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Dissipative decomposition of ordinary differential equations

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Synopsis

A general decomposition theorem that allows one to express uniquely arbitrary differential polynomials in one independent and one dependent variable as a combination of conservative, dissipative and higher order dissipative pieces is proved. The decomposition generalises the Rayleigh dissipation law for linear equations.

1. Introduction

In classical mechanics, a conservative system of differential equations is represented by the Euler–Lagrange equations of some variational principle. Not every differential equation can be represented as a conservative system; the Helmholtz conditions [**5**, Theorem 5.68] give necessary and sufficient conditions for a differential equation to be the Euler–Lagrange equation for some variational problem. For equations which are not Euler–Lagrange equations, one is left with the problem of seeing how "close" they are to Euler–Lagrange equations. A precise measure of "closeness" should include an algorithmic way of determining the "conservative part" of the equation, the remainder playing the role of dissipation or frictional forces. In this paper we propose a general decomposition for polynomial ordinary differential equations in one independent and one dependent variable into conservative, dissipative and higher order dissipative pieces. Subject to certain homogeneity requirements, the decomposition is unique; in particular it determines a unique conservative component of such an equation.

Consider the classical case of a linear ordinary differential equation $\Delta[u]=0$. Here x is the independent and $u=u(x)$ the dependent variable. The equation is an Euler–Lagrange equation if and only if the defining differential operator Δ is self-adjoint: $\Delta=\Delta^*$. A general linear ordinary differential equation can always be written *uniquely* in the form

$$\Delta_0[u]+\mathbf{D}\Delta_1[u]=0,$$

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where both Δ_0 and Δ_1 are self-adjoint differential operators, and $\mathbf{D}$ denotes the total derivative with respect to x. Consequently, there exist two quadratic variational problems $\mathcal{L}_0 = \int L_0[u]\,dx$ and $\mathcal{L}_1 = \int L_1[u]\,dx$, whose Euler–Lagrange expressions

$$\Delta_0[u] = Q_0 = \mathbf{E}[L_0], \quad \Delta_1[u] = Q_1 = \mathbf{E}[L_1],$$

form the two components of the equation. (Here $\mathbf{E}$ denotes the Euler operator or variational derivative.) Therefore the differential equation has the "dissipative decomposition"

$$Q_0 + \mathbf{D}Q_1 = 0, \quad \text{where } Q_0 \text{ and } Q_1 \in \operatorname{im} \mathbf{E}. \tag{1}$$

The Euler–Lagrange expressions Q_0 and Q_1 are *uniquely determined,* hence the corresponding Lagrangians L_0 and L_1 are uniquely determined up to a divergence. In physical problems, $\mathcal{L}_0$ can be identified with the Lagrangian for the conservative (i.e. self-adjoint) component of the problem, while $\mathcal{L}_1$ is closely related to the *Rayleigh dissipation,* and measures the rate of dissipation in the system, [**3**, p. 24]; see Section 7. Thus any linear ordinary differential equation can be uniquely decomposed into a conservative part, Q_0, and a dissipative part, Q_1.

The goal of this paper is to investigate to what extent the conservative/dissipative decomposition of a linear equation generalises to nonlinear ordinary differential equations. In general, the representation (1) is no longer valid; for instance, the simple equation

$$uu'' + 2u'^2 = 0$$

cannot be written in this form. However, for polynomial ordinary differential equations, there is a natural generalisation of this decomposition which incorporates "higher order dissipation" terms.

The fundamental theorem to be proved is the following decomposition theorem.

THEOREM 1. *Let P be a homogeneous differential polynomial of degree n. Then there exist unique differential polynomials Q_j, $0 \leqq j \leqq N$, with $Q_j = \mathbf{E}(L_j)$ for some differential polynomial L_j, such that P can be decomposed as*

$$P = \sum_{j=0}^{n} \mathbf{D}^j Q_j = \sum_{j=0}^{n} \mathbf{D}^j \mathbf{E}(L_j). \tag{2}$$

For example, in the case of a quadratic differential equation $P = 0$, there are three Lagrangians, L_0, L_1, L_2, each uniquely determined up to a divergence, and the equation can be written uniquely in the form

$$P = Q_0 + DQ_1 + D^2Q_2 = \mathbf{E}[L_0] + \mathbf{DE}[L_1] + \mathbf{D}^2\mathbf{E}[L_2] = 0,$$

which is the proper generalisation of the Rayleigh form (1) for a linear ordinary differential equation. For example, we find

$$uu'' + 2u'^2 = (-2uu'' - u'^2) + (3uu'' + 3u'^2) = \mathbf{E}[uu'^2] + \mathbf{D}^2\mathbf{E}[\tfrac{1}{2}u^3],$$

hence

$$Q_0 = -2uu'' - u'^2, \quad Q_1 = 0, \quad Q_2 = 3uu'' + 3u'^2,$$
$$L_0 = uu'^2, \quad L_1 = 0, \quad L_2 = \tfrac{1}{2}u^3.$$

We interpret $Q_0 = \mathbf{E}[L_0]$ as the *conservative* piece of the equation, with L_0 the associated Lagrangian, while $Q_1 = \mathbf{E}[L_1]$ and $Q_2 = \mathbf{E}[L_2]$ represent "first and second order dissipation", respectively. For this reason, we name (2) the *dissipative decomposition* of a differential polynomial P. See Section 7 for a further discussion of the role of the dissipative pieces. An important direction for further research is to relate this decomposition to physical examples of dissipation, both positive and negative, in nonlinear ordinary differential equations arising in applications.

Unfortunately, the method of proof of Theorem 1 is existential, relying on combinatorial formulae for partitions, and we do not yet have a closed formula for computing the functionals $\mathcal{L}_j = \int L_j[u]\,dx$ or Euler–Lagrange expressions $Q_j = \mathbf{E}[L_j]$ directly from the differential equation P. However, in Section 6 we will discuss a simple algorithm which will readily provide the decomposition. A table of representative dissipative decompositions can also be found in this section. Calculations were performed with the aid of the symbolic manipulation language SMP on an Apollo workstation at the University of Minnesota.

2. Some differential algebra

We begin by deriving some simple formulae from the formal calculus of variations of use later; the basic reference for these results is [**5**, Chaps 4 and 5]. We will work with a single dependent variable u and a single independent variable x. We shall use the notation

$$u_i = \frac{d^i u}{dx^i}, \quad i = 0, 1, 2, \ldots,$$

for derivatives of u throughout the paper. Let $\mathcal{A} = \mathcal{A}\{u, x\}$ denote the space of differential polynomials in the dependent variable u and the independent variable x. Thus, $\mathcal{A}$ consists of all polynomials in the variables u_i, and we shall allow arbitrary smooth (C^∞) functions of x as coefficients. We will also have occasion to use the differential subalgebra $\mathcal{A}_0$ consisting of all constant coefficient differential polynomials. By the *degree* of a differential polynomial we mean the degree in the u_i's; for example the differential monomial $x^2uu_x^3$ has degree 4. The (total) derivative operator on $\mathcal{A}$ is given by

$$\mathbf{D} = \frac{d}{dx} = \frac{\partial}{\partial x} + \sum_{i=0}^{\infty} u_{i+1}\partial_i,$$

where

$$\partial_i = \frac{\partial}{\partial u_i}, \quad i = 0, 1, 2, \ldots.$$

Note that only finitely many terms in the sum for $\mathbf{D}$ are needed when applying it to any specific differential polynomial.

LEMMA 2. *If $P \in \mathscr{A}$ is a differential polynomial, then $\mathbf{D}P = 0$ if and only if P is a constant. In particular, if P is a homogeneous differential polynomial of degree $n \geqq 1$, then $\mathbf{D}P = 0$ if and only if $P = 0$.*

Suppose
$$\mathscr{L}[u] = \int L(x, u, u_1, \ldots, u_n)\, dx$$
is a (polynomial) variational problem with Lagrangian $L \in \mathscr{A}$. The extremals of $\mathscr{L}$ satisfy the well-known Euler–Lagrange differential equation
$$\mathbf{E}(L) = 0,$$
where $\mathbf{E}$ is the Euler operator or variational derivative with respect to u:
$$\mathbf{E} = \mathbf{E}_u = \sum_{i=0}^{\infty} (-\mathbf{D})^i \partial_i.$$

THEOREM 3 [**5**, Theorem 4.7]. *Let $L \in \mathscr{A}$ be a differential polynomial. Then $\mathbf{E}(L) \equiv 0$ if and only if $L = \mathbf{D}P$ for some $P \in \mathscr{A}$.*

Therefore, two Lagrangians are *equivalent,* meaning they give rise to the same Euler–Lagrange equations, if and only if they differ by a total derivative:
$$\mathbf{E}(L) = \mathbf{E}(\tilde{L}) \quad \text{if and only if} \quad L = \tilde{L} + \mathbf{D}P. \tag{3}$$

The characterisation of null Lagrangians in Lemma 2 is the first stage in the "variational complex". At the next stage, one solves the so-called inverse problem of the calculus of variations which is to characterise all Euler–Lagrange equations. See [**5**, Section 5.4].

DEFINITION 4. Let $P \in \mathscr{A}$ be a differential polynomial. The *Fréchet derivative* of P is the differential operator
$$\mathbf{D}_P = \sum_{i=0}^{n} P_i \cdot \mathbf{D}^i, \quad \text{where} \quad P_i = \partial_i P.$$

THEOREM 5 [**5**, Theorem 5.68]. *A differential polynomial $P \in \mathscr{A}$ is the Euler-Lagrange expression for some Lagrangian $L \in \mathscr{A}$, i.e. $P = E(L)$, if and only if its Fréchet derivative is a self-adjoint differential operator:*
$$\mathbf{D}_P = \mathbf{D}_P^*. \tag{4}$$

The condition (4) is sometimes referred to as the *Helmholtz conditions.* Finally, we note a useful computational formula.

