Corrections to the First Printing of

Olver, P.J., *Applications of Lie Groups to Differential Equations*,
Second Edition, Springer–Verlag, New York, 1993.

Last updated: May 7, 2019

⋆⋆⋆ *On page 5, lines 24–27, change*

Thus T^2 can be covered by two coordinate charts

$$U_1 = \{(\theta,\rho) : 0 < \theta < 2\pi,\ 0 < \rho < 2\pi\},$$
$$U_2 = \{(\theta,\rho) : \pi < \theta < 3\pi,\ \pi < \rho < 3\pi\},$$

with overlap function ...

to

Thus T^2 can be covered by three coordinate charts, e.g.

$$U_1 = \{(\theta,\rho) : 0 < \theta < 2\pi,\ 0 < \rho < 2\pi\},$$
$$U_2 = \{(\theta,\rho) : \pi < \theta < 3\pi,\ \pi < \rho < 3\pi\},$$
$$U_3 = \left\{\, (\theta,\rho) : \tfrac12\pi < \theta < \tfrac52\pi,\ \tfrac12\pi < \rho < \tfrac52\pi \,\right\}.$$

The first overlap function is ...

⋆⋆⋆ *On page 10, line 6, change*

$\phi\circ\tilde\phi^{-1}\colon \mathbb{R}\to\mathbb{R}$

to

$\phi^{-1}\circ\tilde\phi\colon \mathbb{R}\to\mathbb{R}$

⋆⋆⋆ *On page 19, line 17, change*

$x\in V_0 = \left\{\, x : |\,x\,| < \tfrac12 \,\right\}$

to

$x\in V_0 = \left\{\, -1 < x < \tfrac13 \,\right\}$

⋆⋆⋆ *On page 36, line 9, change*

for all $\varepsilon,\theta\in\mathbb{R}$, $x\in M$, such that both sides are defined, if and only if

to

for all $x\in M$, and $(\varepsilon,\theta)\in V$, where $V\subset\mathbb{R}^2$ is a connected open subset containing $(0,0)$ such that both sides of (1.34) are defined at all points therein, if and only if

$\star\star\star$ *On page 37, lines 7–9, change*

. . . plane:

$$V = \{(\theta,\varepsilon) : \text{both sides of (1.34) are defined at } (\theta,\varepsilon)\}$$

and

$$U = \{(\theta,\varepsilon) : \text{both sides of (1.34) are defined and equal at } (\theta,\varepsilon)\}$$

to

. . . plane: first V is the connected component of

$$\widehat{V} = \{(\theta,\varepsilon) : \text{both sides of (1.34) are defined at}(\theta,\varepsilon)\}$$

containing the origin; second $U = \widehat{U} \cap V$, where

$$\widehat{U} = \{(\theta,\varepsilon) : \text{both sides of (1.34) are defined and equal at}(\theta,\varepsilon)\}$$

$\star\star\star$ *On page 37, line 10, delete the sentence*

Note that $U \subset V$, and that V is a connected subset of the (θ,ε) plane.

$\star\star\star$ *On page 37, line 14, add the following sentence after the final* $U = V$.

Warning: It is not, in general, true that $\widehat{U} = \widehat{V}$!

Remark: The preceding corrections are because Theorem 1.34 had a subtle flaw in it, first pointed out to me by James Devlin. The following exercise gives a counterexample to the original version. More details can be found in my paper: Olver, P.J., Non-associative local Lie groups, *J. Lie Theory* **6** (1996) 23–51. Thanks also to Hans Lundmark for comments.

Exercise: Let $M = \{\,(r,\theta) \mid r > 0\,\}$. Prove that the two vector fields

$$\mathbf{v} = \cos\theta\,\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\,\frac{\partial}{\partial\theta}\,, \qquad \mathbf{w} = \sin\theta\,\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\,\frac{\partial}{\partial\theta}\,,$$

commute, $[\mathbf{v},\mathbf{w}] = 0$ on M, but their flows do not globally commute. *Hint*: Consider r,θ as polar coordinates.

$\star\star\star$ *On page 43, line 23, change*

smoth vector fields

to

smooth vector fields

$\star\star\star$ *On page 52, Example 1.58, change*

Lie proved that

to

Lie proved, [4], that

$\star\star\star$ *On page 64, line 10, delete the middle terms between the two = signs. Thus, the equation should read*

$$exp(\varepsilon \mathbf{v}_0)^* \left[\, \omega|_{exp(\varepsilon \mathbf{v}_0)x} \,\right] = \sum_I \alpha_I(e^{\varepsilon} x) e^{k\varepsilon} dx^I,$$

$\star\star\star$ *On page 64, line 18, change*

$$\int_{\log \varepsilon}^{1}$$

to

$$\int_{\exp \varepsilon}^{1}$$

$\star\star\star$ *On page 64, line 19, change*

$\lambda = \log \widetilde{\varepsilon}$

to

$\lambda = e^{\tilde{\varepsilon}}$

$\star\star\star$ *On page 67, line 15, change*

sort

to

short

$\star\star\star$ *On page 70, Exercise 1.8, change*

in polar coordinates,

to

in polar coordinates with $r > 0$,

$\star\star\star$ *On page 72, Exercise 1.24(b), change*

$\mathfrak{h} \subset \mathfrak{g}$ has the property

to

$\mathfrak{h} \subset \mathfrak{g}$ is a *normal subalgebra* (or *ideal*), meaning that it has the property

