \in A \) we have

\[ (p^{2n}(x, 1), p^{2n}(x, 2)) \to \pi = (\pi(1), \pi(2)), \]

where \( \pi \) satisfies

\[ (\pi(1), \pi(2)) = (\pi(1), \pi(2)) \begin{pmatrix} 7/20 & 13/20 \\ 9/20 & 11/20 \end{pmatrix}. \]

This is system (2) and from (4) and \( \pi_1 + \pi_2 = 1 \) we get \( c_1 = 9/22, \pi(1) = 9/22, \pi(2) = 13/22 \). As a conclusion,

\[ (p^{2n}(x, 1), p^{2n}(x, 2)) \to (9/22, 13/22) \text{ for } x = 1, 2. \]

Similarly, for \( x \in B \) we have \( (p^{2n}(x, 3), p^{2n}(x, 4)) \to (\pi(3), \pi(4)), \)

where \( (\pi(3), \pi(4)) \) satisfies (3) and according to (5) and \( \pi(3) + \pi(4) = 1 \) we obtain \( c_2 = 4/11, \pi(3) = 4/11, \pi(4) = 7/11 \), and

\[ (p^{2n}(x, 3), p^{2n}(x, 4)) \to (4/11, 7/11) \text{ for } x = 3, 4. \]

In all remaining cases \( p^{2n}(x, y) = 0 \to 0 \).
1.38. Observe that in both cases if she goes from home to work or back, the conditional probability \( P(X_{n+1} = 0 | X_n = 0) \) is the probability that the number of umbrellas at the destination once the person reaches it becomes 0 given that the number of umbrellas at the starting point is 0. But the total number of umbrellas is 3. Therefore, \( P(X_{n+1} = 0 | X_n = 0) = 0 \). Similarly,

\[
P(X_{n+1} = 1 | X_n = 0) = P(X_{n+1} = 2 | X_n = 0) = 0.
\]

However, \( P(X_{n+1} = 3 | X_n = 0) = 1 \) since all 3 umbrellas are at the destination. Also because of the total number of umbrellas is 3 we have

\[
P(X_{n+1} = 0 | X_n = 1) = P(X_{n+1} = 1 | X_n = 1) = 0,
\]

\[
P(X_{n+1} = 0 | X_n = 2) = P(X_{n+1} = 3 | X_n = 2) = 0,
\]

\[
P(X_{n+1} = 2 | X_n = 3) = P(X_{n+1} = 3 | X_n = 3) = 0.
\]

The probability \( P(X_{n+1} = 2 | X_n = 1) \) is the probability that it is not raining during the trip given that there is one umbrella at the starting point. The rain and positions of the umbrellas are independent so \( P(X_{n+1} = 2 | X_n = 1) = 0.8 \). In the same way \( P(X_{n+1} = 1 | X_n = 2) = 0.8 \). The probability \( P(X_{n+1} = 3 | X_n = 1) \) is the probability that it is raining during the trip given that there is one umbrella at the starting point. Therefore, \( P(X_{n+1} = 3 | X_n = 1) = 0.2 \).

The probability \( P(X_{n+1} = 2 | X_n = 2) \) is the probability that it is raining during the trip given that there is 2 umbrella at the starting point. Therefore, \( P(X_{n+1} = 2 | X_n = 2) = 0.2 \).

Also is easy to see that

\[
P(X_{n+1} = 0 | X_n = 3) = 0.8, \quad P(X_{n+1} = 1 | X_n = 3) = 0.2.
\]

As a result the transition matrix is (the row number \( i \) corresponds to \( i - 1 \) umbrellas, the same for columns)

\[
p = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 0.8 & 0.2 \\
0 & 0.8 & 0.2 & 0 \\
0.8 & 0.2 & 0 & 0
\end{pmatrix}.
\]

