Joint Differential Invariants of Binary and Ternary Forms

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Abstract

We construct and classify the joint invariants and joint differential invariants of binary and ternary forms. In particular, we prove that the differential invariant algebra of ternary forms is generated by a single third order differential invariant. To connect our results with earlier analysis of Kogan, we develop a general method for relating differential invariants associated with different choices of cross-section.

Key words: *binary form, classical invariant theory, differential invariant, joint differential invariant, joint invariant, moving frame, syzygy, ternary form*

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1 Introduction

Classical invariant theory, [9, 11, 23], refers to the study of invariants, equivalence, and symmetry of homogeneous polynomials, referred to as "forms", under the action of the general linear group. Usually, one introduces projective coordinates and studies the behavior of the corresponding inhomogeneous polynomial under projective transformations. The terminology incorporates the number of variables, so that a *binary form* refers to a homogeneous polynomial in two variables or its equivalent inhomogeneous univariate version, while a homogeneous *ternary form* depends on three variables or, equivalently a projectively transformed inhomogeneous polynomial in two variables. The goal of this paper is to apply the method of equivariant moving frames, first developed in [7] — see also [23] — to study the differential invariants, joint invariants, and joint differential invariants of binary and ternary forms.

Lie himself, in [16, Chapter 23], advocated the application of Lie group methods and differential invariants to the problems of classical invariant theory. This received a significant boost with the discovery that applying the Cartan equivalence method, [22], to a problem in the calculus of variations produced a new solution to the equivalence problem for binary forms based on the *signature curve*[1] traced out by two particular differential invariants, which can be viewed as absolute rational covariants, [20]. In essence, the result says that two binary forms are equivalent under a projective transformation if and only if they have identical signatures. Moreover, the symmetries or self-equivalences of a binary form are determined by the signature's index, meaning the number of times it is retraced.

These results, which have no counterpart in the classical literature, were rederived and extended using the method of equivariant moving frames in [23, Chapter 8]. Further developments appear in Irina Kogan (Berchenko)'s thesis, [5, 14], which includes a Maple package for computing the symmetries (both discrete and continuous) of binary forms. Kogan then analyzed the case of ternary forms, using the moving frame to produce a complete system of differential invariants; see also [15] for further results, including the application to equivalent and symmetry properties of elliptic curves. In his thesis, [31], Kogan's student Thomas Wears revisited the equivalence problem for binary forms, ternary forms, and forms in many independent variables. Wears' moving frame-based analysis is performed in homogeneous coordinates, and so his results must be translated into the projective context analyzed here. Recently, Görlach, Hubert, and Papadopoulo, [8], have analyzed the rational invariants and equivalence of ternary forms under the action of the orthogonal group.

In this paper, we revisit Kogan's results. By choosing a different cross-section to construct the moving frame, we are able to prove that the entire differential invariant algebra of a ternary form is generated by a single invariant by repeatedly applying the operators of invariant differentiation. This result is based on the general "commutator trick" method used to establish similar recent results for the differential invariant algebra associated with Euclidean, equi-affine, conformal, and projective surfaces in $\mathbb{R}^3$, cf. [12, 25, 27]. In the Euclidean case, the differential invariants are entirely generated by the mean curvature, and hence, in particular, for suitably generic surfaces, the Gauss curvature can be written as a universal rational function of the invariant derivatives of the mean curvature. In the equi-affine case, the differential invariant algebra is

[1] In [20, 22], the older term "classifying curve" is used.

generated by the third order Pick invariant.

In addition, we present complete classifications and explicit formulas for the joint invariants and joint differential invariants for both binary and ternary forms. In general, a joint invariant is an invariant of the Cartesian product or joint action of the underlying transformation group, while a joint differential invariant is an invariant of the joint prolonged action of the transformation group on the Cartesian product jet space. As far as we know, these invariants of binary and ternary forms have not appeared or been classified in the literature to date.

Our main tool is the powerful new constructive method of equivariant moving frames, [7, 17, 26], that has seen a remarkably broad range of applications including geometry, differential equations, calculus of variations, computer vision, numerical analysis, and classical invariant theory. The theory rests on a reinterpretation of the classical formulation of a moving frame due to Élie Cartan, [6, 10], as an equivariant map to the transformation group, leading to an algorithmic tool for studying the geometric properties of submanifolds and their invariants under the action of a Lie group. The equivariant approach can be systematically applied to general Lie transformation groups, [7, 17], including infinite-dimensional Lie pseudo-groups, [28]. Among the significant new applications of the equivariant method of moving frames is the derivation and classification of joint invariants and joint differential invariants, [24]. These have been used in computer vision applications, [19, 30], and in the design of invariant numerical approximation schemes, [13, 18, 29].

One of the key applications of differential invariants and joint differential invariants is to the equivalence and symmetry properties of submanifolds — in this case binary and ternary forms. The Cartan equivalence method, [22], tells us that two suitably non-degenerate smooth submanifolds are equivalent under a group transformation if and only if they have identical syzygies among all their differential invariants. While, for any finite-dimensional Lie group action, there are an infinite number of functionally independent differential invariants, and hence an infinite number of syzygies, the higher order ones can always be generated from a finite number of low order syzygies through invariant differentiation. (Recall that the *order* of a differential invariant is that of the the highest order jet coordinate upon which it depends.) This motivates the definition of the signature manifold, [23, 26], which is parametrized by the low order differential invariants appearing in these generating syzygies. Cartan's result thus implies that two suitably non-degenerate smooth submanifolds are equivalent if and only if they have identical signatures. Moreover, the codimension of the signature prescribes the dimension of the (local) symmetry group, while in the maximal dimension case, the symmetry group is discrete, whose cardinality equals the index of the signature, meaning the number of times it is retraced, [27]. In [14, 5], this fact was used to design a computational algorithm, implemented in MAPLE, for explicitly constructing the discrete symmetries of binary forms. The methodology can be extended in the obvious manner to joint differential invariant signatures, [24]. Here, we explain how to construct differential invariant signatures and joint differential invariant signatures for both binary and ternary forms, thus extending the results in [5, 14, 20, 23].

The outline of the paper is as follows. Section 2 summarizes some important preliminaries about the moving frame method, and includes a new general formula relating the (differential, joint, etc.) invariants associated with different cross-sections. Section 3 is devoted to the joint invariants and joint differential invariants of binary forms. In Section 4, we determine the differential invariants of ternary forms, exhibit the relationship between our differential invariants and those of Kogan, and prove that the differential invariant algebra is generated by a single invariant. Finally, Section 5 classifies the joint differential invariants of ternary forms.

2 The Method of Moving Frames

In this section, we review the equivariant moving frame construction proposed in [7, 23, 26]. Let G be an r-dimensional Lie group that acts on an m-dimensional manifold M. A *moving frame* is, by definition, a smooth, G-equivariant map $\rho\colon M \to G$ with respect to either left or right multiplication on G. Clearly, if $\rho(z)$ is any right equivariant moving frame, its group inverse $\rho(z)^{-1}$ is a left equivariant moving frame. All classical moving frames, as in [10], can be reinterpreted as left equivariant maps, but the right equivariant versions are often easier to compute.

Existence of an equivariant moving frame map requires that the action of G be free and regular. Here, *freeness* requires that all points have trivial isotropy, so $g \cdot z_0 = z_0$ for some $z_0 \in M$ if and only if $g = e$. *Regularity* is the global condition that the orbits of G form a regular foliation, and does not play any role in any applications to date. Of course, many interesting transformation groups do not act freely, and hence to construct a moving frame one needs to prolong the action to a higher dimensional space in some canonical manner — either to a jet bundle, or a Cartesian product space, or even a Cartesian product of jet bundles, [24]. The invariant functions in these cases are known, respectively, as differential invariants, joint invariants, and joint differential invariants.

If G acts freely and regularly on M, then a (locally defined) equivariant moving frame is constructed through the choice of a cross-section $\mathcal{K} \subset M$ to the group orbits, meaning that $\mathcal{K}$ intersects each orbit at most once and transversally. Let $z = (z_1, \ldots, z_m)$ be local coordinates on M and $w(g, z) = g \cdot z$ be the explicit local coordinate formula for the group transformations. For simplicity, assume $\mathcal{K} = \{ z_1 = c_1, \ldots, z_r = c_r \}$ is a *coordinate cross-section* prescribed by setting the first r coordinates to suitable constants. The associated right-equivariant moving frame map $g = \rho(z)$ is then obtained by solving the *normalization equations*

$$w_1(g, x) = c_1, \quad \ldots \quad w_r(g, x) = c_r, \tag{2.1}$$

for the group parameters $g = (g_1, \ldots, g_r)$ in terms of the coordinates $z = (z_1, \ldots, z_m)$, as guaranteed by the Implicit Function Theorem.

Substituting the moving frame formulae for the group parameters into the remaining transformation rules yields a complete system of functionally independent invariants:

$$I_1(z) = w_{r+1}(\rho(z), z), \quad \ldots \quad I_{m-r}(z) = w_m(\rho(z), z). \tag{2.2}$$

This is a special case of the process of *invariantization* with respect to the moving frame. Given any object – function, differential form, differential operator, etc. — one invariantizes it by first transforming it according the the action of G and then replacing all group parameters g by their moving frame formulas $g = \rho(z)$. The invariantization process is denoted by ι. Geometrically, the invariantization $J = \iota(F)$ is the unique invariant that agrees with F on the cross-section: $J \mid \mathcal{K} = F \mid \mathcal{K}$. In particular, invariantizing the coordinate functions yields

$$\iota(z_1) = c_1, \ \ldots\ , \iota(z_r) = c_r, \quad \iota(z_{r+1}) = I_1(z), \ \ldots\ , \iota(z_m) = I_{m-r}(z).$$

Thus, invariantizing the first r coordinates that define the cross-section reproduces the normalization constants — because $g = \rho(z)$ solves the normalization equations (2.1) — while invariantizing the remaining $m - r$ coordinates produces the fundamental invariants. The first r trivial constant invariants are sometimes referred to as the *phantom invariants*. Invariantization clearly respects all algebraic operations, and hence the invariantization of a function $F(z)$ is

$$\iota[F(z_1, \ldots, z_m)] = F(\iota(z_1), \ldots, \iota(z_m)) = F(c_1, \ldots, c_r, I_1(z), \ldots, I_{m-r}(z)). \qquad (2.3)$$

Moreover invariantization does not affect an invariant: $\iota(J) = J$, which, in view of (2.3), implies the powerful *Replacement Rule*

$$J(z_1, \ldots, z_m) = J(c_1, \ldots, c_r, I_1(z), \ldots, I_{m-r}(z)) \qquad (2.4)$$

that allows one to immediately rewrite any invariant J in terms of the fundamental invariants, thus proving their completeness.

It will be important here to connect the moving frame invariants associated with two different cross-sections. Suppose the Lie group G acts freely on M, with $z = (z_1, \ldots, z_m) \in M$ being local coordinates. Let $\mathcal{K}, \widehat{\mathcal{K}} \subset M$ be two different cross-sections. Let $\rho, \widehat{\rho}$ the corresponding right moving frames, and $\iota, \widehat{\iota}$ the associated invariantization maps, producing the respective invariants $I = (I_1, \ldots, I_m) = (\iota(z_1), \ldots, \iota(z_m))$ and $\widehat{I} = \big(\widehat{I}_1, \ldots, \widehat{I}_m\big) = \big(\widehat{\iota}(z_1), \ldots, \widehat{\iota}(z_m)\big)$. By completeness, we know that we can write each $\widehat{I}_k$ as a function of the I_j's, and vice versa. To this end, we assume $\mathcal{K} \subset \operatorname{dom} \widehat{\rho}$ and $\widehat{\mathcal{K}} \subset \operatorname{dom} \rho$. The following result enables us to easily determine the required expressions.

Theorem 1 *The normalized invariants corresponding to the two right-equivariant moving frames $\rho, \widehat{\rho}$ are related by the formula*

$$I = \rho(\widehat{I}\,) \cdot \widehat{I} \qquad (2.5)$$

Proof: It suffices to recall that each I_k is the unique invariant that agrees with the coordinate function z_k on the cross-section. Thus, $I \mid \mathcal{K} = z \mid \mathcal{K}$, and, similarly, $\widehat{I} \mid \widehat{\mathcal{K}} = z \mid \widehat{\mathcal{K}}$. Given a point $\widehat{z} \in \widehat{\mathcal{K}}$, the right moving frame determines the unique group element $g = \rho(\widehat{z}) \in G$ that maps $\widehat{z}$ to a point z in the cross-section $\mathcal{K}$. In other words,

$$\widehat{z} \in \widehat{\mathcal{K}} \qquad \longmapsto \qquad z = \rho(\widehat{z}) \cdot \widehat{z} \in \mathcal{K}. \qquad (2.6)$$

By the first remark, this implies that the left and right hand sides of (2.5) agree on the cross-section $\mathcal{K}$. Since they are constant on orbits, these two invariants must agree everywhere, which proves that formula (2.5) holds where defined. *Q.E.D.*

As noted earlier, one typically needs to prolong the group action in order to make it free and regular, and hence admit a moving frame. We assume the reader is familiar with the basic constructions of jet bundles, [21, 22]. We let $x = (x_1, \ldots, x_p)$ denote the independent variables and $u = (u^1, \ldots, u^q)$ the dependent variables, defining local coordinates on a manifold M of dimension $m = p + q$, so that one can identify the graphs of functions $u = f(x)$ as p-dimensional submanifolds of M. The corresponding local coordinates u^α_K, for $\alpha = 1, \ldots, q$, $K = (k_1, \ldots, k_i)$, $1 \le k_\sigma \le p$, $i = \#K \le k$, on the order k jet bundle $\mathrm{J}^k = \mathrm{J}^k(M, p)$ represent the derivatives of the u's with respect to the x's.