LEMMA 6. *If $L \in \mathscr{A}$, then*
$$\mathbf{E}(\mathbf{E}(L)) = \partial_0 \mathbf{E}(L) = \mathbf{E}(\partial_0 L). \tag{5}$$

Proof. Using Theorem 3, we see that
$$\mathbf{E}(\mathbf{E}(L)) = \mathbf{E}\Big(\sum_{i=1}^{\infty} (-\mathbf{D})^i \partial_i L\Big) = \mathbf{E}(\partial_0 L)$$
since by (3) all the other summands are annihilated by $\mathbf{E}$. Moreover, it is easily

seen that the partial derivative $\partial_0 = \partial/\partial u$ commutes with all the ∂_i's and the total derivative $\mathbf{D}$, hence it also commutes with the Euler operator $\mathbf{E}$, completing the proof of (5).

3. Uniqueness of the decomposition

The proof of the decomposition theorem (Theorem 1) will be divided into two stages. We first use the algebraic results from the previous section to prove that, provided such a decomposition exists, it *is* unique. In Section 4 we prove the existence of the decomposition for constant coefficient differential polynomials using combinatorial techniques. Section 5 completes the proof for variable coefficient differential polynomials. We begin with a simple version of the uniqueness result, which holds for general differential functions (i.e. general smooth functions of x, u and derivatives of u), not just polynomials, and has an interesting application to the theory of biHamiltonian systems.

PROPOSITION 7. *Suppose $k > 0$ and L, M, are differential polynomials, or, more generally, smooth differential functions. Let $P = \mathbf{E}(L)$, $Q = \mathbf{E}(M)$. Then*

$$P = \mathbf{D}^k Q \tag{6}$$

if and only if k is even and P and Q are affine functions of the form

$$Q = \sum_{i=0}^{n} c_i u_{2i} + f(x), \quad P = \sum_{i=0}^{n} c_i u_{2i+k} + f^{(k)}(x), \tag{7}$$

where each coefficient c_i is a constant, and f is an arbitrary smooth function of x.

Proof. We use the Helmholtz conditions (4) characterising Euler–Lagrange expressions. Taking the Fréchet derivative of both sides of (6), we find that

$$\mathbf{D}_P = \mathbf{D}^k \cdot \mathbf{D}_Q.$$

Since both P and Q are Euler–Lagrange expressions, (4) implies that

$$\mathbf{D}^k \cdot \mathbf{D}_Q = \mathbf{D}_P = \mathbf{D}_P^* = (-1)^k \mathbf{D}_Q^* \cdot \mathbf{D}^k = (-1)^k \mathbf{D}_Q \cdot \mathbf{D}^k,$$

hence Q must satisfy the equation

$$\mathbf{D}^k \cdot \mathbf{D}_Q = (-1)^k \mathbf{D}_Q \cdot \mathbf{D}^k. \tag{8}$$

Suppose Q depends on $u, u_1, \ldots, u_n$, with $Q_n = \partial_n Q \neq 0$. Expanding both sides of (8), we see that

$$\begin{aligned}\mathbf{D}^k \cdot \mathbf{D}_Q &= \sum_{i=0}^{n} \mathbf{D}^k \cdot Q_i \cdot \mathbf{D}^i = \sum_{i=0}^{n} \{Q_i \mathbf{D}^{i+k} + k\mathbf{D}Q_i \cdot \mathbf{D}^{i+k-1} + \ldots\} \\ &= Q_n \mathbf{D}^{n+k} + (Q_{n-1} + k\mathbf{D}Q_n) \cdot \mathbf{D}^{n+k-1} + \ldots,\end{aligned}$$

whereas

$$\mathbf{D}_Q \cdot \mathbf{D}^k = \sum_{i=0}^{n} Q_i \mathbf{D}^{i+k} = Q_n \mathbf{D}^{n+k} + Q_{n-1} \mathbf{D}^{n+k-1} + \ldots.$$

Comparing the coefficients of $\mathbf{D}^{n+k}$ on both sides of (8), we deduce that k must be even. Then the coefficient of $\mathbf{D}^{n+k-1}$ shows that

$$k \cdot \mathbf{D}Q_n = 0,$$

hence Q_n must be constant. Continuing to compare the coefficients of the lower order powers of $\mathbf{D}$, we find that all the derivatives $Q_i = \partial_i Q$ must be constant, hence Q must be of the form (7). This proves the proposition.

COROLLARY 8. *Suppose $j \neq m$ and $L, M \in \mathcal{A}$. Then*

$$\mathbf{D}^j\mathbf{E}(L) = \mathbf{D}^m\mathbf{E}(M)$$

if and only if $k = j - m$ is even and $P = \mathbf{E}(L)$, $Q = \mathbf{E}(M)$ are of the form given by (7).

This last result has a direct application to biHamiltonian systems, cf. [**5**, Section 7.3]:

COROLLARY 9. *An evolution equation $u_t = K[u]$ in one spatial variable is a biHamiltonian system with respect to two constant coefficient Hamiltonian operators $\mathbf{D}^j$ and $\mathbf{D}^m$, j, m odd, if and only if it takes the form* (7) *of a linear constant coefficient equation plus a potential.*

We can extend this result to more general constant coefficient Hamiltonian operators; we leave it to the reader to state the relevant theorem.

Proceeding to the general uniqueness result of Theorem 1, we need to prove the following:

PROPOSITION 10. *Let $L_0, \ldots, L_n$ be homogeneous differential polynomials of degree $n+1$. Suppose*

$$\sum_{j=0}^{n} \mathbf{D}^j\mathbf{E}(L_j) = 0. \tag{9}$$

Then

$$\mathbf{E}(L_j) = 0, \quad j = 0, \ldots, n,$$

and hence $L_j = \mathbf{D}P_j$ for some $P_j \in \mathcal{A}$, $0 \leqq j \leqq n$.

Proof. We work by induction on the degree n of the differential polynomials. The case $n = 1$ is the classical case of a linear differential polynomial, and follows directly from Proposition 7. Further, let u_k denote the highest order derivative of u which appears in the summands $\mathbf{D}^j\mathbf{E}(L_j)$ in (9), and for each n we do a further induction on k. The case $k = 0$ is completely trivial for all n since only the first term in the sum can depend on u alone.

To prove the inductive step, let $n \geqq 2$. Apply the Euler operator to (9), and use (3) and (5). We find

$$0 = \mathbf{E}\Big(\sum_{j=0}^{n} \mathbf{D}^j\mathbf{E}(L_j)\Big) = \mathbf{E}(\mathbf{E}(L_0)) = \partial_0\mathbf{E}(L_0),$$

hence $\mathbf{E}(L_0)$ cannot depend on u, although it can still depend on derivatives u_i for

$i \geqq 1$. Next we apply the operator ∂_0 to (9), and use its commutativity properties:

$$0 = \partial_0\left(\sum_{j=0}^{n} \mathbf{D}^j\mathbf{E}(L_j)\right) = \sum_{j=1}^{n} \mathbf{D}^j\mathbf{E}(\partial_0 L_j)$$
$$= \mathbf{D}\left(\sum_{j=0}^{n-1} \mathbf{D}^j\mathbf{E}(\hat{L}_j)\right),$$

where

$$\hat{L}_j = \partial_0 L_{j+1}, \quad j = 0, \ldots, n-1.$$

The last expression in parentheses is a homogeneous differential polynomial of degree $n-1 \geqq 1$, hence according to Lemma 2,

$$\sum_{j=0}^{n-1} \mathbf{D}^j\mathbf{E}(\hat{L}_j) = 0.$$

By the induction hypothesis (on n), this is possible if and only if

$$0 = \mathbf{E}(\hat{L}_j) = \partial_0\mathbf{E}(L_{j+1}), \quad j = 0, \ldots, n-1.$$

Therefore none of the Euler–Lagrange expressions $\mathbf{E}(L_j)$ can depend on u, i.e. they are all functions of $u_1, u_2, \ldots, u_k$.

We now use a result proved in [**4**, Lemma 2.15] characterising Euler–Lagrange expressions which only depend on derivatives of the dependent variable.

LEMMA 11. *Suppose $Q = \mathbf{E}_u(L)$ satisfies $\partial_0 Q = 0$. Then there is an equivalent Lagrangian $\tilde{L} + \mathbf{D}P$ such that $Q = \mathbf{E}_u(\tilde{L})$, and $\tilde{L}$ also satisfies $\partial_0\tilde{L} = 0$. Moreover, if we make the substitution $v = u_1 = du/dx$, (hence $v_j = u_{j+1}$), then both Q and $\tilde{L}$ become functions of v and its derivatives, and*

$$Q = \mathbf{D}(\mathbf{E}_v(\tilde{L})). \tag{10}$$

Applying this result to our situation, we see that we can replace the Lagrangians L_j by equivalent Lagrangians $\tilde{L}_j$, which are functions of $v = u_1$, $v_1 = u_2$, etc. Moreover, if we use (10), then (9) becomes

$$\sum_{j=0}^{n} \mathbf{D}^{j+1}\mathbf{E}_v(\tilde{L}_j) = \mathbf{D}\left(\sum_{j=0}^{n} \mathbf{D}^j\mathbf{E}_v(\tilde{L}_j)\right) = 0,$$

where we are now viewing everything as a function of v. Again, by Lemma 2, this implies that

$$\sum_{j=0}^{n} \mathbf{D}^j\mathbf{E}_v(\tilde{L}_j) = 0.$$

Furthermore, each summand $\mathbf{D}^j\mathbf{E}_v(\tilde{L}_j)$ depends on $v, v_1, \ldots, v_{k-1}$, and so we are back in the same situation as before, but with the order of the highest derivative reduced to $k-1$. Therefore, we can use our induction hypothesis on k to complete the proof.