The chain is obviously irreducible and therefore the law of large numbers is applicable and the stationary distribution is unique. This distribution satisfies \( \pi = \pi p \) that is

\[
\begin{align*}
\pi(0) &= 0.8\pi(3) \\
\pi(1) &= 0.8\pi(2) + 0.2\pi(3) \\
\pi(2) &= 0.8\pi(1) + 0.2\pi(2) \\
\pi(3) &= \pi(0) + 0.2\pi(1)
\end{align*}
\]
Denote \( c = \pi(3) \), then the first equation means that \( \pi(0) = c0.8 \), the last one that \( \pi(1) = c \) and the third equation yields \( \pi(2) = \pi(1) = c \). From \( \sum \pi(i) = 1 \) we get \( c3.8 = 1, c = 5/19 \),

\[
\pi(0) = 0.8 \cdot 5/19 = 4/19, \quad \pi(1) = \pi(2) = \pi(3) = 5/19.
\]

The person gets wet as many times as \( X_n = 0 \) and it rains. By the law of large numbers, for large \( m \), the number of times \( n = 0, 1, ..., m \) such that \( X_n = 0 \) is approximately \( \pi(0)m \) and each time the person can get wet with probability 0.2. Hence, the number of times \( n = 0, 1, ..., m \) such that the person gets wet is approximately \( \pi(0)m0.2 \) and the limit fraction of time he gets wet is

\[
\pi(0)0.2 = 0.2 \cdot 4/19 = 4/95 = 0.04210526316.
\]

By the way, the probability that it rains is 0.2, so without having any umbrella the person gets wet in 20 percent out of all trips, and using three of them in a disorderly fashion decreases this percentage only about 5 times (= 19/4). It is better to have one and carry it with you all the time.

Grade distribution for Assignment 2: 36, 36, 36, 36, 36, 36, 36, 36, 36, 36, 36, 36, 36, 36, 36, 36, 36, 36, 36, 36, 36, 34, 33, 33, 31, 31, 29, 29, 28, 28, 25, 24, 8.

Grade distribution after Assignments 1+2: 72, 72, 72, 72, 72, 72, 72, 72, 72, 72, 72, 69, 67, 66, 65, 64, 64, 63, 61, 60, 58, 54, 44, 44, 17.

Midterm Feb 19.

1. A basketball player gives consecutive tries to hits the basket. If the player missed in the \( n-1 \)st and the \( n \)th shots, the probability to hit in the \( n+1 \)st shot is 1/2. If the player missed in the \( n-1 \)st and hit in the \( n \)th shots, the probability to hit in the \( n+1 \)st shot is 2/3. If the player hit in the \( n-1 \)st and missed in the \( n \)th shots, the probability to hit in the \( n+1 \)st shot is 2/3. If the player hit both in the \( n-1 \)st and the \( n \)th shots, the probability to hit in the \( n+1 \)st shot is 3/4.

If the player tries \( m \) times, find the limit as \( m \to \infty \) of the fraction of time the player hits a shot.

(Hint: The chain has four states \( MM, MH, HM, HH \). To start the chain take by definition that the \((-1)\)st shot was a hit.)

2 (similar to 1.52) Statement. At the New York State Fair in Syracuse, David Durrett encounters a carnival game where for one dollar he may buy a single coupon allowing him to play the game of Dummo. On each play of Dummo, David has an even chance of
winning another coupon or losing a coupon. When he runs out of coupons he loses the game. However, if he collects three coupons, he wins a surprise.

(a) What is the probability David will win the surprise?
(b) What is the expected number of plays he needs to win or lose the game?
(c) Answer (a) and (b) when the goal is \( N \) coupons.

1.40. Consider the points 1,2,3,4 to be marked on a straight line. Let \( X_n \) be a Markov chain that moves to the right with probability 2/3 and to the left with probability 1/3, but subject to the rule that if \( X_n \) tries to go to the right from 4 or to the left from 1 it stays put (until the next step). Find (a) the transition matrix for the chain, and (b) the limit of the fraction of time spent at each site.

**Solutions.**

1. Let

\( X_n = MM \) if the player missed in the \( n-1 \)st and the \( n \)th shots,

\( X_n = MH \) if the player missed in the \( n-1 \)st and hit in the \( n \)th shots,

\( X_n = HM \) if the player hit in the \( n-1 \)st and missed in the \( n \)th shots,

\( X_n = HH \) if the player hit both in the \( n-1 \)st and the \( n \)th shots.