Suppose G acts freely and regularly on an open subset of J^k – which, for k sufficiently large, holds in all examples of interest. (However, Scot Adams has recently constructed rather intricate counterexamples for both smooth, [2], and analytic, [3], actions. On the other hand, prolonged freeness and regularity is true for algebraic actions, [1]. See also [4] where a version of this result is proved for connected analytic actions.) Let $\rho(x, u^{(k)})$ be the right-equivariant moving frame constructed through the choice of cross-section $\mathcal{K} \subset \mathrm{J}^k$. A complete system of differential invariants is then found by invariantizing the jet coordinates: $H_i = \iota(x_i)$, $I_K^\alpha = \iota(u_K^\alpha)$. Of course r of them — namely those corresponding to the cross-section coordinates — are constant phantom differential invariants, while the remainder provide a complete system of functionally independent differential invariants of order $\leq k$. Any other differential invariant can be immediately expressed in terms of these fundamental differential invariants though the Replacement Rule (2.4).

An alternative means of constructing differential invariants is through the process of invariant differentiation. For $i = 1, \ldots, p$, let $\omega_i = \iota(dx_i)$ denoted the invariantized horizontal one-forms, and $\mathcal{D}_i = \iota(D_i)$ the dual invariantized differential operator obtained from the total derivative with respect to the independent variable x_i, [21]. Each $\mathcal{D}_i$ represents an invariant derivation, so that if J is any differential invariant so is $\mathcal{D}_i J$. The Fundamental Basis Theorem, [7, 22], states that all differential invariants can be found by repeated invariant differentiation of a finite number of low order differential invariants. In fact, if the moving frame is of order n, then the fundamental differential invariants of order $\leq n+1$ provide a basis, generating the entire differential invariant algebra. However, these bases are typically far from minimal, and indeed, determination of a minimal basis is often not easy. Indeed, there are, as yet, no known algorithms for testing for minimality, except in the trivially minimal case when the algebra is generated by a single differential invariant.

Although invariantization respects all algebraic operations, it does not respect differentiation. However, its effect on derivatives can be explicitly determined through the powerful *recurrence formulae*. Let $F(x, u^{(k)})$ be a differential function and $\iota(F)$ its moving frame invariantization. Then

$$\mathcal{D}_i[\iota(F)] = \iota[D_i(F)] + \sum_{\sigma=1}^{r} R_i^\sigma \, \iota\big[\, \mathrm{pr}\, \mathbf{v}_\sigma(F)\, \big]. \tag{2.7}$$

Here

$$\mathbf{v}_\sigma = \xi_\sigma(x,u)\frac{\partial}{\partial x} + \varphi_\sigma(x,u)\frac{\partial}{\partial u}, \qquad \sigma = 1, \ldots, r, \tag{2.8}$$

are a basis for the Lie algebra of infinitesimal generators of the action of G, while $\mathrm{pr}\,\mathbf{v}_\sigma$ denotes their jet bundle prolongations, whose explicit formulae are well known, [21, 22]. The R_i^σ are certain differential invariants known as the *Maurer-Cartan invariants*[2], and are determined by setting F in (2.7) to be the cross-section variables corresponding to the phantom differential invariants. Since each such F has constant invariantization, the left hand side of (2.7) is 0, and hence we obtain a system of r linear equations which can be uniquely solved for the Maurer–Cartan invariants. Substituting the resulting

[2]The Maurer–Cartan invariants can be intrinsically characterized as the coefficients of the invariantized horizontal forms ω_i in the pulled-back Maurer–Cartan forms via the moving frame map, [7]. However, while this characterization is important for the underlying proofs, it need not concern us here since, as we will see, the Maurer–Cartan invariants can all be explicitly determined directly from the recurrence formulae.

expressions back into (2.7) produces the complete set of recurrence formulae for the remaining differential invariants, which then prescribes the complete structure of the algebra of differential invariants. The recurrence formula (2.7) extends to differential forms as stated, where the differential operators and vector fields act as Lie derivatives when F represents a differential form.

The general moving frame constructions can be readily adapted to Cartesian product actions of groups, as well as their prolongations to Cartesian products of jet bundles. The invariantization process produces complete bases of joint invariants and joint differential invariants. The corresponding recurrence formulae will determine the complete structure of the algebra of joint differential invariants. Explicitly, given a Lie group G acting on M, consider the "joint action" of G on the $(n+1)$-fold Cartesian product $M^{\times(n+1)} = M \times \cdots \times M$ given by

$$g \cdot (z^0, \ldots, z^n) = (g \cdot z^0, \ldots, g \cdot z^n), \qquad g \in G \qquad z^0, \ldots, z^n \in M. \tag{2.9}$$

An invariant $I(z^0, \ldots, z^n)$ of this product action is known as an $(n+1)$-*point joint invariant* of the original transformation group. The product action (2.9) induces a prolonged action on the k-th order Cartesian product jet bundle $(J^k)^{\times n+1}$, which coincides with the n-fold Cartesian product of the action on $J^k = J^k(M,p)$. The invariants of the induced action of G on $(J^k)^{\times n}$ are known as *joint differential invariants*, [24].

3 Binary Forms

To study binary forms, as in [23], we consider the planar action

$$X = \frac{\alpha x + \beta}{\gamma x + \delta}, \qquad U = (\gamma x + \delta)^{-n} u, \tag{3.1}$$

of the general linear group $G = \mathrm{GL}(2)$ on functions $u = f(x)$ whose graphs are identified as plane curves. In our analysis, we will focus on the real case, although all our results apply equally well to complex forms under $\mathrm{GL}(2,\mathbb{C})$. When n is a positive integer, this induces the transformation rules for binary forms (polynomials) $u = q(x)$ of degree n, whose invariance properties form the focus of classical invariant theory. In the following, we will assume $n \neq 0, 1$. (These cases can also be handled by the moving frame method, but the formulas are slightly different.) Since the differential invariants of binary forms are well known, [5, 14, 23], we will study their joint invariants and joint differential invariants. Here the variables $z^0 = (x^0, u^0), \ldots, z^n = (x^n, u^n)$ are simultaneously subjected to the same transformation rule (3.1), as in (2.9).

For the joint action of $\mathrm{GL}(2)$ on $M^{\times n} \simeq \mathbb{R}^{2n}$, we choose the simple cross-section

$$x^0 = 0, \qquad u^0 = x^1 = u^1 = 1, \tag{3.2}$$

noting that setting any $u^k = 0$ does not produce a valid cross-section. The corresponding normalization equations are

$$X^0 = 0, \qquad U^0 = 1, \qquad X^1 = 1, \qquad U^1 = 1. \tag{3.3}$$

Substituting the formulas based on (3.1), and then solving the resulting normalization equations for the group parameters produces the right moving frame $\rho \colon M^{\times 2} \to \mathrm{GL}(2)$, given by[3]

$$\alpha = \frac{\sqrt[n]{u^1}}{x^1 - x^0}, \qquad \beta = \frac{x^0\,\sqrt[n]{u^1}}{x^0 - x^1}, \qquad\qquad (3.4)$$
$$\gamma = \frac{\sqrt[n]{u^0} - \sqrt[n]{u^1}}{x^0 - x^1}, \qquad \delta = \frac{x^0\,\sqrt[n]{u^1} - x^1\,\sqrt[n]{u^0}}{x^0 - x^1}.$$

The moving frame formulas are then used to invariantize the remaining variables to produce the normalized joint invariants

$$y^k \longmapsto I^k = \iota(x^k) = \frac{(x^k - x^0)\,\sqrt[n]{u^1}}{A^k}, \qquad\qquad (3.5)$$
$$v^k \longmapsto J^k = \iota(u^k) = u^k \left(\frac{x^1 - x^0}{A^k}\right)^n,$$

where

$$A^k = (x^k - x^0)\,\sqrt[n]{u^1} + (x^1 - x^k)\,\sqrt[n]{u^0}. \qquad\qquad (3.6)$$

We note that, for all distinct i, j, k,

$$I^k_{ij} = \frac{x^j - x^k}{x^i - x^k}\;\frac{\sqrt[n]{u^i}}{\sqrt[n]{u^j}} = \frac{I^j - I^k}{I^i - I^k}\;\frac{\sqrt[n]{J^i}}{\sqrt[n]{J^j}}, \qquad\qquad (3.7)$$

are also joint invariants, whose expressions in terms of the fundamental invariants (3.5) are immediate consequences of the Replacement Rule (2.4). Also,

$$I^k = \frac{1}{1 - I^k_{01}}, \qquad J^k = \left(\frac{1}{I^0_{1k} - I^1_{0k}}\right)^n. \qquad\qquad (3.8)$$

The prolonged group transformations are obtained by implicit differentiation of (3.1); in particular,

$$v_y = \frac{(\gamma\,x + \delta)u_x - n\,\gamma\,u}{(\gamma\,x + \delta)^{n-1}(\alpha\,\delta - \beta\,\gamma)}. \qquad\qquad (3.9)$$

Plugging (3.4) into the preceding formula yields the fundamental normalized first order joint differential invariants

$$K^0 = \iota(u^0_x) = \frac{(x^1 - x^0)u^0_x\,\sqrt[n]{u^0} + n\,u^0(\,\sqrt[n]{u^0} - \sqrt[n]{u^1})}{u^0\,\sqrt[n]{u^1}},$$
$$K^1 = \iota(u^1_x) = \frac{(x^1 - x^0)u^1_x\,\sqrt[n]{u^1} + n\,u^1(\,\sqrt[n]{u^0} - \sqrt[n]{u^1})}{u^1\,\sqrt[n]{u^0}}, \qquad\qquad (3.10)$$
$$K^k = \iota(u^k_x) = -\,\frac{\big[\,(x^0 - x^k)u^k_x + n\,u^k\,\big]\,\sqrt[n]{u^1} - \big[\,(x^1 - x^k)u^k_x + n\,u^k\,\big]\,\sqrt[n]{u^0}}{\sqrt[n]{u^0u^1}}\left(\frac{x^1 - x^0}{A^k}\right)^{n-1}.$$

The invariantization process can be used to produce the higher order joint differential invariants. However, we will avoid the long explicit expressions by employing invariant differentiation and the recurrence formulae. The invariantized one forms that play the role of *joint arc length forms* are

$$dX^k = \frac{(\alpha\,\delta - \beta\,\gamma)\,dx^k}{(\gamma\,x^k + \delta)^2} \quad\longmapsto\quad \omega^k = \iota(dx^k) = \frac{(x^1 - x^0)\,\sqrt[n]{u^0u^1}\,dx^k}{(A^k)^2}, \qquad\qquad (3.11)$$

[3]For simplicity, we do not try to deal with the ambiguity in the n-th root and consequential local equivariance of the moving frame map, that stems from the local freeness of the product action on $M^{\times 2}$. Furthermore, if any $u^k < 0$, one can replace it with $-u^k$.

with corresponding invariant differential operators

$$D_{X^k} \quad \longmapsto \quad \mathcal{D}^k = \iota(D_{x^k}) = \frac{(A^k)^2}{(x^1 - x^0)\sqrt[n]{u^0 u^1}} D_{x^k}. \tag{3.12}$$

In particular,

$$\mathcal{D}^0 = \frac{(x^1 - x^0)\sqrt[n]{u^0}}{\sqrt[n]{u^1}} D_{x^0}, \qquad \mathcal{D}^1 = \frac{(x^1 - x^0)\sqrt[n]{u^1}}{\sqrt[n]{u^0}} D_{x^1}. \tag{3.13}$$

The invariant differential operators satisfy the following commutation relations:

$$\begin{aligned} &[\,\mathcal{D}^0, \mathcal{D}^1\,] = \left(1 - \frac{K^1}{n}\right)\mathcal{D}^0 + \left(1 + \frac{K^0}{n}\right)\mathcal{D}^1, \qquad &&[\,\mathcal{D}^0, \mathcal{D}^k\,] = \left((2x^k - 1)\frac{K^0}{n} + 1\right)\mathcal{D}^k, \\ &[\,\mathcal{D}^1, \mathcal{D}^k\,] = -\left((2\,x^k - 1)\frac{K^1}{n} + 1\right)\mathcal{D}^k, &&[\,\mathcal{D}^j, \mathcal{D}^k\,] = 0, \qquad \text{where } j, k \geq 2. \end{aligned} \tag{3.14}$$

These can be derived directly from the explicit formulae (3.12), (3.13), or symbolically through use of the recurrence formulae for the differentials of the invariantized horizontal one-forms, [7, 25]; see also (4.14) below.