4. Counting dimensions – restricted partitions

We now turn to the existence of the dissipative decomposition for differential polynomials. We begin by considering the simpler case of constant coefficient differential polynomials, and so work with the slightly smaller differential algebra $\mathcal{A}_0$. By keeping track of degrees of homogeneity and orders of derivatives, we can reduce the formula to a result on the direct sum decomposition of certain finite-dimensional subspaces of the full space of differential polynomials. We can then use combinatorial methods to count the dimensions of these subspaces, and thereby prove our result.

Let $\mathcal{S}$ denote the set of all multi-indices of the form $I = (i_0, i_1, \ldots, i_m)$, where $m \geqq 0$ is arbitrary (but finite), and where each integer entry is non-negative, $i_\nu \geqq 0$, and the last entry is strictly positive, $i_m > 0$. We let u^I denote the corresponding differential monomial

$$u^I = u^{i_0} \cdot u_1^{i_1} \cdot u_2^{i_2} \cdot \ldots \cdot u_m^{i_m},$$

so there is a one-to-one correspondence between differential monomials, and multi-indices $I \in \mathcal{S}$. Given such a multi-index $I = (i_0, i_1, \ldots, i_m)$, define

$$\sum I = i_0 + i_1 + \ldots + i_m, \quad |I| = i_1 + 2i_2 + \ldots + mi_m.$$

Note that $\sum I$ counts the degree of the monomial u^I, while $|I|$ counts the number of derivatives appearing in it. For $n, k > 0$, let

$$\mathcal{S}_k = \{I \in \mathcal{S} : |I| = k\}, \quad \mathcal{S}^n = \{I \in \mathcal{S} : \textstyle\sum I = n\}, \quad \mathcal{S}_k^n = \mathcal{S}^n \cap \mathcal{S}_k.$$

Note that $\mathcal{S}_k^n = \varnothing$ when $k > n$; we also set $\mathcal{S}_k = \mathcal{S}_k^n = \varnothing$ when $k \leqq 0$. Furthermore, let $\mathcal{A}_k$ (respectively $\mathcal{A}^n$, $\mathcal{A}_k^n$) be the subspace of $\mathcal{A}_0$ spanned by all the monomials u^I where $I \in \mathcal{S}_k$ (respectively $\mathcal{S}^n$, $\mathcal{S}_k^n$). (For $k \leqq 0$, we set $\mathcal{A}_k = \{0\}$.) Note that $\mathcal{A}^n$ is the space of all constant coefficient homogeneous differential polynomials of degree n, while $\mathcal{A}_k$ is the space of all constant coefficient differential polynomials in which exactly k derivatives of u appear in each monomial. The intersection $\mathcal{A}_k^n = \mathcal{A}^n \cap \mathcal{A}_k$ is easily seen to be a finite dimensional vector space over $\mathbb{R}$. (Note, if $\mathcal{S}_k^n = \varnothing$, then $\mathcal{A}_k^n = \{0\}$.) One of our principal objectives is to determine a formula for the dimension of this vector space, which we denote by

$$N_k^n = \dim \mathcal{A}_k^n = \operatorname{card} \mathcal{S}_k^n,$$

which is the same as the cardinality of the corresponding set of multi-indices $\mathcal{S}_k^n$. It is not difficult to identify this dimension with a standard combinatorial quantity.

Lemma 12. *The set $\mathcal{S}_k^n$ can be identified with the set of partitions of the integer k into at most n parts, so N_k^n is the number of these (restricted) partitions.*

See **[1]** for a survey of the theory of partitions, in which N_k^n, the number of partitions of k into at most n parts, is denoted by $p(\infty, n, k)$; cf. [**1**, Section 3.2].

We can now give a more precise statement of the main decomposition theorem for the space $\mathcal{A}^n$ of homogeneous constant coefficient differential polynomials of degree n. Rather than work with the entire infinite dimensional space $\mathcal{A}^n$ all at once, it is easier to work with the subspaces $\mathcal{A}_k^n$, so we need only prove the following finite dimensional version.

THEOREM 13. *Let P be a homogeneous differential polynomial in the space $\mathscr{A}_k^n$. Then there exist unique differential polynomials $Q_j \in \mathscr{A}_{k-j}^n$, $0 \leqq j \leqq n$, with $Q_j = \mathbf{E}(L_j)$ for some differential polynomial $L_j \in \mathscr{A}_{k-j}^{n+1}$ such that*

$$P = \sum_{j=0}^{n} \mathbf{D}^j Q_j = \sum_{j=0}^{n} \mathbf{D}^j \mathbf{E}(L_j).$$

Theorem 13 can be restated in terms of vector spaces as saying that the finite dimensional vector space $\mathscr{A}_k^n$ is the direct sum of the subspaces determined by the operators $\mathbf{D}^j \cdot \mathbf{E}$, $j = 0, \ldots, n$:

$$\mathscr{A}_k^n = \bigoplus_{j=0}^{n} \operatorname{im} \mathbf{D}^j \mathbf{E} \cap \mathscr{A}_k^n. \tag{11}$$

The uniqueness result of Proposition 10 shows that the right-hand side of (11) is indeed a direct sum, so to prove Theorem 13 we need only check that the dimensions of the subspaces on the right-hand side of (11) add up to the dimension of the full space. In other words, we need only prove

$$N_k^n = \dim \mathscr{A}_k^n = \sum_{j=0}^{n} \dim (\operatorname{im} \mathbf{D}^j \mathbf{E} \cap \mathscr{A}_k^n). \tag{12}$$

We begin with the following elementary observations:

LEMMA 14. *For all n, $k \geqq 1$, and $i \geqq 0$,*

(a) $\mathbf{D}[\mathscr{A}_k^n] \subset \mathscr{A}_{k+1}^n$,
(b) $\partial_i[\mathscr{A}_k^n] \subset \mathscr{A}_{k-i}^{n-1}$,
(c) $\mathbf{E}[\mathscr{A}_k^n] \subset \mathscr{A}_k^{n-1}$.

The easy proofs are omitted; in particular (c) follows from (a) and (b).

Combining Lemma 2 and Theorem 3 with this lemma, we see that for $n \geqq 1$, $k \geqq 0$, the sequence

$$0 \to \mathscr{A}_{k-1}^n \xrightarrow{\mathbf{D}} \mathscr{A}_k^n \xrightarrow{\mathbf{E}} \mathscr{A}_k^{n-1}$$

is exact, i.e. $\mathbf{D}$ is an injection, and $\operatorname{im} \mathbf{D} = \ker \mathbf{E}$. Therefore

$$\dim \mathbf{E}[\mathscr{A}_k^n] = \dim \mathscr{A}_k^n - \dim \mathbf{D}[\mathscr{A}_{k-1}^n] = \dim \mathscr{A}_k^n - \dim \mathscr{A}_{k-1}^n;$$

in other words, we have the important formula

$$\dim \mathbf{E}[\mathscr{A}_k^n] = N_k^n - N_{k-1}^n. \tag{13}$$

The subspaces appearing in the desired identity (11) are just

$$\operatorname{im} \mathbf{D}^j \mathbf{E} \cap \mathscr{A}_k^n = \mathbf{D}^j \mathbf{E}[\mathscr{A}_{k-j}^{n+1}],$$

and so have dimensions

$$\begin{aligned}\dim \{\operatorname{im} \mathbf{D}^j \mathbf{E} \cap \mathscr{A}_k^n\} &= \dim \{\mathbf{D}^j \mathbf{E}[\mathscr{A}_{k-j}^{n+1}]\}, \\ &= \dim \{\mathbf{E}[\mathscr{A}_{k-j}^{n+1}]\}, \\ &= N_{k-j}^{n+1} - N_{k-j-1}^{n+1}.\end{aligned}$$

Therefore, formula (12) reduces to

$$N_k^n = \sum_{j=0}^{n} (N_{k-j}^{n+1} - N_{k-j-1}^{n+1}) = N_k^{n+1} - N_{k-n-1}^{n+1},$$

since the summation collapses. Consequently, to prove Theorem 13, we need only verify the following combinatorial lemma.

LEMMA 15. *For all n, $k > 0$*

$$N_k^n = N_k^{n+1} - N_{k-n-1}^{n+1}. \tag{14}$$

Proof. This is an elementary result from the theory of partitions, cf. [**1**, formula (3.2.6)]. Define the injections

$$\iota_1\colon \mathscr{S}_k^n \to \mathscr{S}_k^{n+1}, \quad \iota_1(i_0, i_1, \ldots, i_m) = (i_0 + 1, i_1, \ldots, i_m),$$

$$\iota_2\colon \mathscr{S}_{k-n-1}^{n+1} \to \mathscr{S}_k^{n+1}, \quad \iota_2(j_0, j_1, \ldots, j_m) = (0, j_0, j_1, \ldots, j_m).$$

Then the image of ι_1 is $\{I \in \mathscr{S}_k^{n+1}: i_0 > 0\}$, whereas the image of ι_2 is $\{I \in \mathscr{S}_k^{n+1}: i_0 = 0\}$. Thus $\mathscr{S}_k^{n+1}$ is the disjoint union of these two images, which have respective cardinalities N_k^n and N_{k-n-1}^{n+1}. The lemma follows immediately.