To start the chain running from \( n = 0 \) we take by definition that the \((-1)\)st shot was a hit. We are given that \( X_n \) is a Markov chain with transition matrix

\[
p = \begin{pmatrix}
    MM & MH & HM & HH \\
    1/2 & 1/2 & 0 & 0 \\
    0 & 0 & 1/3 & 2/3 \\
    0 & 0 & 1/4 & 3/4 \\
\end{pmatrix}
\]

The chain is easily seen to be irreducible, so there is a unique stationary distribution \( \pi = (\pi(1), \pi(2), \pi(3), \pi(4)) \) which satisfies

\[
\begin{align*}
\pi(MM) &= \pi(MM)/2 + \pi(HM)/3 \\
\pi(MH) &= \pi(MH)/2 + \pi(HM)/2/3 \\
\pi(HM) &= \pi(MH)/3 + \pi(HH)/4 \\
\pi(HH) &= \pi(MH)/2/3 + \pi(HH)/3/4 \\
\end{align*}
\]

If we denote \( \pi(MM) = c \), then the first equation yields \( \pi(HM) = 3c/2 \), then from the second we see that \( \pi(MH) = 3c/2 \). Finally the third equation shows that \( 3c/2 = c/2 + \pi(HH)/4 \), that is \( \pi(HH) = 4c \).
From $\sum \pi(i) = 1$ we get $8c = 1$, $c = 1/8$, 

$$
\pi(MM) = 1/8, \quad \pi(MH) = \pi(HM) = 3/16, \quad \pi(HH) = 1/2.
$$

For large $m$ we need to find the ratio to $m$ of the number of $n$’s in the sequence $n = 0, 1, \ldots, m$ for which the player hit the basket. The player hits on his $n$th shot iff either $X_n = MH$ or $X_n = HH$. Therefore, the fraction of time the player hits a shot in the limit as $m \to \infty$ is the sum of average number of visits to $MH$ plus the average number of visits to $HH$. In other words, the answer is $\pi(MH) + \pi(HH) = 3/16 + 1/2 = 11/16$.

2. This is a fair game and directly from the book we get that the probability to win a surprise is $x/N$ where $x = 1$. If $N = 3$ this is 1/3. The average number of plays before winning a surprise or losing all money is $x(N - x) = N - 1$ which is 2 if $N = 3$.

1.40. The transition matrix

$$
\begin{pmatrix}
\frac{1}{3} & \frac{2}{3} & 0 & 0 \\
\frac{1}{3} & 0 & \frac{2}{3} & 0 \\
0 & \frac{1}{3} & 0 & \frac{2}{3} \\
0 & 0 & \frac{1}{3} & \frac{2}{3}
\end{pmatrix}
$$

Stationary distribution satisfies

$$
\pi(1) = (1/3)\pi(1) + (1/3)\pi(2), \quad \pi(2) = (2/3)\pi(1) + (1/3)\pi(3), \\
\pi(3) = (2/3)\pi(2) + (1/3)\pi(4), \quad \pi(4) = (2/3)\pi(3) + (2/3)\pi(4).
$$

Starting from the first equation we find $\pi(2) = 2\pi(1)$. The second one says $\pi(3) = 4\pi(1)$, the third $\pi(4) = 8\pi(1)$. You find $\pi(1)$ from

$$(1 + 2 + 4 + 8)\pi(1) = \pi(1) + \pi(2) + \pi(3) + \pi(4) = 1$$

and then use that the limiting fraction of the time spent in $i$ is $\pi(i)$.

Grade distribution for Midterm 1:

36, 36, 36, 36, 36, 36, 36, 36, 36, 36, 36, 36, 36, 36, 36, 34, 34, 33, 33, 33, 31, 31, 30, 30, 26, 26, 24, 22, 18.

Assignment 3 due Feb 26: 1.23 (assume that each woman will have a daughter), 1.61, 1.67, 1.77 (assume $p > 0$, 1.55 mentioned in the statement is Example 1.55).