The recurrence relations (2.7) for the two-point joint invariants I have the form

$$\mathcal{D}^k \iota(I) = \iota(D_{x^k} I) + \sum_{\sigma=1}^{4} R_k^\sigma \iota(\mathbf{v}_\sigma(I)), \qquad k = 0, 1, \tag{3.15}$$

where

$$\mathbf{v}_1 = \partial_x, \qquad \mathbf{v}_2 = x\,\partial_x, \qquad \mathbf{v}_3 = u\,\partial_u, \qquad \mathbf{v}_4 = x^2\partial_x + n\,x\,u\,\partial_u. \tag{3.16}$$

are the infinitesimal generators of the action (3.1), while R_k^σ are the Maurer-Cartan invariants. The latter are found by solving the recurrence formulae for the phantom invariants corresponding to the cross-section (3.2), which, by (3.15), (3.16), are

$$\begin{aligned} 0 &= \mathcal{D}^0 \iota(x^0) = \iota(1) + R_0^1 \iota(1) + R_0^2\, \iota(x^0) + R_0^3 \iota(0) + R_0^4\, \iota((x^0)^2) = 1 + R_0^1, \\ 0 &= \mathcal{D}^1 \iota(x^0) = \iota(0) + R_1^1 \iota(1) + R_1^2\, \iota(x^0) + R_1^3\, \iota(0) + R_1^4\, \iota((x^0)^2) = R_1^1, \\ 0 &= \mathcal{D}^0 \iota(u^0) = \iota(u_x^0) + R_0^1\, \iota(0) + R_0^2\, \iota(0) + R_0^3\, \iota(u^0) + R_0^4\, \iota(n\,x^0 u^0) = K^0 + R_0^3, \\ 0 &= \mathcal{D}^1 \iota(u^0) = \iota(0) + R_1^1\, \iota(0) + R_1^2\, \iota(0) + R_1^3\, \iota(u^0) + R_1^4\, \iota(n\,x^0 u^0) = R_1^3, \\ 0 &= \mathcal{D}^0 \iota(x^1) = \iota(0) + R_0^1\, \iota(1) + R_0^2\, \iota(x^1) + R_0^3\, \iota(0) + R_0^4\, \iota((x^1)^2) = R_0^1 + R_0^2 + R_0^4, \\ 0 &= \mathcal{D}^1 \iota(x^1) = \iota(1) + R_1^1\, \iota(1) + R_1^2\, \iota(x^1) + R_1^3\, \iota(0) + R_1^4\, \iota((x^1)^2) = 1 + R_1^2 + R_1^4, \\ 0 &= \mathcal{D}^0 \iota(u^1) = \iota(0) + R_0^1\, \iota(0) + R_0^2\, \iota(0) + R_0^3\, \iota(u^1) + R_0^4\, \iota(n\,x^1 u^1) = R_0^3 + n\,R_0^4, \\ 0 &= \mathcal{D}^1 \iota(u^1) = \iota(u_x^1) + R_1^1\, \iota(0) + R_1^2\, \iota(0) + R_1^3\, \iota(u^1) + R_1^4\, \iota(n\,x^1 u^1) = K^1 + n\,R_1^4. \end{aligned} \tag{3.17}$$

Thus,

$$\begin{aligned} &R_0^1 = -1, \qquad &&R_0^2 = 1 - \frac{K^0}{n}, \qquad &&R_0^3 = -K^0, \qquad &&R_0^4 = \frac{K^0}{n}, \\ &R_1^1 = 0, &&R_1^2 = \frac{K^1}{n} - 1, &&R_1^3 = 0, &&R_1^4 = -\frac{K^1}{n}. \end{aligned} \tag{3.18}$$

Substituting back into (3.15), the final recurrence formulae are

$$\begin{aligned}
&\mathcal{D}^0 I^k = (I^k - 1)\left(\frac{I^k K^0}{n} + 1\right), && \mathcal{D}^1 I^k = I^k\left(\frac{(1-I^k)K^1}{n} - 1\right),\\
&\mathcal{D}^0 J^k = (I^k - 1) J^k K^0, && \mathcal{D}^1 J^k = -I^k J^k K^1,\\
&\mathcal{D}^j I^k = \begin{cases} 1, & j = k,\\ 0, & j \neq k,\end{cases} && \mathcal{D}^j J^k = \begin{cases} K^k, & j = k,\\ 0, & j \neq k,\end{cases}\\
&\mathcal{D}^0 K^0 = L^0 + \left(\frac{1}{n} - 1\right)(K^0)^2, && \mathcal{D}^1 K^0 = -\frac{K^0 K^1}{n} + K^0 - K^1,\\
&\mathcal{D}^0 K^1 = -\frac{K^0 K^1}{n} + K^0 - K^1, && \mathcal{D}^1 K^1 = L^1 + \left(\frac{1}{n} - 1\right)(K^1)^2,\\
&\mathcal{D}^j K^k = \begin{cases} L^k, & j = k,\\ 0, & j \neq k,\end{cases}
\end{aligned} \tag{3.19}$$

where $j \neq k$ are not 0 or 1, and $L^k = \iota(u^k_{xx})$ are the second order normalized invariants. In particular, assuming we have at least 3 points, by invariantly differentiating the order 0 joint invariants I^k and J^k for $k \geq 2$, we can obtain all the first order joint differential invariants K^k, including K^0, K^1, and hence all the higher order differential invariants. On the other hand, there are no two-point joint invariants, and hence in this case the joint differential invariant algebra is generated by the first order invariants K^0, K^1. Finally recall that the ordinary (1 point) differential invariant algebra is generated by a single third order differential invariant, [20, 23].

As is well known, [5, 23], the ordinary differential invariant signature of a binary form is parametrized by the fundamental differential invariants of order 3 and 4. We call the maximum of these orders, namely 4, the *order* of the signature. According to our recurrence formulae, we can now specify the appropriate joint invariant signatures for binary forms:

1 point:	Fourth order signature;
2 point:	Second order signature parametrized by K^0, K^1, L^0, L^1;
3 point:	First order signature parametrized by I^2, J^2, K^0, K^1, K^2;
4 point:	First order signature parametrized by $I^2, J^2, I^3, J^3, K^0, K^1, K^2, K^3$;
≥ 5 point:	Zero-th order signature parametrized by $I^k, J^k, \ k \geq 2$.

Applications of these signatures to the equivalence and symmetry properties of binary forms will be the subject of future investigations.

4 Differential Invariants of Ternary Forms

Now, we turn our attention to ternary forms. As above, we work in projective coordinates. In this section, we shall revisit Kogan's analysis, [14], of their differential invariants.

We are thus interested in the action

$$X = \frac{\alpha x + \beta y + \gamma}{\rho x + \sigma y + \tau}, \qquad Y = \frac{\lambda x + \mu y + \nu}{\rho x + \sigma y + \tau}, \qquad U = (\rho x + \sigma y + \tau)^{-n} u, \tag{4.1}$$

of the general linear group $G = \mathrm{GL}(3)$ on two dimensional surfaces $S \subset M = \mathbb{R}^3$ representing the graphs of functions $u = q(x,y)$. When n is a positive integer, this action encodes the transformation rules for ternary forms of degree n, [9, 11, 14, 23].

Let

$$\Delta = \det \begin{pmatrix} \alpha & \beta & \gamma \\ \lambda & \mu & \nu \\ \rho & \sigma & \tau \end{pmatrix}. \tag{4.2}$$

We prolong the action (4.1) to the surface jet spaces $\mathrm{J}^n(M,2)$, coordinatized by x, y and the derivatives

$$u_{jk} = D_x^j D_y^k u \qquad \text{for} \qquad 0 \leq j+k \leq n. \tag{4.3}$$

Explicitly, the action is given by

$$u_{jk} \longmapsto U_{jk} = D_X^j D_Y^k U,$$

where

$$\begin{aligned} D_X &= \frac{\rho x + \sigma y + \tau}{\Delta} \left(\big[(\mu \rho - \lambda \sigma) x + \mu \tau - \nu \sigma \big] D_x + \big[(\mu \rho - \lambda \sigma) y - \lambda \tau + \nu \rho \big] D_y \right), \\ D_Y &= \frac{\rho x + \sigma y + \tau}{\Delta} \left(\big[(\alpha \sigma - \beta \rho) x - \beta \tau + \gamma \sigma \big] D_x + \big[(\alpha \sigma - \beta \rho) y + \alpha \tau - \gamma \rho \big] D_y \right), \end{aligned} \tag{4.4}$$

are the operators of implicit differentiation. They are dual to the transformed horizontal one-forms

$$\begin{aligned} dX &= \frac{\big[(\alpha \sigma - \beta \rho) y + \alpha \tau - \gamma \rho \big] dx + \big[(\beta \rho - \alpha \sigma) x + \beta \tau - \gamma \sigma \big] dy}{(\rho x + \sigma y + \tau)^2}, \\ dY &= \frac{\big[(\lambda \sigma - \mu \rho) y + \lambda \tau - \nu \rho \big] dx + \big[(\mu \rho - \lambda \sigma) x + \mu \tau - \nu \sigma \big] dy}{(\rho x + \sigma y + \tau)^2}, \end{aligned} \tag{4.5}$$

meaning that[4]

$$dF = (D_x F)\, dx + (D_y F)\, dy = (D_X F)\, dX + (D_Y F)\, dY,$$

for any differential function F.

Let us construct the moving frame based on the minimal order cross-section

$$\mathcal{K} = \left\{ x = y = 0, \quad u = 1, \quad u_x = u_y = u_{xx} = u_{yy} = 0, \quad u_{xy} = u_{xxx} = 1 \right\}. \tag{4.6}$$

The moving frame formulas obtained by solving the associated normalization equations are quite complicated, and will be discussed at the end of this section. Let $I_{jk} = \iota(u_{jk})$ denote the resulting differential invariants obtained by invariantization of the jet coordinates using the moving frame; these are obtained by substituting the moving frame formulas into the prolonged transformations. Again the expressions are quite complicated. In particular, the phantom invariants are

$$\begin{gathered} H_1 = \iota(x) = 0, \qquad H_2 = \iota(y) = 0, \\ I_{00} = \iota(u) = 1, \qquad I_{10} = \iota(u_x) = 0, \qquad I_{01} = \iota(u_y) = 0, \\ I_{20} = \iota(u_{xx}) = 0, \qquad I_{11} = \iota(u_{xy}) = 1, \qquad I_{02} = \iota(u_{yy}) = 0, \qquad I_{30} = \iota(u_{xxx}) = 1. \end{gathered} \tag{4.7}$$

[4]When F depends on jet coordinates, dF denotes the horizontal differential, [22].