For the reader's convenience, a short table of the dimensions N_k^n follows. For instance, the dimension of the space $\mathscr{A}_6^3$ is 7, and in this particular case, basis elements are provided by

$$u^2u_6,\ uu_1u_5,\ uu_2u_4,\ uu_3^2,\ u_1^2u_4,\ u_1u_2u_3,\ u_2^3,$$

corresponding to the multi-indices

$$(2, 0, 0, 0, 0, 0, 1),\ (1, 1, 0, 0, 0, 1),\ (1, 0, 1, 0, 1),$$
$$(1, 0, 0, 2),\ (0, 2, 0, 1),\ (0, 1, 1, 1),\ (0, 0, 3),$$

which make up $\mathscr{S}_6^3$.

TABLE 1
Dimensions N_k^n

$k\backslash n$	1	2	3	4	5	6
0	1	1	1	1	1	1
1	1	1	1	1	1	1
2	1	2	2	2	2	2
3	1	2	3	3	3	3
4	1	3	4	5	5	5
5	1	3	5	6	7	7
6	1	4	7	9	10	11
7	1	4	8	11	13	14
8	1	5	10	15	18	20
9	1	5	12	18	23	26
10	1	6	14	23	30	35
11	1	6	16	27	37	44
12	1	7	19	34	47	58

5. Proof of the decomposition theorem

We now generalise the considerations of Section 4 for the constant coefficient case to the full differential algebra $\mathscr{A}$, and thereby complete the proof of Theorem 1. According to Theorem 13, we know that if $M \in \mathscr{A}_k^n$ is any constant coefficient differential monomial of degree n, then there are uniquely determined (up to divergence) differential polynomials $L_j \in \mathscr{A}_{k-j}^{n+1}$, $j = 1, \ldots, n$, such that

$$M = \mathbf{E}[L_0] + \mathbf{DE}[L_1] + \ldots + \mathbf{D}^n\mathbf{E}[L_n]. \tag{15}$$

To prove the theorem for variable coefficient differential polynomials, then, it suffices to show that we can effect a similar decomposition for any differential monomial of the form $f(x)M$, where $f(x)$ is an arbitrary smooth function of x. We prove the result by induction on k, the number of derivatives of u appearing in M.

The case $k=0$ is trivial, since the only such monomial is $M=u^n$, and we have

$$f(x)u^n = \mathbf{E}\left(f(x)\cdot\frac{u^{n+1}}{n+1}\right).$$

Now, suppose we have proved the result for all differential monomials $M \in \mathscr{A}_j^n$, $0 \leqq j < k$, and let m be a monomial in $\mathscr{A}_k^n$, with dissipative decomposition (15). Consider the expression

$$K = \mathbf{E}[f(x)L_0] + \mathbf{DE}[f(x)L_1] + \ldots + \mathbf{D}^n\mathbf{E}[f(x)L_n].$$

When we expand the Euler operators and total derivatives, we find that K is a linear combination of derivatives of the function f of the form

$$K = \sum_{i=0}^{k} f^{(i)}(x)R_i, \tag{16}$$

where each constant coefficient differential polynomial R_i is a linear combination of the derivatives of the Lagrangians L_j, and lies in the space $\mathscr{A}_{k-i}^n$. Furthermore, the $i=0$ summand in (16) is the same as fM, i.e. $R_0 = M$, hence the difference

$$f(x)M - K = \sum_{i=1}^{k} f^{(i)}(x)R_i, \quad \text{where } R_i \in \mathscr{A}_{k-i}^n.$$

Now use the inductive hypothesis on k to write each term $f^{(i)}(x)R_i$ in the form

$$f^{(i)}(x)R_i = \sum_{j=0}^{n} \mathbf{D}^j\mathbf{E}(\tilde{L}_{ij}),$$

for some $\tilde{L}_{ij} \in \mathscr{A}$. Therefore

$$f(x)M = \sum_{j=0}^{n} \mathbf{D}^j\mathbf{E}(\hat{L}_j),$$

where

$$\hat{L}_j = f(x)L_j + \sum_{i=0}^{k} \tilde{L}_{ij}.$$

This completes the induction step, and hence the proof of Theorem 1.

6. An algorithm for finding the dissipative decomposition

According to the decomposition theorem, the number of the basis elements for $\mathscr{A}_k^n$ is equal to the number of basis elements for the direct sum

$$\left.\begin{array}{ll}\mathbf{E}(\mathscr{A}_k^{n+1}) \oplus \mathbf{DE}(\mathscr{A}_{k-1}^{n+1}) \oplus \mathbf{D}^2\mathbf{E}(\mathscr{A}_{k-2}^{n+1}) \oplus \ldots \oplus \mathbf{D}^n\mathbf{E}(\mathscr{A}_{k-n}^{n+1}), & k>n \\ \mathbf{E}(\mathscr{A}_k^{n+1}) \oplus \mathbf{DE}(\mathscr{A}_{k-1}^{n+1}) \oplus \mathbf{D}^2\mathbf{E}(\mathscr{A}_{k-2}^{n+1}) \oplus \ldots \oplus \mathbf{D}^k\mathbf{E}(\mathscr{A}_0^{n+1}), & k \leqq n.\end{array}\right\} \tag{17}$$

We can find a decomposition for all the basis elements of $\mathscr{A}_k^n$ simultaneously, by using the following algorithm. The first step is to find canonical basis elements for the spaces $\mathbf{E}(\mathscr{A}_k^{n+1})$, $\mathbf{E}(\mathscr{A}_{k-1}^{n+1})$, etc. This task is made easier by use of the following lemma.

LEMMA 16. *A basis for* $\mathbf{E}(\mathscr{A}_k^n)$ *is given by the differential polynomials* $\mathbf{E}(u^I)$, *where* $I=(i_0, i_1, \ldots, i_m)$ *ranges over all multi-indices in* $\mathscr{S}_k^n$ *such that* $i_m \geqq 2$.

In other words, to find a complete set of independent Euler–Lagrange expressions, we need only look at Lagrangians in which the highest order derivative in each monomial occurs at least quadratically. For example, a basis for $\mathbf{E}(\mathscr{A}_8^4) \subset \mathscr{A}_8^3$ is provided by the four differential polynomials

$$\begin{aligned}\mathbf{E}(u^2u_4^2) &= 2u^2u_8 + 16uu_1u_7 + 24uu_2u_6 + 24u_1^2u_6 + 16uu_3u_5 \\ &\quad + 48u_1u_2u_5 + 6uu_4^2 + 16u_1u_3u_4 + 12u_2^2u_4, \\ \mathbf{E}(u_1^2u_3^2) &= -2u_1^2u_6 - 12u_1u_2u_5 - 20u_1u_3u_4 - 12u_2^2u_4 - 14u_2u_3^2, \\ \mathbf{E}(uu_2u_3^2) &= -2uu_2u_6 - 6uu_3u_5 - 6u_1u_2u_5 - 4uu_4^2 - 14u_1u_3u_4 - 6u_2^2u_4 - 6u_2u_3^2, \\ \mathbf{E}(u_2^4) &= 12u_2^2u_4 + 24u_2u_3^2,\end{aligned}$$

corresponding to the multi-indices $(2,0,0,0,2)$, $(0,2,0,2)$, $(1,0,1,2)$, $(0,0,4)$. The proof of Lemma 16 is a straightforward integration by parts.

Note that this lemma concurs with formula (13). Indeed, if we define the injection

$$\iota: \mathscr{S}_{k-1}^n \to \mathscr{S}_k^n, \quad \iota(j_0, j_1, \ldots, j_m) = (j_0, j_1, \ldots, j_m - 1, 1),$$

then the image of ι is $\{I=(i_0, i_1, \ldots, i_m) \in \mathscr{S}_k^{n+1}: i_m = 1\}$, so $\mathscr{S}_k^n$ is the disjoint union of the image of ι and the subset of multi-indices indicated in the lemma. Indeed, in the above example, $\dim \mathbf{E}(\mathscr{A}_8^4) = 4$, while from our table, $N_8^4 = 15$, $N_7^4 = 11$.

Once we have determined canonical basis elements for the relevant subspaces $\mathbf{E}(\mathscr{A}_{k-j}^{n+1})$ appearing in the decomposition formula (20), it is then a simple matter to rewrite any constant coefficient differential polynomial P in $\mathscr{A}_k^n$ in terms of these basis elements. Let $r = N_k^n = \dim \mathscr{A}_k^n$, and let M_ν, $\nu = 1, \ldots, r$, denote the canonical basis of $\mathscr{A}_k^n$ given by the monomials u^I, $I \in \mathscr{S}_k^n$. Further, let P_μ, $\mu = 1, \ldots, r$, denote the basis elements formed from the decomposition, i.e. the differential polynomials $\mathbf{D}^j\mathbf{E}(u^K)$, where the u^K are the basis elements of $\mathscr{A}_{k-j}^{n+1}$ given by Lemma 16. By inspection, we then determine the coefficient matrix $C = (c_{\mu\nu})$ for the basis P_μ in terms of the monomial basis M_ν, writing

$$P_\mu = \sum_{\nu=1}^{r} c_{\mu\nu} M_\nu.$$

The inverse matrix $B = C^{-1}$ will then provide the dissipative decomposition of all the basis monomials in $\mathscr{A}_k^n$:

$$M_\nu = \sum_{\mu=1}^{r} b_{\nu\mu} P_\mu.$$

EXAMPLE 17. Find the Euler decomposition for the basis elements of $\mathscr{A}_2^2$:

According to (17), we have

$$\mathscr{A}_2^2 = \mathbf{E}(\mathscr{A}_2^3) \oplus \mathbf{D}\mathbf{E}(\mathscr{A}_1^3) \oplus \mathbf{D}^2\mathbf{E}(\mathscr{A}_0^3).$$

The monomial basis for $\mathscr{A}_2^2$ is given by

$$M_1 = uu_2 \quad \text{and} \quad M_2 = u_1^2,$$

corresponding to the multi-indices $(1, 0, 1)$ and $(0, 2)$ in $\mathscr{S}_2^2$.