Solutions

1.23. States: $W$ (working daughter) and $N$. The transition matrix

$$
\begin{pmatrix}
0.8 & 0.2 \\
0.3 & 0.7
\end{pmatrix}
$$
Stationary distribution satisfies

\[ \pi(W) = \pi(W)0.8 + \pi(N)0.3, \quad \pi(N) = \pi(W)0.2 + \pi(N)0.7. \]

From the second equation \( \pi(N) = (2/3)\pi(W) \) and since \( \pi(W) + \pi(N) = 1, \) \( \pi(W) = 3/5 = 0.6. \)

1.61. (a) Let starting position be 12 and denote \( m(i) \) the expected number of steps to reach 12 starting from state \( i. \) Then \( m(12) = 0, \) \( m(i) = 1 + (m(i + 1) + m(i - 1))/2 \) for \( i \neq 12 \) with the agreement that \( m(0) = 0. \) We see that it is a simple random walk with absorption. Hence, \( m(i) = i(12 - i) \) and the answer is \( (1/2)(m(1) + m(11)) = 22. \)

(b) The event in question is the event that \( X_1 = 1 \) or \( X_1 = 11 \) and if \( X_1 = 1, \) the \( X_n \) never reaches 11 before reaching the starting point, or \( X_1 = 11, \) the \( X_n \) never reaches 1 before reaching the starting point. The probability of the first event is \( 1/2 \) times the probability that simple random walk on 0, 1, ..., 11 starting from 1 reaches 0 before 11. This probability is \( i/(12 - i) = 1/12. \) The answer is \( 2 \times (1/2)(1/12) = 1/12. \)

1.67. The state space is 0, 1, 2, ..., 6. If we saw \( i \) numbers then the probability to see \( i + 1 \) numbers on the next step is the probability that none of the \( i \) numbers we have reappeared. This probability is \( (6 - i)/6: \) \( p(i, i + 1) = (6 - i)/6, \) \( p(i, i) = i/64, \) \( i = 0, 1, 2, ..., 6. \) All other entries of \( p \) are zeros. Let \( m(i) \) be the average number of steps to see all 6 numbers if we saw \( i \) of them. Then \( m(i) = 1 + m(i)i/6 + m(i+1)(1 - i/6), \) \( i < 6, \) \( m(6) = 0. \) That is

\[ m(i) = \frac{6}{6 - i} + m(i + 1), \quad m(5) = \frac{6}{5}, \quad m(4) = \frac{6}{4} + \frac{6}{5}, \ldots. \]

1.77. By Lemma 1.31 the probability of extinction \( \bar{p} \) is the least root of

\[ \bar{p} = \sum_{k=0}^{\infty} p(1-p)^k \bar{p}^k = \frac{p}{1 - (1-p)\bar{p}}, \quad \bar{p} - \bar{p}^2 + p(\bar{p}^2 - 1) = 0, \]

\[ (1 - \bar{p})(\bar{p} - p(1 + \bar{p}) = 0, \quad (1 - \bar{p})(\bar{p}(1 - p) - p) = 0 \]

If \( p \geq 1/2, \) then \( \bar{p} = 1, \) if \( p < 1/2, \) \( p = p/(1-p) < 1. \)

Grade distribution for Assignment 3: 36, 36, 36, 36, 36, 36, 36, 36, 36, 36, 32, 30, 30, 30, 30, 30, 30, 27, 27, 21, 21, 21, 18, 16, 14.

Assignment 4 due March 18:

1. Assume that $S$ and $T$ are independent exponentially distributed, set $U = \min(S, T), V = \max(S, T)$. Prove that $U$ and $W = V - U$ are independent.

Also 2.17, 2.20 (waiting time is uniform on $(0,1)$ in hours), 2.44.

1. Let $\lambda$ be the rate of $S$ and $\mu$ the rate of $T$. Observe that $W = |S - T|$. Note that, for any $u, w > 0$, the set $\{s, t \geq 0 : \min(s, t) \leq u, |s - t| \leq w\}$ is the union of $\{s, t \geq 0 : t \leq u, t \leq s \leq t + w\}$ and $\{s, t \geq 0 : s \leq u, s \leq t \leq s + w\}$. We know that the joint density of $(S, T)$ is $\lambda \mu e^{-\lambda s - \mu t}$. Therefore,

\[
P(U \leq u, V - U \leq w) = \lambda \mu \int_0^u dt \int_t^{t+w} e^{-\lambda s - \mu t} ds + \lambda \mu \int_0^u ds \int_s^{s+w} e^{-\lambda s - \mu t} dt,
\]