Let us next construct the recurrence formulae (2.7) for the differentiated invariants. We let $\mathcal{D}_1 = \iota(D_x)$, $\mathcal{D}_2 = \iota(D_y)$, be the invariant differential operators. The infinitesimal generators of the group action (4.1) are

$$\begin{aligned} &\mathbf{v}_1 = \partial_x, \quad \mathbf{v}_2 = \partial_y, \quad \mathbf{v}_3 = x\,\partial_x, \quad \mathbf{v}_4 = y\,\partial_y, \quad \mathbf{v}_5 = u\,\partial_u, \quad \mathbf{v}_6 = y\,\partial_x, \\ &\mathbf{v}_7 = x\,\partial_y, \quad \mathbf{v}_8 = x^2\partial_x + x\,y\,\partial_y + n\,x\,u\,\partial_u, \quad \mathbf{v}_9 = x\,y\,\partial_x + y^2\partial_y + n\,y\,u\,\partial_u. \end{aligned} \tag{4.8}$$

Thus, according to (2.7), the recurrence formulae for the differential invariants are

$$\mathcal{D}_1 I_{jk} = I_{j+1,k} + \sum_{\sigma=1}^{9} \iota(\varphi_\sigma^{jk})\, R_1^\sigma, \qquad \mathcal{D}_2 I_{jk} = I_{j,k+1} + \sum_{\sigma=1}^{9} \iota(\varphi_\sigma^{jk})\, R_2^\sigma, \tag{4.9}$$

where R_i^σ are the Maurer–Cartan invariants and φ_σ^{jk} is the coefficient of $\partial/\partial u_{jk}$ in the prolongation of the infinitesimal generator $\mathbf{v}_\sigma$ obtained by the standard prolongation formula, [21]. Solving the resulting phantom recurrence formulae

$$\begin{aligned} 0 &= \mathcal{D}_1 H_1 = 1 + R_1^1, & 0 &= \mathcal{D}_2 H_1 = R_2^1, \\ 0 &= \mathcal{D}_1 H_2 = R_1^2, & 0 &= \mathcal{D}_2 H_2 = 1 + R_2^2, \\ 0 &= \mathcal{D}_1 I_{00} = I_{10} + R_1^5, & 0 &= \mathcal{D}_2 I_{00} = I_{01} + R_2^5, \\ 0 &= \mathcal{D}_1 I_{10} = I_{20} + n\,R_1^8, & 0 &= \mathcal{D}_2 I_{10} = I_{11} + n\,R_2^8, \\ 0 &= \mathcal{D}_1 I_{01} = I_{11} + n\,R_1^9, & 0 &= \mathcal{D}_2 I_{01} = I_{02} + n\,R_2^9, \\ 0 &= \mathcal{D}_1 I_{20} = I_{30} - 2\,R_1^7, & 0 &= \mathcal{D}_2 I_{20} = I_{21} - 2\,R_2^7, \\ 0 &= \mathcal{D}_1 I_{11} = I_{21} - R_1^3 - R_1^4, & 0 &= \mathcal{D}_2 I_{11} = I_{12} - R_2^3 - R_2^4, \\ 0 &= \mathcal{D}_1 I_{02} = I_{12} - 2\,R_1^6, & 0 &= \mathcal{D}_2 I_{02} = I_{03} - 2\,R_2^6, \\ 0 &= \mathcal{D}_1 I_{30} = I_{40} - 3\,R_1^3 - 3\,I_{21} R_1^7, & 0 &= \mathcal{D}_2 I_{30} = I_{31} - 3\,R_2^3 - 3\,I_{21} R_2^7, \end{aligned}$$

produces the Maurer-Cartan invariants

$$\begin{aligned} &R_1^1 = -1, && R_1^2 = 0, && R_1^3 = \tfrac{1}{3}\,I_{40} - \tfrac{1}{2}\,I_{21}, \\ &R_1^4 = \tfrac{3}{2}\,I_{21} - \tfrac{1}{3}\,I_{40}, && R_1^5 = 0, && R_1^6 = \tfrac{1}{2}\,I_{12}, \\ &R_1^7 = \tfrac{1}{2}, && R_1^8 = 0, && R_1^9 = -1/n, \\ &R_2^1 = 0, && R_2^2 = -1, && R_2^3 = \tfrac{1}{3}\,I_{31} - \tfrac{1}{2}\,I_{21}^2, \\ &R_2^4 = I_{12} - \tfrac{1}{3}\,I_{31} + \tfrac{1}{2}\,I_{21}^2, && R_2^5 = 0, && R_2^6 = \tfrac{1}{2} I_{03}, \\ &R_2^7 = \tfrac{1}{2}\,I_{21}, && R_2^8 = -1/n, && R_2^9 = 0. \end{aligned} \tag{4.10}$$

Substituting these expressions back into (4.9) produces all the non-phantom recurrence formulae that completely prescribe the structure of the differential invariant algebra. The non-phantom third order recurrence formulae are

$$\begin{aligned} \mathcal{D}_1 I_{21} &= I_{31} - \tfrac{1}{3}\,I_{21} I_{40} - \tfrac{1}{2}\,I_{21}^2 - \tfrac{3}{2}\,I_{12}, \\ \mathcal{D}_2 I_{21} &= I_{22} - \tfrac{1}{3}\,I_{21} I_{31} + \tfrac{1}{2}\,I_{21}^3 - 2\,I_{21} I_{12} - \tfrac{1}{2}\,I_{03} - 2 + 4/n, \\ \mathcal{D}_1 I_{12} &= I_{22} + \tfrac{1}{3}\,I_{12} I_{40} - \tfrac{7}{2}\,I_{21} I_{12} - \tfrac{1}{2}\,I_{03} - 2 + 4/n, \\ \mathcal{D}_2 I_{12} &= I_{13} + \tfrac{1}{3}\,I_{12} I_{31} - \tfrac{1}{2}\,I_{21}^2 I_{12} - 2\,I_{12}^2 - \tfrac{3}{2}\,I_{21} I_{03}, \\ \mathcal{D}_1 I_{03} &= I_{13} + I_{03} I_{40} - \tfrac{9}{2}\,I_{21} I_{03} - \tfrac{3}{2}\,I_{12}^2, \\ \mathcal{D}_2 I_{03} &= I_{04} + I_{03} I_{31} - \tfrac{9}{2}\,I_{12} I_{03} - \tfrac{3}{2}\,I_{21}^2 I_{03}, \end{aligned} \tag{4.11}$$

while those of the fourth order are

$$\begin{aligned}
\mathcal{D}_1 I_{40} &= I_{50} - \tfrac{4}{3} I_{40}^2 + 2\, I_{21} I_{40} - 2\, I_{31}, \\
\mathcal{D}_2 I_{40} &= I_{41} - \tfrac{4}{3} I_{31} I_{40} - 2\, I_{21}^2 I_{40} - 2\, I_{21} I_{31} - 4 + 12/n, \\
\mathcal{D}_1 I_{31} &= I_{41} - \tfrac{2}{3} I_{40} I_{31} - \tfrac{1}{2} I_{12} I_{40} - \tfrac{3}{2} I_{22} - 1 + 3/n, \\
\mathcal{D}_2 I_{31} &= I_{32} - \tfrac{1}{2} I_{03} I_{40} - \tfrac{2}{3} I_{31}^2 + I_{21}^2 I_{31} - I_{12} I_{31} - \tfrac{3}{2} I_{21} I_{22} - (3 - 9/n)\, I_{21},
\end{aligned}$$

$$\begin{aligned}
\mathcal{D}_1 I_{22} &= I_{32} - 2\, I_{22} I_{21} - I_{31} I_{12} - I_{13} - (2 - 6/n)\, I_{21}, \\
\mathcal{D}_2 I_{22} &= I_{23} - 2\, I_{22} I_{12} - I_{31} I_{03} - I_{13} I_{21} - (2 - 6/n) I_{12}, \\
\mathcal{D}_1 I_{13} &= I_{23} + \tfrac{2}{3} I_{13} I_{40} - 4 I_{21} I_{13} - \tfrac{3}{2} I_{12} I_{22} - \tfrac{1}{2} I_{04} - (3 - 9/n)\, I_{12}, \\
\mathcal{D}_2 I_{13} &= I_{14} + \tfrac{2}{3} I_{13} I_{31} - I_{21} I_{13} - 3\, I_{12} I_{13} - \tfrac{3}{2} I_{03} I_{22} - \tfrac{1}{2} I_{21} I_{04} - (1 - 3/n)\, I_{03}, \\
\mathcal{D}_1 I_{04} &= I_{14} - 6\, I_{21} I_{04} + \tfrac{4}{3} I_{40} I_{04} - 2\, I_{12} I_{13} - (4 - 12/n)\, I_{03}, \\
\mathcal{D}_2 I_{04} &= I_{05} - 4\, I_{12} I_{04} + \tfrac{4}{3} I_{04} I_{31} - 2\, I_{21}^2 I_{04} - 2\, I_{03} I_{13}.
\end{aligned} \tag{4.12}$$

Finally, the invariant differential operators $\mathcal{D}_1$ and $\mathcal{D}_2$ satisfy the *commutator relation*

$$[\, \mathcal{D}_1, \mathcal{D}_2 \,] = Y_1\, \mathcal{D}_1 + Y_2\, \mathcal{D}_2, \tag{4.13}$$

whose coefficients are known as the *commutator invariants*. As a consequence of the general recurrence formulae for the invariant horizontal differential one-forms, [7, 25], we find

$$\begin{aligned}
Y_1 &= \sum_{\sigma=1}^{9} \big[\, R_2^\sigma \iota(D_x \xi_\sigma) - R_1^\sigma \iota(D_y \xi_\sigma) \,\big] = R_2^3 - R_1^6, \\
Y_2 &= \sum_{\sigma=1}^{9} \big[\, R_2^\sigma \iota(D_x \eta_\sigma) - R_1^\sigma \iota(D_y \eta_\sigma) \,\big] = R_2^7 - R_1^4,
\end{aligned} \tag{4.14}$$

in which ξ_σ, η_σ are the coefficients of ∂_x, ∂_y, respectively, in the infinitesimal generator $\mathbf{v}_\sigma$. Substituting our formulas (4.10) for the Maurer–Cartan invariants yields

$$Y_1 = \tfrac{1}{3} I_{31} - \tfrac{1}{2} I_{21}^2 - \tfrac{1}{2} I_{12}, \qquad Y_2 = \tfrac{1}{3} I_{40} - I_{21}. \tag{4.15}$$

Now we investigate the structure of the algebra of differential invariants of a ternary form, with a particular interest in generating sets. To proceed, let us abbreviate

$$I = I_{21}, \qquad J = I_{12}, \qquad K = I_{03}. \tag{4.16}$$

The recurrence formulae (4.11), (4.12), combined with a general theorem, [7], imply that the differential invariant algebra $\mathcal{I}$ of ternary forms is generated by I, J, K by repeatedly applying the invariant differentiation operators $\mathcal{D}_1, \mathcal{D}_2$. In fact, we can prove more:

Theorem 2 *The single differential invariant $I = I_{21}$ generates the entire differential invariant algebra $\mathcal{I}$ of a general ternary form through invariant differentiation.*

Proof: During the proof, we will use lower case Greek letters — $\alpha, \beta, \ldots, \zeta$ — to denote functions of I and its invariant derivatives. Thus, they will not denote group parameters in what follows.

First recall the "commutator trick", [25]. In view of the commutator formula (4.13), we can write

$$\begin{aligned}
\mathcal{D}_1 \mathcal{D}_2 I - \mathcal{D}_2 \mathcal{D}_1 I &= Y_1 \mathcal{D}_1 I + Y_2 \mathcal{D}_2 I, \\
\mathcal{D}_1 \mathcal{D}_2 \mathcal{D}_N I - \mathcal{D}_2 \mathcal{D}_1 \mathcal{D}_N I &= Y_1 \mathcal{D}_1 \mathcal{D}_N I + Y_2 \mathcal{D}_2 \mathcal{D}_N I,
\end{aligned} \tag{4.17}$$

where $\mathcal{D}_N I = \mathcal{D}_{n_1}\mathcal{D}_{n_2}\cdots\mathcal{D}_{n_k} I$, with $n_j = 1$ or 2, is any invariant derivative of I. Treating (4.17) as a pair of linear equations for Y_1, Y_2, and assuming non-vanishing of the determinant

$$\mathcal{D}_1 I\,\mathcal{D}_2\mathcal{D}_N I - \mathcal{D}_2 I\,\mathcal{D}_1\mathcal{D}_N I \neq 0, \tag{4.18}$$

allows us to write the the commutator invariants as rational combinations of I and its invariant derivatives. In particular, setting $N = (1)$, say, yields

$$\begin{aligned} Y_1 = \eta_1 &= \frac{(\mathcal{D}_1\mathcal{D}_2 I - \mathcal{D}_2\mathcal{D}_1 I)\mathcal{D}_2\mathcal{D}_1 I - (\mathcal{D}_1\mathcal{D}_2\mathcal{D}_1 I - \mathcal{D}_2\mathcal{D}_1^2 I)\mathcal{D}_2 I}{\mathcal{D}_1 I\,\mathcal{D}_2\mathcal{D}_1 I - \mathcal{D}_2 I\,\mathcal{D}_1^2 I}, \\ Y_2 = \eta_2 &= \frac{(\mathcal{D}_1\mathcal{D}_2\mathcal{D}_1 I - \mathcal{D}_2\mathcal{D}_1^2 I)\mathcal{D}_1 I - (\mathcal{D}_1\mathcal{D}_2 I - \mathcal{D}_2\mathcal{D}_1 I)\mathcal{D}_1^2 I}{\mathcal{D}_1 I\,\mathcal{D}_2\mathcal{D}_1 I - \mathcal{D}_2 I\,\mathcal{D}_1^2 I}, \end{aligned} \tag{4.19}$$

bearing mind our convention about lower case Greek letters. A routine but tedious calculation shows that the denominators in (4.19) are non-vanishing differential functions.

Remark: By the same argument as was used for Euclidean surfaces, [27], the only ternary forms for which all the determinants (4.18) vanish identically, are so are not amenable to the commutator trick, are those for which

$$\mathcal{D}_1 I = \phi_1(I), \qquad \mathcal{D}_2 I = \phi_2(I), \tag{4.20}$$

for scalar functions ϕ_i. It would be interesting to explicitly characterize such degenerate ternary forms.