Next, note that the middle subspace $\mathbf{E}(\mathscr{A}_1^3) = \{0\}$, since $N_2^3 = N_1^3 = 1$, cf. (13). Canonical basis elements for the other two subspaces are

$$\begin{aligned} &\mathbf{E}(\mathscr{A}_2^3): \quad P_1 = \mathbf{E}(uu_1^2) = -2uu_2 - u_1^2. \\ &\mathbf{D}^2\mathbf{E}(\mathscr{A}_0^3): \ P_2 = \mathbf{D}^2\mathbf{E}(u^3) = \mathbf{D}^2(3u^2) = 6uu_2 + 6u_1^2. \end{aligned}$$

Therefore the coefficient matrix is $C = \begin{pmatrix} -2 & -1 \\ 6 & 6 \end{pmatrix}$, with inverse

$$B = \begin{pmatrix} -2 & -1 \\ 6 & 6 \end{pmatrix}^{-1} = \begin{pmatrix} -1 & -\frac{1}{6} \\ 1 & \frac{1}{3} \end{pmatrix}.$$

We thus find the dissipative decomposition for the basis monomials of $\mathscr{A}_2^2$ to be

$$uu_2 = \mathbf{E}(-uu_1^2) + \mathbf{D}^2\mathbf{E}(-\tfrac{1}{6}u^3), \quad u_1^2 = \mathbf{E}(uu_1^2) + \mathbf{D}^2\mathbf{E}(\tfrac{1}{3}u^3).$$

A table of constant coefficient dissipative decompositions for basis monomials for $n = 2, 3, 4$, and $k \leqq 5$ appears at the end of this section. The computations were performed using the symbolic manipulation language SMP.

For variable coefficient differential polynomials, one needs to implement the algorithm described in the proof of Section 5.

EXAMPLE 18. Consider the differential polynomial

$$P = f(x)u_1^2,$$

where $f(x)$ is an arbitrary smooth function of x. Using the decomposition of u_1^2 given above, we find that P differs from the decomposition terms

$$\mathbf{E}(fuu_1^2) + \mathbf{D}^2\mathbf{E}(\tfrac{1}{3}fu^3)$$

by lower order terms:

$$P = \mathbf{E}(fuu_1^2) + \mathbf{D}^2\mathbf{E}(\tfrac{1}{3}fu^3) - 2f'uu_1 - f''u^2.$$

We then decompose these lower order terms, again using the table:

$$\begin{aligned} f''u^2 &= \mathbf{E}(\tfrac{1}{3}f''u^3), \\ f'uu_1 &= \mathbf{DE}(\tfrac{1}{6}f'u^3) - \tfrac{1}{2}f''u^2. \end{aligned}$$

We conclude that this monomial has the decomposition

$$f(x)u_1^2 = \mathbf{E}(fuu_1^2) + \mathbf{DE}(-\tfrac{1}{3}f'u^3) + \mathbf{D}^2\mathbf{E}(\tfrac{1}{3}fu^3).$$

A similar computation shows that

$$f(x)uu_2 = \mathbf{E}(-fuu_1^2 + \tfrac{1}{6}f''u^3) + \mathbf{D}^2\mathbf{E}(-\tfrac{1}{6}fu^3).$$

TABLE 2
Dissipative decompositions: $n = 2$

k	
0	$u^2 = u^2$
	$= \mathbf{E}(\frac{1}{3}u^3)$
1	$uu_1 = uu_1$
	$= \mathbf{DE}(\frac{1}{6}u^3)$
2	$uu_2 = (2uu_2 + u_1^2) + (-uu_2 - u_1^2)$
	$= \mathbf{E}(-uu_1^2) + \mathbf{D}^2\mathbf{E}(-\frac{1}{6}u^3)$
	$u_1^2 = (-2uu_2 - u_1^2) + (2uu_2 + 2u_1^2)$
	$= \mathbf{E}(uu_1^2) + \mathbf{D}^2\mathbf{E}(\frac{1}{3}u^3)$
3	$uu_3 = (-2u_1u_2) + (uu_3 + 2u_1u_2)$
	$= \mathbf{E}(\frac{1}{3}u_1^3) + \mathbf{DE}(-\frac{1}{2}uu_1^2)$
	$u_1u_2 = u_1u_2$
	$= \mathbf{E}(-\frac{1}{6}u_1^3)$
4	$uu_4 = (2uu_4 + 4u_1u_3 + 3u_2^2) + (-u_1u_3 - u_2^2) + (-uu_4 - 3u_1u_3 - 2u_2^2)$
	$= \mathbf{E}(uu_2^2) + \mathbf{DE}(\frac{1}{6}u_1^3) + \mathbf{D}^2\mathbf{E}(\frac{1}{2}uu_1^2)$
	$u_1u_3 = (-2uu_4 - 4u_1u_3 - 3u_2^2) + (-u_1u_3 - u_2^2) + (2uu_4 + 6u_1u_3 + 4u_2^2)$
	$= \mathbf{E}(-uu_2^2) + \mathbf{DE}(\frac{1}{6}u_1^3) + \mathbf{D}^2\mathbf{E}(-uu_1^2)$
	$u_2^2 = (2uu_4 + 4u_1u_3 + 3u_2^2) + (2u_1u_3 + 2u_2^2) + (-2uu_4 - 6u_1u_3 - 4u_2^2)$
	$= \mathbf{E}(uu_2^2) + \mathbf{DE}(-\frac{1}{3}u_1^3) + \mathbf{D}^2\mathbf{E}(uu_1^2)$
5	$uu_5 = (-4u_1u_4 - 8u_2u_3) + (uu_5 + 3u_1u_4 + 5u_2u_3) + (u_1u_4 + 3u_2u_3)$
	$= \mathbf{E}(-2u_1u_2^2) + \mathbf{DE}(\frac{1}{2}uu_2^2) + \mathbf{D}^2\mathbf{E}(-\frac{1}{6}u_1^3)$
	$u_1u_4 = (3u_1u_4 + 6u_2u_3) + (-2u_1u_4 - 6u_2u_3)$
	$= \mathbf{E}(\frac{3}{2}u_1u_2^2) + \mathbf{D}^2\mathbf{E}(\frac{1}{3}u_1^3)$
	$u_2u_3 = (-u_1u_4 - 2u_2u_3) + (u_1u_4 + 3u_2u_3)$
	$= \mathbf{E}(-\frac{1}{2}u_1u_2^2) + \mathbf{D}^2\mathbf{E}(-\frac{1}{6}u_1^3)$
6	$uu_6 = (2uu_6 + 6u_1u_5 + 14u_2u_4 + 9u_3^2) + (-2u_1u_5 - 6u_2u_4 - 4u_3^2)$
	$+ (-uu_6 - 4u_1u_5 - 8u_2u_4 - 5u_3^2)$
	$= \mathbf{E}(-uu_3^2 + \frac{4}{3}u_2^3) + \mathbf{DE}(-u_1u_2^2) + \mathbf{D}^2\mathbf{E}(-\frac{1}{2}uu_2^2)$
	$u_1u_5 = (-2uu_6 - 6u_1u_5 - 13u_2u_4 - 8u_3^2) + (-u_1u_5 - 3u_2u_4 - 2u_3^2)$
	$+ (2uu_6 + 8u_1u_5 + 16u_2u_4 + 10u_3^2)$
	$= \mathbf{E}(uu_3^2 - \frac{7}{6}u_2^3) + \mathbf{DE}(-\frac{1}{2}u_1u_2^2) + \mathbf{D}^2\mathbf{E}(uu_2^2)$
	$u_2u_4 = (2uu_6 + 6u_1u_5 + 11u_2u_4 + 6u_3^2) + (2u_1u_5 + 6u_2u_4 + 4u_3^2)$
	$+ (-2uu_6 - 8u_1u_5 - 16u_2u_4 - 10u_3^2)$
	$= \mathbf{E}(-uu_3^2 + \frac{5}{6}u_2^3) + \mathbf{DE}(u_1u_2^2) + \mathbf{D}^2\mathbf{E}(-uu_2^2)$
	$u_3^2 = (-2uu_6 - 6u_1u_5 - 10u_2u_4 - 5u_3^2) + (-2u_1u_5 - 6u_2u_4 - 4u_3^2)$
	$+ (2uu_6 + 8u_1u_5 + 16u_2u_4 + 10u_3^2)$
	$= \mathbf{E}(uu_3^2 - \frac{2}{3}u_2^3) + \mathbf{DE}(-u_1u_2^2) + \mathbf{D}^2\mathbf{E}(uu_2^2)$

[redacted]

TABLE 3
Dissipative decompositions: $n=3$

k

0
$$\begin{aligned} u^3 &= u^3 \\ &= \mathbf{E}(\tfrac{1}{4}u^4) \end{aligned}$$

1
$$\begin{aligned} u^2u_1 &= u^2u_1 \\ &= \mathbf{DE}(\tfrac{1}{12}u^4) \end{aligned}$$