where the first term equals

\[
\mu \int_0^u e^{-\mu t} (e^{-\lambda t} - e^{-\lambda (t+w)}) dt = \frac{\mu}{\lambda + \mu} (1 - e^{-(\lambda + \mu)u}) (1 - e^{-\lambda w})
\]

and the second term is (just interchange $\lambda$ and $\mu$ in the above)

\[
\frac{\lambda}{\lambda + \mu} (1 - e^{-(\lambda + \mu)u}) (1 - e^{-\mu w}).
\]

It follows that

\[
P(U \leq u, V - U \leq w) = (1 - e^{-(\lambda + \mu)u}) \left(\frac{\mu}{\lambda + \mu} (1 - e^{-\lambda w}) + \frac{\lambda}{\lambda + \mu} (1 - e^{-\mu w})\right).
\]

As $w \to \infty$ we see (what we already know)

\[
P(U \leq u) = 1 - e^{-(\lambda + \mu)u},
\]

and as $u \to \infty$ we get

\[
P(V - U \leq w) = \frac{\mu}{\lambda + \mu} (1 - e^{-\lambda w}) + \frac{\lambda}{\lambda + \mu} (1 - e^{-\mu w}).
\]

Hence,

\[
P(U \leq u, V - U \leq w) = P(U \leq u)P(V - U \leq w)
\]

and we are done.
Second solution. Bulb interpretation. We have green and red tiny bulbs on the real line each occupying $dt$ of space. Greens light up with probability $\lambda dt$ and the reds with probability $\mu dt$ and all events are independent. Then we know that the distance of the first shining green light to the origin is $\text{expon}(\lambda)$ and the distance of the first shining red light to the origin is $\text{expon}(\mu)$. Then if $S < T$, $V - U$ is the distance between the first shining green and the first shining red. Since everything is independent and works the same way in all places on the line, this distance is independent of $S$ and is $\text{expon}(\mu)$. That is

$$P(W > w \mid S < T) = e^{-\mu w}, \quad P(W \leq w \mid S < T) = 1 - e^{-\mu w}$$

Similarly,

$$P(W \leq w \mid T < S) = 1 - e^{-\lambda w}.$$ 

Next,

$$P(U \leq u, W \leq w) = P(S \leq u, S < T, W \leq w) + P(S \leq u, T < S, W \leq w),$$

where by using the above independence

$$P(S \leq u, S < T, W \leq w) = P(W \leq w \mid S < T, S \leq u)P(S < T, S \leq u)$$

$$= (1 - e^{-\mu w})P(S < T, U \leq u) = (1 - e^{-\mu w})\frac{\lambda}{\lambda + \mu}(1 - e^{-(\lambda + \mu)u}),$$

where the last equality is obtained by using Theorem 2.1. Similarly,

$$P(S \leq u, T < S, W \leq w) = (1 - e^{-\mu w})\frac{\mu}{\lambda + \mu}(1 - e^{-(\lambda + \mu)u})$$

and we arrive at (6) once more.

2.17. Let $N(t)$ be the total number of calls during first $i$ hours.

(a) Here the question is to find $P(N(1) < 2)$. Recalling that $P(N(t) = k) = ((4t)^k/k!e^{-4t}$ we have

$$P(N(1) < 2) = P(N(1) = 0) + P(N(1) = 1) = e^{-4} + 4e^{-4} = 5e^{-4}.$$ 

(b) Here we are asked to find $P(N(2) - N(1) < 2 \mid N(1) - N(0) = 6)$. Since the increments of the Poisson process are independent and time homogeneous, the answer is the same as in (a), namely $5e^{-4}$.

(d) The first break occurs at the first time $\tau$ when $N(\tau) = 10$. After the operator comes back the whole process starts afresh. Therefore we only need to find $E\tau$. Notice that $\tau = \tau_1 + \tau_2 + ... + \tau_{10}$, where $\tau_k$ is the time between the $k - 1$st and the $k$th calls. These times have the same exponential(4) distribution. Hence,

$$E\tau = 10E\tau_1 = 10/4 \text{ (hours)}.$$
However you may like to add to this 1/4 (hours) the expected time the first call comes after she returns (she returned but was not working until the first after her return call came in).