Thus, according to (4.15),

$$I_{31} = \tfrac{3}{2} J + \widehat{\eta}_1, \quad I_{40} = \widehat{\eta}_2, \qquad \text{where} \qquad \widehat{\eta}_1 = 3\eta_1 + \tfrac{3}{2} I^2, \quad \widehat{\eta}_2 = 3\eta_2 + 3I. \tag{4.21}$$

Next, the second recurrence formula in (4.11) for $I_{21} = I$ implies

$$\mathcal{D}_2 I = I_{22} - \tfrac{1}{2} K - \tfrac{5}{2} I J + \varphi, \qquad \text{where} \qquad \varphi = -\tfrac{1}{3} I\,\widehat{\eta}_1 + \tfrac{1}{2} I^3 - 2 + \frac{4}{n}.$$

Hence,

$$I_{22} = \tfrac{1}{2} K + \tfrac{5}{2} I J + \widehat{\varphi}, \qquad \text{where} \qquad \widehat{\varphi} = -\varphi + \mathcal{D}_2 I.$$

Substituting this expression into the third formula in (4.11) for $I_{12} = J$ yields

$$\mathcal{D}_1 J = I_{22} - \tfrac{1}{2} K - \tfrac{7}{2} I J + \tfrac{1}{3} J\,\widehat{\eta}_2 - 2 + \frac{4}{n} = \alpha J + \beta, \quad \text{where} \quad \alpha = \tfrac{1}{3}\widehat{\eta}_2 - I, \quad \beta = \widehat{\varphi} - 2 + \frac{4}{n}. \tag{4.22}$$

Next, differentiating (4.22) yields

$$\mathcal{D}_1 I_{31} - \mathcal{D}_2 I_{40} = \tfrac{3}{2}\mathcal{D}_1 J + \mathcal{D}_1\widehat{\eta}_1 - \mathcal{D}_2\widehat{\eta}_2 = \tfrac{3}{2}\alpha J + \zeta, \quad \text{where} \quad \zeta = \mathcal{D}_1\widehat{\eta}_1 - \mathcal{D}_2\widehat{\eta}_2 + \tfrac{3}{2}\beta. \tag{4.23}$$

On the other hand, the fourth order recurrence formulae (4.12) imply that

$$\begin{gathered} \mathcal{D}_1 I_{31} - \mathcal{D}_2 I_{40} = \tfrac{2}{3} I_{31} I_{40} - \tfrac{3}{2} I_{22} + 2 I I_{31} - \tfrac{1}{2} J I_{40} - 2 I^2 I_{40} + 3 - \frac{9}{n} = -\tfrac{3}{4} K + \lambda J + \mu, \\ \text{where} \qquad \lambda = \tfrac{1}{2}\widehat{\eta}_2 - \tfrac{3}{4} I, \qquad \mu = \tfrac{2}{3}\widehat{\eta}_1\widehat{\eta}_2 - \tfrac{3}{2}\widehat{\varphi} + 2I\widehat{\eta}_1 - 2I^2\widehat{\eta}_2 + 3 - \frac{9}{n}. \end{gathered} \tag{4.24}$$

Comparing (4.23), (4.24) yields

$$K = \widehat{\lambda}\, J + \widehat{\mu}, \qquad \text{where} \qquad \widehat{\lambda} = \tfrac{4}{3}\lambda - 2\alpha, \qquad \widehat{\mu} = \tfrac{4}{3}(\mu - \zeta). \tag{4.25}$$

Next, again using (4.11),

$$\begin{aligned} \mathcal{D}_1 K - \mathcal{D}_2 J &= \bigl(I_{13} + K\, I_{40} - \tfrac{9}{2} I\, K - \tfrac{3}{2} J^2 \bigr) - \bigl(I_{13} + \tfrac{1}{3}\, J\, I_{31} - \tfrac{3}{2} I\, K - 2\, J^2 - \tfrac{1}{2} I^2\, J \bigr) \\ &= -\, J\, Y_1 + 3\, K\, Y_2 = \rho\, J + \sigma, \end{aligned} \tag{4.26}$$

where, in view of (4.19), (4.25),

$$\rho = 3\, \widehat{\lambda}\, \eta_2 - \eta_1, \qquad \sigma = 3\, \widehat{\mu}\, \eta_2.$$

On the other hand, differentiating (4.25) and using (4.22),

$$\mathcal{D}_1 K = \kappa\, J + \tau, \qquad \text{where} \qquad \kappa = \mathcal{D}_1 \widehat{\lambda} + \widehat{\lambda}\, \alpha, \qquad \tau = \mathcal{D}_1 \widehat{\mu} + \widehat{\lambda}\, \beta.$$

Thus, comparing with (4.26), we find

$$\mathcal{D}_2 J = \gamma\, J + \delta, \qquad \text{where} \qquad \gamma = \kappa - \rho, \qquad \delta = \tau - \sigma. \tag{4.27}$$

Finally, cross differentiating (4.22), (4.27) yields

$$\mathcal{D}_1 \mathcal{D}_2 J - \mathcal{D}_2 \mathcal{D}_1 J = (\mathcal{D}_1 \gamma - \mathcal{D}_2 \alpha)\, J + \mathcal{D}_1 \delta - \mathcal{D}_2 \beta + \beta\, \gamma - \alpha\, \delta.$$

On the other hand, (4.13), (4.19) combined with (4.22), (4.27) says

$$\mathcal{D}_1 \mathcal{D}_2 J - \mathcal{D}_2 \mathcal{D}_1 J = Y_1\, \mathcal{D}_1 J + Y_2\, \mathcal{D}_2 J = (\alpha\, \eta_1 + \gamma\, \eta_2) J + \beta\, \eta_1 + \delta\, \eta_2.$$

These last two identities imply

$$\omega\, J + \theta = 0, \qquad \text{where} \qquad \begin{aligned} \omega &= \mathcal{D}_1 \gamma - \mathcal{D}_2 \alpha - \alpha\, \eta_1 - \gamma\, \eta_2, \\ \theta &= \mathcal{D}_1 \delta - \mathcal{D}_2 \beta + \beta\, \gamma - \alpha\, \delta - \beta\, \eta_1 - \delta\, \eta_2. \end{aligned}$$

Thus, provided $\omega \neq 0$, we can solve for

$$J = -\, \theta / \omega \tag{4.28}$$

as a rather complicated but explicit rational function of I and its derivatives. Substituting this formula into (4.25) proves the same is true of K, thereby proving the claim. To verify that ω is not identically zero, we merely note that, when the third order expressions (4.19) are employed, $\mathcal{D}_1 \gamma$ evidently depends upon fifth order derivatives of I, while all the other terms in ω involve at most fourth order derivatives. On the other hand, ω may well vanish for special forms; again, their explicit characterization would be of interest. *Q.E.D.*

It is also of interest to establish, by a slightly different argument, an alternative generating differential invariant.

Theorem 3 *The single differential invariant $K = I_{03}$ generates the entire differential invariant algebra $\mathcal{I}$ of a general ternary form through invariant differentiation.*

Remark: We believe the same is also true for $J = I_{12}$ although we have not tried to work through the details.

Remark: The one difference between Theorem 2 and Theorem 3 is that, whereas I generates within the category of rational differential invariant algebras — see the remarks after (4.28) — the proof that K generates relies on the solution to a system of quadratic equations and the resulting expressions for the other differential invariants are algebraic functions of K and its invariant derivatives.

Proof: Applying the preceding "commutator trick", replacing I by K in (4.17) and (4.19), allows us to alternatively express the commutator invariants Y_1, Y_2 as rational combinations of invariant derivatives of K:

$$\begin{aligned} Y_1 &= \frac{(\mathcal{D}_1\mathcal{D}_2K - \mathcal{D}_2\mathcal{D}_1K)\mathcal{D}_2\mathcal{D}_1K - (\mathcal{D}_1\mathcal{D}_2\mathcal{D}_1K - \mathcal{D}_2\mathcal{D}_1^2K)\mathcal{D}_2K}{\mathcal{D}_1K\,\mathcal{D}_2\mathcal{D}_1K - \mathcal{D}_2K\,\mathcal{D}_1^2K}, \\ Y_2 &= \frac{(\mathcal{D}_1\mathcal{D}_2\mathcal{D}_1K - \mathcal{D}_2\mathcal{D}_1^2K)\mathcal{D}_1K - (\mathcal{D}_1\mathcal{D}_2K - \mathcal{D}_2\mathcal{D}_1K)\mathcal{D}_1^2K}{\mathcal{D}_1K\,\mathcal{D}_2\mathcal{D}_1K - \mathcal{D}_2K\,\mathcal{D}_1^2K}, \end{aligned} \tag{4.29}$$

Thus, from (4.15), we can write

$$I_{40} = 3\,Y_2 + 3\,I = \rho_1(I,J;K^{(3)}), \qquad I_{31} = 3\,Y_1 + \tfrac{3}{2}\,I^2 + \tfrac{3}{2}\,J = \rho_2(I,J;K^{(3)}), \tag{4.30}$$

where ρ_1, ρ_2 are quadratic (or linear) functions of the invariants I, J whose coefficients are rational combinations of K and its invariant derivatives. In subsequent calculations during this proof, all the indicated functions ρ_j will be of a similar form.

The last two recurrence relations in (4.11) combined with (4.30) allow us to write

$$\begin{aligned} I_{13} &= \mathcal{D}_1K - K\,I_{40} + \tfrac{9}{2}I\,K + \tfrac{3}{2}J^2 &&= \rho_3(I,J;K^{(3)}), \\ I_{04} &= \mathcal{D}_2K - K\,I_{31} + \tfrac{9}{2}J\,K + \tfrac{3}{2}I^2K &&= \rho_4(I,J;K^{(3)}). \end{aligned} \tag{4.31}$$

Then, by use of (4.30), (4.31), the first four recurrence relations in (4.11) can be rewritten in the form

$$\begin{aligned} \mathcal{D}_1I &= \rho_5(I,J;K^{(3)}), & \mathcal{D}_1J &= I_{22} + \rho_7(I,J;K^{(3)}), \\ \mathcal{D}_2I &= I_{22} + \rho_6(I,J;K^{(3)}), & \mathcal{D}_2J &= \rho_8(I,J;K^{(3)}). \end{aligned} \tag{4.32}$$

Next, according to (4.12), and applying the same substitutions (4.30), (4.31), we have

$$\mathcal{D}_2I_{40} - \mathcal{D}_1I_{31} = \tfrac{3}{2}\,I_{22} + \rho_9(I,J;K^{(3)}). \tag{4.33}$$

On the other hand, invariantly differentiating the expressions (4.30) and substituting for the derivatives of I, J wherever they occur according to (4.32) produces a formula of the form

$$\mathcal{D}_2I_{40} - \mathcal{D}_1I_{31} = \tfrac{3}{2}\,I_{22} + \rho_{10}(I,J;K^{(4)}). \tag{4.34}$$

Clearly, $\rho_{11} = \rho_{10} - \rho_9 \not\equiv 0$, and so, after substituting for Y_1, Y_2 according to (4.29), we derive a nontrivial rational relation that takes the explicit form

$$0 = \rho_{11}(I,J;K^{(4)}) = -\tfrac{3}{4}I\,J + \tfrac{3}{4}K + 3\,\mathcal{D}_1Y_1 - 3\,\mathcal{D}_2Y_2 - 6\,Y_1Y_2 + 3/n. \tag{4.35}$$

Furthermore, if we invariantly differentiate (4.35) and substitute for the derivatives of I, J according to (4.32), we derive expressions of the form

$$\begin{aligned} 0 &= \mathcal{D}_1\rho_{11} = -\tfrac{3}{4}\,I\,I_{22} + \rho_{12}(I,J;K^{(5)}), \\ 0 &= \mathcal{D}_2\rho_{11} = -\tfrac{3}{4}\,J\,I_{22} + \rho_{13}(I,J;K^{(5)}). \end{aligned} \tag{4.36}$$

Setting $\rho_{14} = J\,\rho_{12} - I\,\rho_{13}$ produces a second quadratic relation, which, after a tedious computation, can be shown to have the explicit form

$$\begin{aligned}\rho_{14}(I,J;K^{(5)}) = \bigl(\tfrac{3}{4}\mathcal{D}_1 K - \tfrac{9}{4}K\,Y_2\bigr)I^2 + \bigl(-3\,\mathcal{D}_2\mathcal{D}_1 Y_1 + 3\,\mathcal{D}_2^2 Y_2 + 6\,\mathcal{D}_2(Y_1Y_2) - \tfrac{3}{4}\mathcal{D}_2 K\bigr)I \\ - \tfrac{9}{4}Y_1 J^2 + \bigl(3\,\mathcal{D}_1^2 Y_1 - 3\,\mathcal{D}_1\mathcal{D}_2 Y_2 - 6\,\mathcal{D}_2(Y_1Y_2) + \tfrac{3}{4}\mathcal{D}_1 K\bigr)J = 0.\end{aligned} \tag{4.37}$$

Clearly ρ_{11}, ρ_{14} are functionally independent as quadratic functions of I, J, and so we can locally solve (4.35), (4.37) for I, J as functions K and its invariant derivatives up to order 5:

$$I = \sigma_1(K^{(5)}), \qquad J = \sigma_2(K^{(5)}). \tag{4.38}$$

In fact, a MATHEMATICA calculation demonstrates that the functions σ_1, σ_2 can be written as explicit complicated rational algebraic combinations of K and its derivatives. Thus, we have shown how to generate both I and J directly from K, which implies that K indeed (algebraically) generates the entire differential invariant algebra. *Q.E.D.*

In Irina Kogan's thesis, [14], a more symmetric, but non-minimal order cross-section was used to construct the differential invariants, namely

$$\widehat{\mathcal{K}} = \bigl\{\, x = y = 0, \quad u = 1, \quad u_x = u_y = u_{xx} = u_{yy} = 0, \quad u_{xxx} = u_{yyy} = 1 \,\bigr\}. \tag{4.39}$$

Let us label the differential invariants resulting from this cross-section as $\widehat{I}_{jk} = \widehat{\iota}(u_{jk})$.