2
$$\begin{aligned} u^2u_2 &= (2u^2u_2+2uu_1^2)+(-u^2u_2-2uu_1^2) \\ &= \mathbf{E}(-u^2u_1^2)+\mathbf{D}^2\mathbf{E}(-\tfrac{1}{12}u^4) \\ uu_1^2 &= (-u^2u_2-uu_1^2)+(u^2u_2+2uu_1^2) \\ &= \mathbf{E}(\tfrac{1}{2}u^2u_1^2)+\mathbf{D}^2\mathbf{E}(\tfrac{1}{12}u^4) \end{aligned}$$

3
$$\begin{aligned} u^2u_3 &= (-6uu_1u_2-2u_1^3)+(u^2u_3+6uu_1u_2+2u_1^3) \\ &= \mathbf{E}(uu_1^3)+\mathbf{D}^3\mathbf{E}(\tfrac{1}{12}u^4) \\ uu_1u_2 &= (3uu_1u_2+u_1^3)+(u^2u_3+4uu_1u_2+u_1^3)+(-u^2u_3-6uu_1u_2-2u_1^3) \\ &= \mathbf{E}(-\tfrac{1}{2}uu_1^3)+\mathbf{DE}(-\tfrac{1}{2}u^2u_1^2)+\mathbf{D}^3\mathbf{E}(-\tfrac{1}{12}u^4) \\ u_1^3 &= (-6uu_1u_2-2u_1^3)+(-3u^2u_3-12uu_1u_2-3u_1^3)+(3u^2u_3+18uu_1u_2+6u_1^3) \\ &= \mathbf{E}(uu_1^3)+\mathbf{DE}(\tfrac{3}{2}u^2u_1^2)+\mathbf{D}^3\mathbf{E}(\tfrac{1}{4}u^4) \end{aligned}$$

4
$$\begin{aligned} u^2u_4 &= (2u^2u_4+8uu_1u_3+6uu_2^2+11u_1^2u_2)+(-2uu_1u_3-2uu_2^2-4u_1^2u_2) \\ &\quad +(-u^2u_4-6uu_1u_3-4uu_2^2-7u_1^2u_2) \\ &= \mathbf{E}(u^2u_2^2-\tfrac{7}{12}u_1^4)+\mathbf{DE}(\tfrac{1}{3}uu_1^3)+\mathbf{D}^2\mathbf{E}(\tfrac{1}{2}u^2u_1^2) \\ uu_1u_3 &= (-u^2u_4-4uu_1u_3-3uu_2^2-5u_1^2u_2)+(-uu_1u_3-uu_2^2-2u_1^2u_2) \\ &\quad +(u^2u_4+6uu_1u_3+4uu_2^2+7u_1^2u_2) \\ &= \mathbf{E}(-\tfrac{1}{2}u^2u_2^2+\tfrac{1}{4}u_1^4)+\mathbf{DE}(\tfrac{1}{6}uu_1^3)+\mathbf{D}^2\mathbf{E}(-\tfrac{1}{2}u^2u_1^2) \\ uu_2^2 &= (u^2u_4+4uu_1u_3+3uu_2^2+3u_1^2u_2)+(2uu_1u_3+2uu_2^2+4u_1^2u_2) \\ &\quad +(-u^2u_4-6uu_1u_3-4uu_2^2-7u_1^2u_2) \\ &= \mathbf{E}(\tfrac{1}{2}u^2u_2^2-\tfrac{1}{12}u_1^4)+\mathbf{DE}(-\tfrac{1}{3}uu_1^3)+\mathbf{D}^2\mathbf{E}(\tfrac{1}{2}u^2u_1^2) \\ u_1^2u_2 &= u_1^2u_2 \\ &= \mathbf{E}(-\tfrac{1}{12}u_1^4) \end{aligned}$$

5
$$\begin{aligned} u^2u_5 &= (-10uu_1u_4-20uu_2u_3-20u_1^2u_3-30u_1u_2^2)+(u_1^2u_3+2u_1u_2^2) \\ &\quad +(2uu_1u_4+6uu_2u_3+6u_1^2u_3+10u_1u_2^2) \\ &\quad +(u^2u_5+8uu_1u_4+14uu_2u_3+13u_1^2u_3+18u_1u_2^2) \\ &= \mathbf{E}(-5uu_1u_2^2)+\mathbf{DE}(-\tfrac{1}{12}u_1^4)+\mathbf{D}^2\mathbf{E}(-\tfrac{1}{3}uu_1^3)+\mathbf{D}^3\mathbf{E}(-\tfrac{1}{2}u^2u_1^2) \\ uu_1u_4 &= (5uu_1u_4+10uu_2u_3+10u_1^2u_3+15u_1u_2^2) \\ &\quad +(u^2u_5+6uu_1u_4+10uu_2u_3+9u_1^2u_3+13u_1u_2^2) \\ &\quad +(-2uu_1u_4-6uu_2u_3-6u_1^2u_3-10u_1u_2^2) \\ &\quad +(-u^2u_5-8uu_1u_4-14uu_2u_3-13u_1^2u_3-18u_1u_2^2) \\ &= \mathbf{E}(\tfrac{5}{2}uu_1u_2^2)+\mathbf{DE}(\tfrac{1}{2}u^2u_2^2-\tfrac{1}{4}u_1^4)+\mathbf{D}^2\mathbf{E}(\tfrac{1}{3}uu_1^3)+\mathbf{D}^3\mathbf{E}(\tfrac{1}{2}u^2u_1^2) \\ uu_2u_3 &= (-uu_1u_4-2uu_2u_3-2u_1^2u_3-3u_1u_2^2)+(-u_1^2u_3-2u_1u_2^2) \\ &\quad +(uu_1u_4+3uu_2u_3+3u_1^2u_3+5u_1u_2^2) \\ &= \mathbf{E}(-\tfrac{1}{2}uu_1u_2^2)+\mathbf{DE}(\tfrac{1}{12}u_1^4)+\mathbf{D}^2\mathbf{E}(-\tfrac{1}{6}uu_1^3) \\ u_1^2u_3 &= (-4uu_1u_4-8uu_2u_3-8u_1^2u_3-12u_1u_2^2) \\ &\quad +(-2u^2u_5-12uu_1u_4-20uu_2u_3-17u_1^2u_3-24u_1u_2^2) \\ &\quad +(2u^2u_5+16uu_1u_4+28uu_2u_3+26u_1^2u_3+36u_1u_2^2) \\ &= \mathbf{E}(-2uu_1u_2^2)+\mathbf{DE}(-u^2u_2^2+\tfrac{5}{12}u_1^4)+\mathbf{D}^3\mathbf{E}(-u^2u_1^2) \\ u_1u_2^2 &= (2uu_1u_4+4uu_2u_3+4u_1^2u_3+6u_1u_2^2) \\ &\quad +(u^2u_5+6uu_1u_4+10uu_2u_3+9u_1^2u_3+13u_1u_2^2) \\ &\quad +(-u^2u_5-8uu_1u_4-14uu_2u_3-13u_1^2u_3-18u_1u_2^2) \\ &= \mathbf{E}(uu_1u_2^2)+\mathbf{DE}(\tfrac{1}{2}u^2u_2^2-\tfrac{1}{4}u_1^4)+\mathbf{D}^3\mathbf{E}(\tfrac{1}{2}u^2u_1^2) \end{aligned}$$

[redacted]