2.20. The number of cars willing to take the professor is obtained by thinning the Poisson process of all cars with rate 6. Thus, it is a Poisson process with rate 2 per hour. The first jump of this process at time $\tau$, having expon(2) distribution should occur after the arrival time $\sigma$ of the bus, that is uniformly distributed on $(0, 1)$, in order for the professor end up riding the bus. Thus we need to find $P(\sigma < \tau)$.

The joint density of $(\sigma, \tau)$ is

$$p(s, t) = 2e^{-2t} \text{ for } s \in (0, 1), t > 0.$$ 

The answer is

$$\int_0^1 ds \int_s^\infty 2e^{-2t} dt = \int_0^1 e^{-2s} ds = (1/2)(1 - e^{-2}).$$

2.44. This is the same problem as in Example 2.9 and the answer is 300x25=7500.

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Grade distribution for Assignment 4: 36, 36, 36, 36, 36, 36, 36, 36, 36, 36, 36, 36, 36, 36, 36, 36, 36, 36, 33, 33, 33, 30, 30, 30, 28, 28, 27.


Assignment 5 due Apr 1:

3.2, 3.8, 3.17 (read ”average cost person” as ”the long-run cost per unit time of the tour for the college”),

3.2 This is an alternating renewal prosesses and by Theorem 3.4 the answer is

$$\frac{11}{11 + 3}.$$

3.8 This is not about queueing theory, there is no queue.

(a) The probability in question is obviously equal to the limiting fraction of time the counter is locked, again an alternating renewal prosesses and the answer

$$\frac{\tau}{\tau + 1/\lambda}.$$
(b) Let \( N(k) \) be the number or all particles which arrived before during the \( k \)th cycle. Then \( k/N_r(k) \) will be the fraction of the number of registered particles and we are asked to find

\[
\lim_{k \to \infty} \frac{k}{N(k)}.
\]

By the law of large numbers

\[
\lim_{k \to \infty} \frac{N(k)}{k}
\]

is the expectation of the number of arriving particles per cycle, which is 1 plus the expected number of particles that arrived during time \( \tau \), which is \( \lambda \tau \). Hence,

\[
\lim_{k \to \infty} \frac{k}{N(k)} = \frac{1}{1 + \lambda \tau}.
\]

Second solution. Treat the registered particle as a reward and use Theorem 3.3. Then the long-run time average of the number of registered is

\[
\frac{1}{\tau + 1/\lambda}.
\]

Then consider the number of rejected as a reward. The average number of rejected particles per cycle is \( EN_\tau \), where \( N_\tau \) is Poisson rate \( \lambda \). Then

\[
\text{the long-run time average of the number of rejected}
\]

\[
\frac{\lambda \tau}{\tau + 1/\lambda}.
\]

The result:

\[
\frac{1}{\tau + 1/\lambda} : \left( \frac{1}{\tau + 1/\lambda} + \frac{\lambda \tau}{\tau + 1/\lambda} \right) = \frac{1}{1 + \lambda \tau}.
\]

3.17. The average duration of the cycle is the average time when the \( k \)th person arrives = \( k\lambda = k \). The average cost of one tour is 20 plus 0.1 times (\( \tau \)'s are interarrival times)

\[
E((\tau_2 + \tau_3 + ... + \tau_k) + (\tau_3 + \tau_4 + ... + \tau_k) + ... + \tau_k)
\]

\[
= (k - 1) + (k - 2) + ... + 1 = (1/2)k(k - 1).
\]

In light of Theorem 3.3. we have to find the min of

\[
\frac{20}{k} + 0.05 \cdot (k - 1).
\]
Durrett is generous, this minimum is attained on an integer \( k = 20 \). Indeed, we know that \( a + a^{-1} \geq 2 \) and the equality is attained only if \( a = 1 \). Since

\[
\frac{20}{k} + 0.05 \cdot (k - 1) = \frac{20}{k} + \frac{k}{20} - 0.05.
\]

\( k = 20 \) provides the minimal cost which is now \$1.95.

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