We shall apply Theorem 1 to determine the relationship between the two sets of differential invariants associated with the preceding cross-sections (4.6) and (4.39). To this end, let us determine the group elements that map $\widehat{\mathcal{K}}$ to $\mathcal{K}$, as in (2.6). A direct calculation[5] shows that a group element sufficiently close to the identity[6] preserves the common equations

$$x = y = 0, \qquad u = 1, \qquad u_x = u_y = u_{xx} = u_{yy} = 0, \qquad u_{xxx} = 1,$$

if and only if it has the diagonal form

$$g = \begin{pmatrix} \alpha & \beta & \gamma \\ \lambda & \mu & \nu \\ \rho & \sigma & \tau \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{4.40}$$

Such elements act by simple scaling of the jet coordinates (4.3):

$$u_{jk} \;\longmapsto\; \frac{u_{jk}}{\mu^k}. \tag{4.41}$$

Thus, g maps $\widehat{\mathcal{K}}$ to $\mathcal{K}$ if and only if

$$u_{xy} \longmapsto u_{xy}/\mu = 1, \qquad \text{and hence} \qquad \mu = u_{xy}.$$

This implies

$$\rho(z^{(3)}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & u_{xy} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{for any } z^{(3)} = (0,0,1,0,0,0,u_{xy},0,1,u_{xxy},u_{xyy},1) \in \widehat{\mathcal{K}}. \tag{4.42}$$

[5] One can determine values for the matrix entries order by order to simplify the computations; for example the fact that both cross-sections have $x = y = 0$ implies $\gamma = \nu = 0$, and one can then restrict to this subgroup when determining the first order prolonged action.

[6] This restriction is needed because of local freeness.

In view of (2.5), (4.41), (4.42), the relationship between the the two sets of normalized differential invariants is given explicitly by

$$I_{jk} = \frac{\widehat{I}_{jk}}{(\widehat{I}_{11})^k}. \tag{4.43}$$

In particular,

$$I_{30} = \widehat{I}_{30} = 1, \qquad I_{21} = \frac{\widehat{I}_{21}}{\widehat{I}_{11}}, \qquad I_{12} = \frac{\widehat{I}_{12}}{(\widehat{I}_{11})^2}, \qquad I_{03} = \frac{\widehat{I}_{03}}{(\widehat{I}_{11})^3} = \frac{1}{(\widehat{I}_{11})^3}. \tag{4.44}$$

Keep in mind that the invariant differential operators and hence the commutator invariants are different for the two cross-sections. Indeed, for a group element of the simple form (4.40),

$$dx \longmapsto dx, \qquad dy \longmapsto \mu\, dy.$$

Invariantizing these horizontal forms and dual differential operators implies that two invariant differential operators are related by

$$\mathcal{D}_1 = \widehat{\mathcal{D}}_1, \qquad \mathcal{D}_2 = \frac{1}{\widehat{I}_{11}}\, \widehat{\mathcal{D}}_2. \tag{4.45}$$

As an immediate consequence of Theorem 3, we deduce that the differential invariant algebra is generated by the differential invariant $\widehat{I}_{11} = I_{03}^{-1/3} = K^{-1/3}$.

The explicit formulas for the lowest order differential invariants of a ternary form were found in Irina Kogan's thesis, [14]; see also [15]. Since these have not all appeared in print, we collect them here. Given a ternary form $u(x,y)$, we begin by defining the following differential functions:

$$\begin{aligned}
Q_{20} &= u^{2(1/n-1)} \left[u\, u_{20} - \frac{n-1}{n}\, u_{10}^2 \right], \\
Q_{11} &= u^{2(1/n-1)} \left[u\, u_{11} - \frac{n-1}{n}\, u_{10}\, u_{01} \right], \\
Q_{02} &= u^{2(1/n-1)} \left[u\, u_{02} - \frac{n-1}{n}\, u_{01}^2 \right], \\
Q_{30} &= u^{3(1/n-1)} \left[u^2\, u_{30} - 3\frac{n-2}{n}\, u\, u_{10}\, u_{20} + 2\frac{(n-1)(n-2)}{n^2}\, u_{10}^3 \right], \\
Q_{21} &= u^{3(1/n-1)} \left[u^2\, u_{21} - \frac{n-2}{n}\, u\,(u_{01}\, u_{20} + 2\, u_{10}\, u_{11}) + 2\, \frac{(n-1)(n-2)}{n^2}\, u_{10}^2\, u_{01} \right], \\
Q_{12} &= u^{3(1/n-1)} \left[u^2\, u_{12} - \frac{n-2}{n}\, u\,(2\, u_{01}\, u_{11} + u_{10}\, u_{02}) + 2\frac{(n-1)(n-2)}{n^2}\, u_{10}\, u_{01}^2 \right], \\
Q_{03} &= u^{3(1/n-1)} \left[u^2\, u_{03} - 3\frac{n-2}{n}\, u\, u_{01} u_{02} + 2\frac{(n-1)(n-2)}{n^2}\, u_{01}^3 \right].
\end{aligned} \tag{4.46}$$

Set

$$
\begin{aligned}
P_2 &= Q_{20}Q_{02} - Q_{11}^2, \\
P_3 &= Q_{30}^2Q_{03}^2 - 6\,Q_{30}Q_{21}Q_{12}Q_{03} + 4\,Q_{30}Q_{12}^3 + 4\,Q_{21}^3Q_{03} - 3\,Q_{21}^2Q_{12}^2, \\
M_1 &= Q_{30}Q_{12}Q_{02} - Q_{30}Q_{03}Q_{11} - Q_{21}^2Q_{02} + Q_{21}Q_{12}Q_{11} + Q_{21}Q_{03}Q_{20} - Q_{12}^2Q_{20}, \\
M_2 &= 5\,Q_{30}^2Q_{02}^3 - 30\,Q_{30}Q_{21}Q_{11}Q_{02}^2 + 6\,Q_{30}Q_{12}Q_{20}Q_{02}^2 + 24\,Q_{30}Q_{12}Q_{11}^2Q_{02} - \\
&\qquad - 6\,Q_{30}Q_{03}Q_{20}Q_{11}Q_{02} - 4\,Q_{30}Q_{03}Q_{11}^3 + 9\,Q_{21}^2Q_{20}Q_{02}^2 + 36\,Q_{21}^2Q_{11}^2Q_{02} - \\
&\qquad - 54\,Q_{21}Q_{12}Q_{20}Q_{11}Q_{02} - 36\,Q_{21}Q_{12}Q_{11}^3 + 6\,Q_{21}Q_{03}Q_{20}^2Q_{02} + 9\,Q_{12}^2Q_{20}^2Q_{02} + \\
&\qquad + 24\,Q_{21}Q_{03}Q_{20}Q_{11}^2 + 36\,Q_{12}^2Q_{20}Q_{11}^2 - 30\,Q_{12}Q_{03}Q_{20}^2Q_{11} + 5\,Q_{03}^2Q_{20}^3.
\end{aligned} \tag{4.47}
$$

Then,

$$
\begin{aligned}
\widehat{I}_1 &= \frac{\widehat{I}_{21}\widehat{I}_{12} - 1}{(\widehat{I}_{11})^3} = \frac{M_1}{P_2^2}\,, \qquad \widehat{I}_2 = -4\,\frac{9\,\widehat{I}_{21}\widehat{I}_{12} + 1}{(\widehat{I}_{11})^3} = \frac{M_2}{P_2^3}\,, \\
\widehat{I}_3 &= \frac{1 - 3\,(\widehat{I}_{21}\widehat{I}_{12})^2 - 6\,\widehat{I}_{21}\widehat{I}_{12} + 4\,(\widehat{I}_{21})^3 + 4\,(\widehat{I}_{12})^3}{(\widehat{I}_{11})^6} = \frac{P_3}{P_2^3}\,.
\end{aligned} \tag{4.48}
$$

The algebraic equations (4.48) can be solved to give the three generating differential invariants $\widehat{I}_{11}, \widehat{I}_{21}, \widehat{I}_{12}$, or, equivalently through (4.44), our differential invariants I_{21}, I_{12}, I_{03}. The resulting formulas are rational algebraic functions of the form u and its derivatives of order ≤ 3. The ambiguities in these functions stem from the fact that the action of GL(3) on J^3 is only locally free; according to [31], the cardinality of the isotropy subgroup of an individual point on the cross-section (4.39) is 18.

The formulas for the invariant differential operators are not explicitly displayed in [14, 15], but can be reconstructed by substituting their moving frame formulas for the group parameters into the implicit differentiation operators (4.4).

5 Joint Differential Invariants of Ternary Forms

This final section is devoted to the study of joint differential invariants of ternary forms. The GL(3) transformations (4.1) now act on multiple points $z^k = (x^k, y^k, u^k)$, $k = 0, 1, \ldots$. To simplify the subsequent notation, we define the quantities

$$
A^{ijk} = x^i(y^j - y^k) + x^j(y^k - y^i) + x^k(y^i - y^j), \tag{5.1}
$$

which are skew symmetric in their indices.

We begin by studying the case when there are ≥ 3 points, leaving the more difficult 2 point case until the end of this section. For the Cartesian product action on $M^{\times(n+1)}$ for $n \geq 2$, a simple cross section is given by

$$
\begin{aligned}
x^0 &= 0, &\qquad y^0 &= 0, &\qquad u^0 &= 1, \\
x^1 &= 0, & y^1 &= 1, & u^1 &= 1, \\
x^2 &= 1, & y^2 &= 1, & u^2 &= 1.
\end{aligned} \tag{5.2}
$$

The solution of the corresponding normalization equations gives the following right moving frame:

$$
\begin{aligned}
\alpha &= \frac{(y^0-y^1)\sqrt[n]{u^2}}{A^{012}}, \qquad \beta = \frac{(x^1-x^0)\sqrt[n]{u^2}}{A^{012}}, \qquad \gamma = \frac{(x^0y^1-x^1y^0)\sqrt[n]{u^2}}{A^{012}},\\
\lambda &= \frac{(y^2-y^0)\sqrt[n]{u^1}+(y^0-y^1)\sqrt[n]{u^2}}{A^{012}}, \qquad \mu = \frac{(x^0-x^2)\sqrt[n]{u^1}+(x^1-x^0)\sqrt[n]{u^2}}{A^{012}},\\
\nu &= \frac{(x^2y^0-x^0y^2)\sqrt[n]{u^1}+(x^0y^1-x^1y^0)\sqrt[n]{u^2}}{A^{012}},\\
\rho &= \frac{(y^1-y^2)\sqrt[n]{u^0}+(y^2-y^0)\sqrt[n]{u^1}+(y^0-y^1)\sqrt[n]{u^2}}{A^{012}},\\
\sigma &= \frac{(x^2-x^1)\sqrt[n]{u^0}+(x^0-x^2)\sqrt[n]{u^1}+(x^1-x^0)\sqrt[n]{u^2}}{A^{012}},\\
\tau &= \frac{(x^1y^2-x^2y^1)\sqrt[n]{u^0}+(x^2y^0-x^0y^2)\sqrt[n]{u^1}+(x^0y^1-x^1y^0)\sqrt[n]{u^2}}{A^{012}}.
\end{aligned}
\tag{5.3}
$$