TABLE 4
Dissipative decompositions: $n = 4$

k	
0	$u^4 = u^4$
	$= \mathbf{E}(\frac{1}{5}u^5)$
1	$u^3u_1 = u^3u_1$
	$= \mathbf{DE}(\frac{1}{20}u^5)$
2	$u^3u_2 = (2u^3u_2 + 3u^2u_1^2) + (-u^3u_2 - 3u^2u_1^2)$
	$= \mathbf{E}(-u^3u_1^2) + \mathbf{D}^2\mathbf{E}(-\frac{1}{20}u^5)$
	$u^2u_1^2 = (-\frac{2}{3}u^3u_2 - u^2u_1^2) + (\frac{2}{3}u^3u_2 + 2u^2u_1^2)$
	$= \mathbf{E}(\frac{1}{3}u^3u_1^2) + \mathbf{D}^2\mathbf{E}(\frac{1}{30}u^5)$
3	$u^3u_3 = (-9u^2u_1u_2 - 6uu_1^3) + (u^3u_3 + 9u^2u_1u_2 + 6uu_1^3)$
	$= \mathbf{E}(\frac{3}{2}u^2u_1^3) + \mathbf{D}^3\mathbf{E}(\frac{1}{20}u^5)$
	$u^2u_1u_2 = (3u^2u_1u_2 + 2uu_1^3) + (\frac{2}{3}u^3u_3 + 4u^2u_1u_2 + 2uu_1^3) + (-\frac{2}{3}u^3u_3 - 6u^2u_1u_2 - 4uu_1^3)$
	$= \mathbf{E}(-\frac{1}{2}u^2u_1^3) + \mathbf{DE}(-\frac{1}{3}u^3u_1^2) + \mathbf{D}^3\mathbf{E}(-\frac{1}{30}u^5)$
	$uu_1^3 = (-3u^2u_1u_2 - 2uu_1^3) + (-u^3u_3 - 6u^2u_1u_2 - 3uu_1^3) + (u^3u_3 + 9u^2u_1u_2 + 6uu_1^3)$
	$= \mathbf{E}(\frac{1}{2}u^2u_1^3) + \mathbf{DE}(\frac{1}{2}u^3u_1^2) + \mathbf{D}^3\mathbf{E}(\frac{1}{20}u^5)$
4	$u^3u_4 = (2u^3u_4 + 12u^2u_1u_3 + 9u^2u_2^2 + 36uu_1^2u_2 + 6u_1^4)$
	$+ (-u^3u_4 - 12u^2u_1u_3 - 9u^2u_2^2 - 36uu_1^2u_2 - 6u_1^4)$
	$= \mathbf{E}(u^3u_2^2 - 2uu_1^4) + \mathbf{D}^4\mathbf{E}(-\frac{1}{20}u^5)$
	$u^2u_1u_3 = (-\frac{2}{3}u^3u_4 - 4u^2u_1u_3 - 3u^2u_2^2 - 12uu_1^2u_2 - 2u_1^4) + (-3u^2u_1u_3 - 3u^2u_2^2 -$
	$- 12uu_1^2u_2 - 2u_1^4) + (\frac{2}{3}u^3u_4 + 8u^2u_1u_3 + 6u^2u_2^2 + 24uu_1^2u_2 + 4u_1^4)$
	$= \mathbf{E}(-\frac{1}{3}u^3u_2^2 + \frac{2}{3}uu_1^4) + \mathbf{DE}(\frac{1}{2}u^2u_1^3) + \mathbf{D}^4\mathbf{E}(\frac{1}{30}u^5)$
	$u^2u_2^2 = (\frac{2}{3}u^3u_4 + 4u^2u_1u_3 + 3u^2u_2^2 + 4uu_1^2u_2) + (-\frac{4}{3}u^3u_4 - 12u^2u_1u_3 - 8u^2u_2^2 -$
	$- 28uu_1^2u_2 - 4u_1^4) + (\frac{2}{3}u^3u_4 + 8u^2u_1u_3 + 6u^2u_2^2 + 24uu_1^2u_2 + 4u_1^4)$
	$= \mathbf{E}(\frac{1}{3}u^2u_2^2) + \mathbf{D}^2\mathbf{E}(\frac{2}{3}u^3u_1^2) + \mathbf{D}^4\mathbf{E}(\frac{1}{30}u^5)$
	$uu_1^2u_2 = (4uu_1^2u_2 + u_1^4) + (3u^2u_1u_3 + 3u^2u_2^2 + 12uu_1^2u_2 + 2u_1^4) + (u^3u_4 + 9u^2u_1u_3 +$
	$+ 6u^2u_2^2 + 21uu_1^2u_2 + 3u_1^4) + (-u^3u_4 - 12u^2u_1u_3 - 9u^2u_2^2 - 36uu_1^2u_2 - 6u_1^4)$
	$= \mathbf{E}(-\frac{1}{3}uu_1^4) + \mathbf{DE}(-\frac{1}{2}u^2u_1^3) + \mathbf{D}^2\mathbf{E}(-\frac{1}{2}u^3u_1^2) + \mathbf{D}^4\mathbf{E}(-\frac{1}{20}u^5)$
	$u_1^4 = (-12uu_1^2u_2 - 3u_1^4) + (-12u^2u_1u_3 - 12u^2u_2^2 - 48uu_1^2u_2 - 8u_1^4)$
	$+ (-4u^3u_4 - 36u^2u_1u_3 - 24u^2u_2^2 - 84uu_1^2u_2 - 12u_1^4)$
	$+ (4u^2u_4 + 48u^2u_1u_3 + 36u^2u_2^2 + 144uu_1^2u_2 + 24u_1^4)$
	$= \mathbf{E}(uu_1^4) + \mathbf{DE}(2u^2u_1^3) + \mathbf{D}^2\mathbf{E}(2u^3u_1^2) + \mathbf{D}^4\mathbf{E}(\frac{1}{5}u^5)$
5	$u^3u_5 = (-15u^2u_1u_4 - 30u^2u_2u_3 - 60uu_1^2u_3 - 90uu_1u_2^2 - 59u_1^3u_2)$
	$+ (3uu_1^2u_3 + 6uu_1u_2^2 + 6u_1^3u_2)$
	$+ (3u^2u_1u_4 + 9u^2u_2u_3 + 18uu_1^2u_3 + 30uu_1u_2^2 + 20u_1^3u_2)$
	$+ (u^3u_5 + 12u^2u_1u_4 + 21u^2u_2u_3 + 39uu_1^2u_3 + 54uu_1u_2^2 + 33u_1^3u_2)$
	$= \mathbf{E}(-\frac{15}{2}u^2u_1u_2^2 + \frac{29}{20}u_1^4) + \mathbf{DE}(-\frac{1}{4}uu_1^4) + \mathbf{D}^2\mathbf{E}(-\frac{1}{2}u^2u_1^3) + \mathbf{D}^3\mathbf{E}(-\frac{1}{2}u^3u_1^2)$
	$u^2u_1u_4 = (5u^2u_1u_4 + 10u^2u_2u_3 + 20uu_1^2u_3 + 30uu_1u_2^2 + \frac{58}{3}u_1^3u_2)$
	$+ (\frac{2}{3}u^3u_5 + 6u^2u_1u_4 + 10u^2u_2u_3 + 18uu_1^2u_3 + 26uu_1u_2^2 + 16u_1^3u_2)$
	$+ (-2u^2u_1u_4 - 6u^2u_2u_3 - 12uu_1^2u_3 - 20uu_1u_2^2 - \frac{40}{3}u_1^3u_2)$
	$+ (-\frac{2}{3}u^3u_5 - 8u^2u_1u_4 - 14u^2u_2u_3 - 26uu_1^2u_3 - 36uu_1u_2^2 - 22u_1^3u_2)$
	$= \mathbf{E}(\frac{5}{2}u^2u_1u_2^2 - \frac{7}{5}u_1^4) + \mathbf{DE}(\frac{1}{3}u^3u_2^2 - \frac{1}{2}uu_1^4) + \mathbf{D}^2\mathbf{E}(\frac{1}{3}u^2u_1^3) + \mathbf{D}^3\mathbf{E}(\frac{1}{3}u^3u_1^2)$
	$u^2u_2u_3 = (-u^2u_1u_4 - 2u^2u_2u_3 - 4uu_1^2u_3 - 6uu_1u_2^2 - \frac{8}{3}u_1^3u_2)$
	$+ (-2uu_1^2u_3 - 4uu_1u_2^2 - 4u_1^3u_2)$

$$
\begin{aligned}
&\quad + (u^2u_1u_4 + 3u^2u_2u_3 + 6uu_1^2u_3 + 10uu_1u_2^2 + \tfrac{20}{3}u_1^3u_2) \\
&= \mathbf{E}(\tfrac{1}{30}u_1^4 - \tfrac{1}{2}u^2u_1u_2^2) + \mathbf{DE}(\tfrac{1}{6}uu_1^4) + \mathbf{D}^2\mathbf{E}(-\tfrac{1}{6}u^2u_1^3) \\
uu_1^2u_3 &= (-2u^2u_1u_4 - 4u^2u_2u_3 - 8uu_1^2u_3 - 12uu_1u_2^2 - 8u_1^3u_2) \\
&\quad + (-\tfrac{2}{3}u^3u_5 - 6u^2u_1u_4 - 10u^2u_2u_3 - 17uu_1^2u_3 - 24uu_1u_2^2 - 14u_1^3u_2) \\
&\quad + (\tfrac{2}{3}u^3u_5 + 8u^2u_1u_4 + 14u^2u_2u_3 + 26uu_1^2u_3 + 36uu_1u_2^2 + 22u_1^3u_2) \\
&= \mathbf{E}(\tfrac{1}{5}u_1^4 - u^2u_1u_2^2) + \mathbf{DE}(\tfrac{5}{12}uu_1^4 - \tfrac{1}{3}u^3u_2^2) + \mathbf{D}^3\mathbf{E}(-\tfrac{1}{3}u^3u_1^2) \\
uu_1u_2^2 &= (u^2u_1u_4 + 2u^2u_2u_3 + 4uu_1^2u_3 + 6uu_1u_2^2 + 3u_1^3u_2) \\
&\quad + (\tfrac{1}{3}u^3u_5 + 3u^2u_1u_4 + 5u^2u_2u_3 + 9uu_1^2u_3 + 13uu_1u_2^2 + 8u_1^3u_2) \\
&\quad + (-\tfrac{1}{3}u^3u_5 - 4u^2u_1u_4 - 7u^2u_2u_3 - 13uu_1^2u_3 - 18uu_1u_2^2 - 11u_1^3u_2) \\
&= \mathbf{E}(-\tfrac{1}{20}u_1^4 + \tfrac{1}{2}u^2u_1u_2^2) + \mathbf{DE}(-\tfrac{1}{4}uu_1^4 + \tfrac{1}{6}u^3u_2^2) + \mathbf{D}^3\mathbf{E}(\tfrac{1}{6}u^3u_1^2) \\
u_1^3u_2 &= u_1^3u_2 \\
&= \mathbf{E}(-\tfrac{1}{20}u_1^4)
\end{aligned}
$$