As before, we ignore the branching ambiguity in the n-th roots that is a consequence of the local freeness of the action; for instance, we can restrict our attention to the case of real, positive forms. Then, substituting (5.3) into (4.1) produces a complete system of joint invariants:

$$
\begin{aligned}
H^k &= \iota(x^k) = \frac{(x^2-x^1)y^k\sqrt[n]{u^0}+(x^0-x^2)y^k\sqrt[n]{u^1}+A^{01k}\sqrt[n]{u^2}}{B^k},\\
I^k &= \iota(y^k) = \frac{A^{0k2}\sqrt[n]{u^1}+A^{01k}\sqrt[n]{u^2}}{B^k}, \qquad J^k = \iota(u^k) = u^k\left(\frac{A^{012}}{B^k}\right)^n,
\end{aligned}
\tag{5.4}
$$

where

$$
B^k = \sqrt[n]{u^0}A^{12k}+\sqrt[n]{u^1}A^{20k}+\sqrt[n]{u^2}A^{01k}. \tag{5.5}
$$

The invariant differential operators for the joint differential invariants are given by invariantization of the total derivatives: $\mathcal{D}^k = \iota(D_{x^k})$, $\mathcal{E}^k = \iota(D_{y^k})$. As usual, these are found by substituting the moving frame expressions (5.3) into the implicit differentiation operators D_{X^k}, D_{Y^k}, which are simply obtained from (4.4) by replacing x, y by x^k, y^k. The explicit formulas are

$$
\begin{aligned}
\mathcal{D}^k &= \frac{B^k}{\sqrt[n]{u^0u^1u^2}A^{012}}\left(\begin{aligned}&\big[\sqrt[n]{u^0u^1}(x^k-x^2)+\sqrt[n]{u^0u^2}(x^1-x^k)\big]D_{x^k}\\ &\qquad +\big[\sqrt[n]{u^0u^1}(y^k-y^2)+\sqrt[n]{u^0u^2}(y^1-y^k)\big]D_{y^k}\end{aligned}\right),\\
\mathcal{E}^k &= \frac{B^k}{\sqrt[n]{u^0u^1u^2}A^{012}}\left(\begin{aligned}&\big[\sqrt[n]{u^0u^2}(x^k-x^1)+\sqrt[n]{u^1u^2}(x^0-x^k)\big]D_{x^k}\\ &\qquad +\big[\sqrt[n]{u^0u^2}(y^k-y^1)+\sqrt[n]{u^1u^2}(y^0-y^k)\big]D_{y^k}\end{aligned}\right).
\end{aligned}
\tag{5.6}
$$

Further, using the phantom recurrence formulae to calculate the Maurer–Cartan invariants, we deduce the following recurrence formulae for the joint differential invariants of ternary forms. Let

$$
K^k = \iota(u^k_x), \qquad L^k = \iota(u^k_y). \tag{5.7}
$$

Then, for $j,k \geq 2$,

$$\begin{aligned}
\mathcal{D}^0 H^k &= (I^k - 1)\left(\frac{1}{n} H^k K^0 + 1\right), & \mathcal{E}^0 H^k &= \frac{1}{n} H^k (I^k - 1) L^0, \\
\mathcal{D}^0 I^k &= \frac{1}{n} I^k (I^k - 1) K^0, & \mathcal{E}^0 I^k &= (I^k - 1)\left(\frac{1}{n} I^k L^0 + 1\right), \\
\mathcal{D}^0 J^k &= (I^k - 1) J^k K^0, & \mathcal{E}^0 J^k &= (I^k - 1) J^k L^0, \\
\mathcal{D}^1 H^k &= (H^k - I^k)\left(\frac{1}{n} H^k K^1 + 1\right), & \mathcal{E}^1 H^k &= \frac{1}{n} H^k (H^k - I^k) L^1, \\
\mathcal{D}^1 I^k &= \frac{1}{n} (H^k - I^k)(I^k - 1) K^1, & \mathcal{E}^1 I^k &= (H^k - I^k)\left(\frac{1}{n} (I^k - 1) L^1 + 1\right), \\
\mathcal{D}^1 J^k &= (H^k - I^k) J^k K^1, & \mathcal{E}^1 J^k &= (H^k - I^k) J^k L^1, \\
\mathcal{D}^2 H^k &= H^k \left(\frac{1}{n} (1 - H^k) K^2 - 1\right), & \mathcal{E}^2 H^k &= \frac{1}{n} H^k (1 - H^k) L^2, \\
\mathcal{D}^2 I^k &= \frac{1}{n} H^k (1 - I^k) K^2, & \mathcal{E}^2 I^k &= \frac{1}{n} H^k (1 - I^k) L^2 - H^k, \\
\mathcal{D}^2 J^k &= - H^k J^k K^2, & \mathcal{E}^2 J^k &= - H^k J^k L^2,
\end{aligned} \tag{5.8}$$

while, for $k \neq 0,1,2$,

$$\begin{aligned}
&\mathcal{D}^j H^k = \begin{cases} 1, & j = k, \\ 0, & j \neq k, \end{cases} \qquad \mathcal{D}^j I^k = 0, \qquad \mathcal{D}^j J^k = \begin{cases} K^k, & j = k, \\ 0, & j \neq k, \end{cases} \\
&\mathcal{E}^j H^k = 0, \qquad \mathcal{E}^j I^k = \begin{cases} 1, & j = k, \\ 0, & j \neq k, \end{cases} \qquad \mathcal{E}^j J^k = \begin{cases} L^k, & j = k, \\ 0, & j \neq k. \end{cases}
\end{aligned} \tag{5.9}$$

Finally, adapting (4.14) to the multi-point case, we calculate the commutators of the invariant differential operators (5.6). For $k,l \geq 3$,

$$\begin{aligned}
&[\,\mathcal{D}^0, \mathcal{E}^0\,] = \left(\frac{L^0}{n} + 1\right)\mathcal{D}^0 - \frac{K^0}{n}\,\mathcal{E}^0, \qquad [\,\mathcal{D}^0, \mathcal{D}^1\,] = -\mathcal{D}^0 + \frac{K^0}{n}\,\mathcal{E}^0, \\
&[\,\mathcal{D}^0, \mathcal{E}^1\,] = \left(\frac{L^1}{n} - 1\right)\mathcal{E}^0 + \mathcal{D}^1 + \frac{K^0}{n}\,\mathcal{E}^1, \qquad [\,\mathcal{D}^0, \mathcal{D}^2\,] = -\left(\frac{L^2}{n} - 1\right)\mathcal{D}^0 - \frac{K^2}{n}\,\mathcal{E}^0, \\
&[\,\mathcal{D}^0, \mathcal{E}^2\,] = -\frac{L^2}{n}\,\mathcal{D}^0 - \left(\frac{L^2}{n} - 1\right)\mathcal{E}^0 + \left(\frac{L^0}{n} + 1\right)\mathcal{D}^2 + \frac{K^0}{n}\,\mathcal{E}^2, \\
&[\,\mathcal{D}^0, \mathcal{D}^k\,] = \frac{(I^k - 1)K^0}{n}\,\mathcal{D}^k, \qquad [\,\mathcal{D}^0, \mathcal{E}^k\,] = \left(\frac{H^k L^0}{n} + 1\right)\mathcal{D}^k + \frac{(2I^k - 1)K^0}{n}\,\mathcal{E}^k, \\
&[\,\mathcal{E}^0, \mathcal{D}^1\,] = \mathcal{D}^0 - \frac{L^1}{n}\mathcal{E}^0, \qquad [\,\mathcal{E}^0, \mathcal{E}^1\,] = -\left(\frac{L^1}{n} - 1\right)\mathcal{E}^0 + \left(\frac{L^0}{n} + 1\right)\mathcal{E}^1, \\
&[\,\mathcal{E}^0, \mathcal{D}^2\,] = 0, \qquad [\,\mathcal{E}^0, \mathcal{E}^2\,] = \frac{L^0}{n}\,\mathcal{D}^2 + \left(\frac{L^0}{n} + 1\right)\mathcal{E}^2, \\
&[\,\mathcal{E}^0, \mathcal{D}^k\,] = \frac{(I^k - 1)L^0}{n}\,\mathcal{D}^k, \qquad [\,\mathcal{E}^0, \mathcal{E}^k\,] = \frac{H^k L^0}{n}\mathcal{D}^k + \left(\frac{(2\,I^k - 1)L^0}{n} + 1\right)\mathcal{E}^k,
\end{aligned} \tag{5.10}$$

$$
\begin{aligned}
[\,\mathcal{D}^1,\mathcal{E}^1\,] &= \left(\frac{L^1}{n}-1\right)\mathcal{D}^1-\left(\frac{K^1}{n}+1\right)\mathcal{E}^1,\\
[\,\mathcal{D}^1,\mathcal{D}^2\,] &= -\left(\frac{K^2}{n}-1\right)\mathcal{D}^1+\left(\frac{K^1}{n}+1\right)\mathcal{D}^2,\\
[\,\mathcal{D}^1,\mathcal{E}^2\,] &= -\frac{L^2}{n}\,\mathcal{D}^1+\mathcal{E}^1-\left(\frac{K^1}{n}+1\right)\mathcal{D}^2,\\
[\,\mathcal{D}^1,\mathcal{D}^k\,] &= \left(\frac{(2\,H^k-I^k)K^1}{n}+1\right)\mathcal{D}^k+\frac{(I^k-1)K^1}{n}\,\mathcal{E}^k,\\
[\,\mathcal{D}^1,\mathcal{E}^k\,] &= -\left(\frac{H^kK^1}{n}+1\right)\mathcal{D}^k+\frac{(H^k-2\,I^k+1)K^1}{n}\mathcal{E}^k,\\
[\,\mathcal{E}^1,\mathcal{D}^2\,] &= \frac{L^1}{n}\mathcal{D}^2+\mathcal{E}^2,\qquad [\,\mathcal{E}^1,\mathcal{E}^2\,] = -\frac{L^1}{n}\mathcal{D}^2-\mathcal{E}^2,\\
[\,\mathcal{E}^1,\mathcal{D}^k\,] &= \frac{(2\,H^k-I^k)L^1}{n}\mathcal{D}^k+\left(\frac{(I^k-1)L^1}{n}+1\right)\mathcal{E}^k,\\
[\,\mathcal{E}^1,\mathcal{E}^k\,] &= -\frac{H^kL^1}{n}\mathcal{D}^k+\left(\frac{(H^k-2\,I^k+1)L^1}{n}-1\right)\mathcal{E}^k,\\
[\,\mathcal{D}^2,\mathcal{E}^2\,] &= \frac{L^2}{n}\mathcal{D}^2-\left(\frac{K^2}{n}-1\right)\mathcal{E}^2,\\
[\,\mathcal{D}^2,\mathcal{D}^k\,] &= \left(\frac{(1-2H^k)K^2}{n}-1\right)\mathcal{D}^k+\frac{(1-I^k)K^2}{n}\,\mathcal{E}^k,\\
[\,\mathcal{D}^2,\mathcal{E}^k\,] &= -\frac{H^kK^2}{n}\mathcal{E}^k,\\
[\,\mathcal{E}^2,\mathcal{D}^k\,] &= \frac{(1-2H^k)L^2}{n}\mathcal{D}^k+\left(\frac{(1-I^k)L^2}{n}-1\right)\mathcal{E}^k,\\
[\,\mathcal{E}^2,\mathcal{E}^k\,] &= -\frac{H^kL^2}{n}\mathcal{E}^k,\\
[\,\mathcal{D}^k,\mathcal{D}^l\,] &= [\,\mathcal{D}^k,\mathcal{E}^l\,] = [\,\mathcal{E}^k,\mathcal{E}^l\,] = 0.
\end{aligned}
$$

In order to determine a minimal system of generators of the resulting joint differential invariant algebra, observe first that the recurrence relations (5.8), (5.9) allow us to generate all first order joint differential invariants K^j, L^j for $j \geq 0$ from the order 0 joint differential invariants H^k, I^k, J^k, $k \geq 2$. Moreover, according to (5.10), the latter all appear as combinations of commutator invariants. Thus, by applying the commutator trick used above, cf. [25], we can generate all the zeroth order joint differential invariants by differentiating any one of them, for example H^2 or I^2 or J^2. We have thus proved the following result.