7. Dissipation laws

Let us begin by reviewing the Rayleigh dissipation function for the simplest classical mechanical system governed by a single linear, constant coefficient, second order ordinary differential equation

$$P = au_{xx} + bu_x + cu = 0.$$

(To maintain the notation used in the rest of the paper, we use x as the independent variable, although in mechanics this is really the time t.) The second and zeroth order terms form the conservative part of the system, and are derived from the Lagrangian

$$L_0 = -\tfrac{1}{2}au_x^2 + \tfrac{1}{2}cu^2.$$

The energy of the system is given by the corresponding Hamiltonian

$$H_0 = \tfrac{1}{2}au_x^2 + \tfrac{1}{2}cu^2;$$

indeed to obtain the energy law, we multiply the equation by u_x, leading to

$$u_xP = \mathbf{D}H_0 + R = 0,$$

where

$$R = \tfrac{1}{2}bu_x^2$$

is the velocity-dependent Rayleigh dissipation function. In particular, if $b > 0$, the above identity shows that the energy H_0 is a decreasing function of x. Moreover, the equation itself has the Rayleigh form

$$\mathbf{E}_u(L_0) + \mathbf{E}_{u_x}(R) = 0,$$

where $\mathbf{E}_{u_x}$ denotes the variational derivative of R with respect to the velocity variable u_x. However, if we use Lemma 11 (or by direct computation), we see that if we replace u_x by u, leading to the Lagrangian

$$L_1 = \tfrac{1}{2}bu^2,$$

then

$$\mathbf{E}_{u_x}(R) = \mathbf{D}\mathbf{E}_u(L_1), \tag{18}$$

and so the equation does have the dissipative decomposition

$$\mathbf{E}_u(L_0) + \mathbf{D}\mathbf{E}_u(L_1).$$

The identity (18) is the essential relation between the classical Rayleigh dissipation function and the present dissipative decomposition for linear ordinary differential equations.

The goal of this section is to obtain an analogue of the Rayleigh law for a general dissipative decomposition (2). To maintain the analogy with classical computations, we seek an identity of the form

$$u_1 P = \mathbf{D}H + R, \tag{19}$$

where H will play the role of the Hamiltonian or energy of the nonlinear system, and R the role of the dissipation. Ideally, the dissipation function R should be positive definite, so that the energy H will be a nonincreasing function of u. The basic approach is to integrate the terms on the left-hand side of (19) by parts, until we have a quadratic or higher degree polynomial in the highest order derivatives of u which appear in the dissipation term R. The computations become slightly complicated, and we carry them through only for second and third order dissipation. The question of when the dissipation function is positive definite will be left to a subsequent publication.

Define the evolutionary vector fields

$$\mathbf{v}_n = \mathbf{v}_{u_n} = \sum_{i=0}^{\infty} u_{n+i}\partial_i. \tag{20}$$

In particular, $\mathbf{v}_0$ multiplies a differential polynomial by its degree, while $\mathbf{v}_1$ agrees with the total derivative $\mathbf{D}$ when applied to constant coefficient differential polynomials.

LEMMA 19. *Let $L \in \mathcal{A}$, and let $n \geqq 0$. Then*

$$u_n\mathbf{E}(L) = \mathbf{v}_n(L) + \mathbf{D}P$$

for some $P \in \mathcal{A}$.

Proof. The proof is an elementary integration by parts:

$$u_n\mathbf{E}(L) = \sum_{i=0}^{\infty} u_n(-\mathbf{D})^i\partial_i L = \sum_{i=0}^{\infty} (\mathbf{D}^i u_n)\partial_i L + \mathbf{D}P.$$
$$= \sum_{i=0}^{\infty} u_{n+i}\partial_i L + \mathbf{D}P = \mathbf{v}_n(L) + \mathbf{D}P.$$

COROLLARY 20. [**2**]. *If $L \in \mathcal{A}_0$ is an x-independent Lagrangian, then*

$$u_1\mathbf{E}(L) = \mathbf{D}H \tag{21}$$

for some $H \in \mathcal{A}_0$, called the Hamiltonian *associated with L.*

The terminology comes from classical mechanics. Indeed, x-independence of L implies that the one-parameter group of translations in x is a variational symmetry group of L, and the Hamiltonian H is the conservation law resulting from Noether's Theorem [**5**, Theorem 4.29].

COROLLARY 21. *Let $L \in \mathscr{A}$, and let $n \geqq 0$. Then*

$$u_1 \mathbf{D}^n \mathbf{E}(L) = \mathbf{v}_{n+1}(L) + \mathbf{D}P, \tag{22}$$

for some $P \in \mathscr{A}$.

Proof. Just integrate the left-hand side by parts:

$$u_1 \mathbf{D}^n \mathbf{E}(L) = (-1)^n u_{n+1} \mathbf{E}(L) + \mathbf{D}Q$$

for some $Q \in \mathscr{A}$, and then use the lemma.

Now consider a differential polynomial P, with dissipative decomposition (2). The goal is to see how "much" of the ordinary differential equation $P=0$ is conservative, and how much contributes to the dissipative behaviour of the solutions. For simplicity, we treat the case when $P \in \mathscr{A}_0$ does not depend explicitly on x. In this case, if there is only one term in the decomposition, i.e.

$$P = \mathbf{E}(L_0)$$

is an Euler–Lagrange equation, then Corollary 20 shows that the Hamiltonian H, defined by

$$u_1 P = \mathbf{D}H$$

is a constant of the motion, since clearly $\mathbf{D}H = 0$ whenever u is a solution to the equation $P=0$. More generally, motivated by the conservative case, we multiply P by u_1, and try to make as many terms as possible total derivatives. According to Corollary 21

$$\begin{aligned} u_1 P &= \sum_{j=0}^{n} u_1 \mathbf{D}^j \mathbf{E}(L_j) \\ &= \sum_{j=1}^{n} (-1)^j \mathbf{v}_{j+1}(L_j) + \mathbf{D}H \end{aligned} \tag{23}$$

for some $H \in \mathscr{A}_0$. (The $j=0$ term has been incorporated into H using (21).) However, the summation terms so far cannot play the role of the Rayleigh dissipation function R since they are all linear functions of the highest order derivatives, and so can never be positive definite. Thus the next step is to analyse the terms in the summation more closely.

For example, consider the case of a two term dissipative decomposition, so

$$P = \mathbf{E}(L_0) + \mathbf{D}\mathbf{E}(L_1).$$

(P is not necessarily linear.) According to (22),

$$u_1 P = -\mathbf{v}_2(L_1) + \mathbf{D}H, \tag{24}$$

and so we need to analyse

$$\mathbf{v}_2(L) = \sum_{i=0}^{\infty} u_{i+2} \partial_i L, \tag{25}$$

where $L = L_1 \in \mathcal{A}_0$. Note that if L depends on $u_0, u_1, \ldots, u_n$, then $\mathbf{v}_2(L)$ depends on $u_0, u_1, \ldots, u_{n+2}$, *and* is linear in the highest order derivative u_{n+2}. Thus $\mathbf{v}_2(L)$ cannot play the role of the Rayleigh dissipation function since it can never be positive definite. However, we can integrate each of the terms in the summation in (25) by parts, leading to a quadratic function

$$\mathbf{v}_2(L) = -\sum_{i=0}^{\infty} u_{i+1}\mathbf{D}\left(\frac{\partial L}{\partial u_i}\right) + \mathbf{D}M = -\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} u_{i+1}u_{j+1}\frac{\partial^2 L}{\partial u_i \partial u_j} + \mathbf{D}M \tag{26}$$

for some $M \in \mathcal{A}_0$. Let $\mathbf{Q}(L)$ denote the double summation in (26), so that (24) becomes

$$u_1 P = \mathbf{Q}(L_1) + \mathbf{D}\hat{H},$$

where $\hat{H} = H - M$. In particular, if $u(x)$ is any solution to the ordinary differential equation $P = 0$, then

$$\mathbf{D}\hat{H} = -\mathbf{Q}(L_1).$$

Note that $\mathbf{Q}(L_1)$ is a quadratic function of the highest order derivative u_{n+1}, and hence we can identify the Rayleigh dissipation function in this case with $\mathbf{Q}(L_1)$, and, indeed, in the classical linear case, these do agree. More generally, we have the interesting open problem of determining necessary or sufficient conditions on the first order dissipative Lagrangian L_1 in order that $\mathbf{Q}(L_1)$ be positive definite. We hope to return to this question in a future publication.

Similar manipulations can be applied to the higher order dissipative terms, but the calculations become quite complicated. We state the second and third order cases, and leave the more general computations to the interested reader.

LEMMA 22. *Let $L \in \mathcal{A}$. Then*

$$u_1\mathbf{D}^2\mathbf{E}(L) = \mathbf{v}_3(L) + \mathbf{D}P = \frac{1}{2}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} u_{i+1}u_{j+1}u_{k+1}\frac{\partial^3 L}{\partial u_i \partial u_j \partial u_k} + \mathbf{D}M \tag{27}$$

for some P, $M \in \mathcal{A}$.

LEMMA 23. *Let $L \in \mathcal{A}$, and let $n \leqq 0$. Then*

$$\begin{aligned} u_1\mathbf{D}^3\mathbf{E}(L) &= \mathbf{v}_4(L) + \mathbf{D}P \\ &= -\frac{1}{3}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} u_{i+1}u_{j+1}u_{k+1}u_{l+1}\frac{\partial^4 L}{\partial u_i \partial u_j \partial u_k \partial u_l} \\ &\quad -\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} u_{i+2}u_{j+2}\frac{\partial^2 L}{\partial u_i \partial u_j} + \mathbf{D}M \end{aligned} \tag{28}$$

for some P, $M \in \mathcal{A}$.

Note that second order dissipation leads to a cubic Rayleigh function (27), whereas third order dissipation leads to a quartic plus quadratic Rayleigh function (28). Thus we observe that in general, only the odd order dissipative terms can be truly dissipative, in the sense of having any chance of being positive definite.

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