Theorem 4 *The algebra of joint differential invariants of a k point ternary form for $k \geq 3$ is generated by a single joint invariant through invariant differentiation.*

Finally , let us investigate the two-point joint differential invariants, where one only has (x^0, y^0, u^0) and (x^1, y^1, u^1). In this case, since $\dim \mathrm{GL}(3) = 9$ the action is not free on $M^{\times 2}$, and so one must prolong to a Cartesian product jet space. The choice of cross-section is a little subtle. The action is transitive on the dense open subset of $M^{\times 2}$ consisting of linearly independent points with non-zero u coordinate, and so, as in (5.2), we can set

$$\begin{aligned} x^0 &= 0, & y^0 &= 0, & u^0 &= 1, \\ x^1 &= 0, & y^1 &= 1, & u^1 &= 1. \end{aligned} \tag{5.11}$$

Solving the resulting algebraic equations for the group parameters $\beta, \gamma, \mu, \nu, \sigma, \tau$ leads to the following partial moving frame:

$$\begin{aligned} \beta &= \frac{x^0 - x^1}{y^1 - y^0}\,\alpha, & \gamma &= \frac{x^1 y^0 - x^0 y^1}{y^1 - y^0}\,\alpha, \\ \mu &= \frac{(x^0 - x^1)\,\lambda + \sqrt[n]{u^1}}{y^1 - y^0}, & \nu &= \frac{(x^1 y^0 - x^0 y^1)\,\lambda - y^0\,\sqrt[n]{u^1}}{y^1 - y^0}, \\ \sigma &= \frac{(x^0 - x^1)\,\rho + \sqrt[n]{u^1} - \sqrt[n]{u^0}}{y^1 - y^0}, & \tau &= \frac{(x^1 y^0 - x^0 y^1)\,\rho + y^1\,\sqrt[n]{u^0} - y^0\,\sqrt[n]{u^1}}{y^1 - y^0}. \end{aligned} \tag{5.12}$$

Surprisingly, even though $\dim \mathrm{J}^1(M,2)^{\times 2} = 10 > \dim \mathrm{GL}(3) = 9$, the joint action of $\mathrm{GL}(3)$ on $\mathrm{J}^1(M,2)^{\times 2}$ is not free. Indeed, by computing the dimension of the subspace of the tangent space $T\mathrm{J}^1(M,2)^{\times 2}$ spanned by the prolonged infinitesimal generators, we find that the generic orbits are 8 dimensional, and hence there are, in fact, two first order joint differential invariants. Moreover, due to a subtle degeneracy, it is not possible to normalize both of the first order derivatives of either u^0 or u^1. Thus, we must prolong to order 2, and a cross-section is provided by combining (5.11) with the further normalizations

$$u^0_x = u^0_{xx} = 0, \qquad u^1_x = 1. \tag{5.13}$$

Substituting the partial moving frame formula (5.12), and solving the two first order normalization equations for α, λ produces

$$\alpha = \frac{(u^1)^{1/n-1}\big[\,\big(\,n^2 u^0 u^1 - S^0 S^1\,\big)\,\rho + W\,\big]}{T}, \qquad \lambda = \frac{\sqrt[n]{u^1}\big(-n\,u^0\rho + \sqrt[n]{u^0}u^0_x\big)}{T}, \tag{5.14}$$

where

$$\begin{aligned} & S^0 = (x^1 - x^0)u^0_x + (y^1 - y^0)u^0_y + n\,u^0, \qquad S^1 = (x^0 - x^1)u^1_x + (y^0 - y^1)u^1_y + n\,u^1, \\ & T = \sqrt[n]{u^0}\,S^0 - n\,u^0\,\sqrt[n]{u^1}, \qquad W = \sqrt[n]{u^0}\,(S^0 u^1_x - n\,u^1 u^0_x) - \sqrt[n]{u^1}(S^1 u^0_x - n\,u^0 u^1_x). \end{aligned} \tag{5.15}$$

Finally, the remaining second order normalization produces

$$\rho = \frac{(y^1 - y^0)\,T\sqrt{X} + n\,u^0\,Y + n(n-1)\,u^0 u^0_x\,\big(\,\sqrt[n]{u^0} - \sqrt[n]{u^1}\,\big)(S^0 - n\,u^0)}{Z}, \tag{5.16}$$

where

$$\begin{aligned} X &= n^2\,(u^0)^2\big[\,(u^0_{xy})^2 - u^0_{xx}u^0_{yy}\,\big] + n(n-1)u^0\big[\,(u^0_y)^2 u^0_{xx} - 2\,u^0_x u^0_y u^0_{xy} + (u^0_x)^2 u^0_{yy}\,\big], \\ Y &= (x^1 - x^0)\big[\,\sqrt[n]{u^0}\,\big(\,(y^1 - y^0)u^0_y + n\,u^0\,\big) - n\,u^0\,\sqrt[n]{u^1}\,\big]u^0_{xx} - (y^1 - y^0)^2\,\sqrt[n]{u^0}u^0_x u^0_{yy} - \\ &\qquad - (y^1 - y^0)\big[\,\sqrt[n]{u^0}\big(\,(x^1 - x^0)u^0_x - (y^1 - y^0)u^0_y - n\,u^0\,\big) + n\,u^0\,\sqrt[n]{u^1}\,\big]u^0_{xy}, \\ Z &= -n^2\,(u^0)^2\big[\,(x^1 - x^0)^2 u^0_{xx} + 2(x^1 - x^0)(y^1 - y^0)u^0_{xy} + (y^1 - y^0)^2 u^0_{yy}\,\big] + \\ &\qquad + n(n-1)\,u^0\big[\,(x^1 - x^0)^2(u^0_x)^2 + 2(x^1 - x^0)(y^1 - y^0)u^0_x u^0_y + (y^1 - y^0)^2(u^0_y)^2\,\big]. \end{aligned} \tag{5.17}$$

Substituting (5.16) back into (5.14) and then the results into (5.12) produces the explicit formulae for the moving frame which, because they are quite complicated, we do not write out.

Let us denote the resulting differential invariants by

$$L^k = \iota(u^k_y), \qquad P^k = \iota(u^k_{xx}), \qquad Q^k = \iota(u^k_{xy}), \qquad R^k = \iota(u^k_{yy}), \qquad k = 0, 1. \tag{5.18}$$

In particular, $P^0 = 0$ is a phantom invariant. The two first order joint differential invariants are

$$L^0 = (u^0)^{1/n-1}(u^1)^{-1/n}\, S^0 - n, \qquad L^1 = -\,(u^0)^{-1/n}(u^1)^{1/n-1}\, S^1 + n. \tag{5.19}$$

Of course, we can drop the constant terms without changing any of the subsequent results. The second order differential invariants are more complicated, but can be derived by invariant differentiation using the recurrence relations. For this, the invariant differential operators are

$$\begin{gathered}
\mathcal{D}^0 = \frac{\sqrt[n]{u^0/u^1}\,u^1}{\big(\,n^2u^0u^1 - S^0S^1\,\big)\,\rho + W}\,\widetilde{\mathcal{D}}^0, \qquad \mathcal{D}^1 = \frac{u^1}{\big(\,n^2u^0u^1 - S^0S^1\,\big)\,\rho + W}\,\widetilde{\mathcal{D}}^1, \\
\mathcal{E}^0 = \sqrt[n]{u^0/u^1}\,\big[\,(x^1-x^0)D_{x^0} + (y^1-y^0)D_{y^0}\,\big], \\
\mathcal{E}^1 = \sqrt[n]{u^1/u^0}\,\big[\,(x^1-x^0)D_{x^1} + (y^1-y^0)D_{y^1}\,\big],
\end{gathered} \tag{5.20}$$

where

$$\begin{aligned}
\widetilde{\mathcal{D}}^0 &= \big[\,n\,u^0\big(\,(x^1-x^0)\rho - \sqrt[n]{u^1}\,\big) + \sqrt[n]{u^0}\big(\,(y^1-y^0)u^0_y + n\,u^0\,\big)\,\big]D_{x^0} \\
&\qquad\qquad + (y^1-y^0)\big[\,n\,u^0\rho - \sqrt[n]{u^0}u^0_x\,\big]D_{y^0}, \\
\widetilde{\mathcal{D}}^1 &= \big[\,S^0\big(\,(x^1-x^0)\rho + \sqrt[n]{u^0}\,\big) - \sqrt[n]{u^1}\,\big(\,(x^1-x^0)u^0_x + n\,u^0\,\big)\,\big]D_{x^0} \\
&\qquad\qquad + (y^1-y^0)\big[\,S^0\rho - \sqrt[n]{u^1}u^0_x\,\big]D_{y^0},
\end{aligned} \tag{5.21}$$

when ρ is replaced by its moving frame formula (5.16). The resulting expressions are then fully symmetrical with respect to to permutation of indices, but in that form are much longer and more cumbersome to write down.

We use the phantom recurrence formulae to calculate the Maurer–Cartan invariants, and then substitute the resulting expressions to derive the first order recurrence relations

$$\begin{aligned}
\mathcal{D}^0L^0 &= Q^0, & \mathcal{E}^0L^0 &= R^0 - \frac{n-1}{n}\,(L^0)^2, \\
\mathcal{D}^1L^0 &= -\frac{1}{n}\,L^0 - 1, & \mathcal{E}^1L^0 &= -\frac{1}{n}\,L^0L^1 + L^0 - L^1, \\
\mathcal{D}^0L^1 &= -1, & \mathcal{E}^0L^1 &= -\frac{1}{n}\,L^0L^1 + L^0 - L^1, \\
\mathcal{D}^1L^1 &= Q^1 - \frac{n-1}{n}\,L^1, & \mathcal{E}^1L^1 &= R^1 - \frac{n-1}{n}\,(L^1)^2.
\end{aligned} \tag{5.22}$$

Thus, starting with the first order joint differential invariant L^0, we can generate the other first order joint differential invariant L^1, and then use those to generate the second order joint differential invariants Q^0, R^0, Q^1, R^1. In order to generate the remaining second order joint differential invariant P^1, we use some of the second order recurrence formulae:

$$
\begin{aligned}
&\mathcal{D}^0 R^0 = U^0 - 2Q^0, \qquad \mathcal{D}^1 U^0 = 2T^0 - \left(P^1 - \frac{n-3}{n}\right) U^0 - \frac{2n-4}{n} L^1 Q^0,\\
&\mathcal{E}^0 Q^0 = U^0 + \left(\frac{(n-1)(L^0)^2 - nR^0}{2nQ^0} + \frac{1}{2n} L^0 L^1 - \frac{1}{2} L^0 + \frac{1}{2} L^1\right) T^0\\
&\qquad\qquad - \left(\frac{1}{n} L^1 - 1\right)(Q^0)^2 - \left(\frac{2n-3}{n} L^0 + 1\right) Q^0,
\end{aligned}
\tag{5.23}
$$

where, for $k = 0, 1$,

$$
S^k = \iota(u^k_{xxx}), \qquad T^k = \iota(u^k_{xxy}), \qquad U^k = \iota(u^k_{xyy}), \qquad V^k = \iota(u^k_{yyy}). \tag{5.24}
$$

Since we have already generated Q^0, R^0, the first two allow us to then generate the third order invariants U^0, T^0. From these, we can then generate P^1 using the third of these formulae. With all the third order joint differential invariants in hand, the general theorem, [7], allows us to establish our final result.

Theorem 5 *The algebra of joint differential invariants of a two point ternary form is generated by the single first order differential invariant $L^0 = \iota(u^0_y)$ through invariant differentiation.*

Finally, the commutator formulas for the invariant differential operators are

$$
\begin{aligned}
&[\,\mathcal{D}^0, \mathcal{E}^0\,] = \left(\frac{(L^0L^1 - nL^0 + nL^1)T^0}{2nQ^0} + Q^0 - \frac{1}{n}L^1 + \frac{1}{n}L^0 + 1\right)\mathcal{D}^0 - \frac{T^0}{2Q^0}\,\mathcal{E}^0,\\
&[\,\mathcal{D}^0, \mathcal{D}^1\,] = \left(\frac{n-1}{n} - P^1\right)\mathcal{D}^0 - \frac{(L^0L^1 - nL^0 + nL^1)S^0}{2nQ^0}\,\mathcal{D}^1 + \frac{(L^0+n)S^0}{2nQ^0}\,\mathcal{E}^1,\\
&[\,\mathcal{D}^0, \mathcal{E}^1\,] = \left(\frac{n-1}{n} L^1 - Q^1\right)\mathcal{D}^0 + \mathcal{D}^1,\\
&[\,\mathcal{E}^0, \mathcal{E}^1\,] = -\left(\frac{1}{n}L^1 - 1\right)\mathcal{E}^0 + \left(\frac{1}{n}L^0 + 1\right)\mathcal{E}^1, \quad [\,\mathcal{D}^1, \mathcal{E}^1\,] = (L^1 - Q^1 - 1)\mathcal{D}^1 - \frac{1}{n}\,\mathcal{E}^1,\\
&[\,\mathcal{E}^0, \mathcal{D}^1\,] = \mathcal{D}^0 - \frac{1}{n}\mathcal{E}^0 + \left(-\frac{(L^0L^1 - nL^0 + nL^1)T^0}{2nQ^0} - Q^0 + \frac{1}{n}L^1\right)\mathcal{D}^1\\
&\qquad\qquad + \left(\frac{(L^0+n)T^0}{2nQ^0} - \frac{1}{n}Q^0\right)\mathcal{E}^1.
\end{aligned}
\tag{5.25}
$$

Thus, since P^1 appears, modulo an additive constant, as one of the commutator invariants, the commutator trick provides an alternative method to generate it by differentiating L^0, and thereby reprove Theorem 5.

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