$\star\star\star$ *On pages 79–81, the ends of the proofs of Proposition 2.6 and Theorem 2.8 that refer back to* (1.40) *are subtly flawed, in that, under our definition of a local group action, even when $g = g_1 \cdot g_2$ with $g, g_1, g_2 \in G_x$, there is no guarantee that $g_1 \in G_{g_2 \cdot x}$, i.e., that $g_1 \cdot (g_2 \cdot x)$ is defined even though $g \cdot x = (g_1 \cdot g_2) \cdot x$ is defined.*

However, a more direct argument, that avoids this difficulty, is to replace the end of the proof of Proposition 2.6 by the following:

... Conversely, if (2.1) holds everywhere, then

$$\frac{d}{d\varepsilon}\, \zeta(\exp(\varepsilon \mathbf{v})\, x) = 0$$

where defined, and hence $\zeta(\exp(\varepsilon\mathbf{v})\,x) = \zeta(x) = c$ is constant for ε in the connected interval containing 0 in $\{\,\varepsilon\in\mathbb{R}\mid \exp(\varepsilon\mathbf{v})\in G_x\,\}$. Using the fact that the exponential map is a local diffeomorphism from a neighborhood of $0\in\mathfrak{g}$ to a neighborhood of $e\in G_x$, we conclude that $\zeta(g\cdot x)=c$ for all g in an open neighborhood of the identity in G_x. Now, set $\widetilde{G}_x = \{\,g\in G_x \mid \zeta(g\cdot x)=c\,\}$. Applying the preceding argument at the point $g\cdot x$ for any $g\in\widetilde{G}_x$ proves that $\widetilde{G}_x$ is open, while continuity proves that it is closed in G_x. Thus, by connectivity, $\widetilde{G}_x = G_x$, and the result follows. □

Similarly, replace the end of the proof of Theorem 2.8 by:

… We have thus shown that if x_0 is a solution to $F(x)=0$, and $\mathbf{v}$ is an infinitesimal generator of G, and ε is sufficiently small, then $\exp(\varepsilon\mathbf{v})\,x_0$ is also a solution. As in the proof of Proposition 2.6, one then shows that $\widetilde{G}_x = \{\,g\in G_x \mid F(g\cdot x_0)=0\,\}$ is both open and closed in G_x, and hence, by connectivity, $\widetilde{G}_x = G_x$. □

- *Thanks to Colin James Stockdale Klaus for correspondence on this point.*

⋆⋆⋆ *On page 83, line 14, change*

folowing

to

following

⋆⋆⋆ *On page 93, line 8, change*

functon

to

function

⋆⋆⋆ *On page 106, line -5, the term inside the parentheses should be $\Xi_\varepsilon(x)$. Thus, the equation should read*

$$\sum_l \frac{\partial^2 \Xi^k_{-\varepsilon}}{\partial\tilde{x}^j\,\partial\tilde{x}^l}\left(\,\Xi_\varepsilon(x)\,\right)\frac{d\,\Xi^l_\varepsilon}{d\varepsilon}(x) = 0,$$

⋆⋆⋆ *On page 111, line 3, change*

$\mathrm{J}\tilde{f}_\varepsilon(x)$

to

$\mathrm{J}\tilde{f}_\varepsilon(\tilde{x})$

⋆⋆⋆ *On page 113, line 4, change*

$\xi = u$

to

$\xi = -u$

⋆⋆⋆ *On page 123, line 5, change*

$$\tau \frac{\partial}{\partial \tau}$$

to

$$\tau \frac{\partial}{\partial t}$$

⋆⋆⋆ *On page 151, line after* (2.110)*, change*

normal subalgebra of $\mathfrak{g}$

to

normal subalgebra (ideal) of $\mathfrak{g}$

⋆⋆⋆ *On page 163, line -3, change*

differential equatons

to

differential equations

⋆⋆⋆ *On page 167, line 2, change*

$\binom{p+k-1}{k} \cdot l$

to

$\binom{p+k}{k} \cdot l$

⋆⋆⋆ *On page 167, Definition 2.83, change*

differential equatons

to

differential equations

⋆⋆⋆ *On page 169, line -9, change*

Exercise 2.32

to

Exercise 2.33

- *Thanks to Mariano Hermida de La Rica for the preceding four corrections.*

⋆⋆⋆ *On page 187, line 8, change*

with respect y

to

with respect to y

⋆⋆⋆ *On page 187, line 14, change*

$$\frac{\partial\delta}{\partial\hat{x}} + \frac{\partial\delta}{\partial y}\frac{\partial\eta}{\partial x} + \frac{\partial\delta}{\partial v}\frac{\partial v}{\partial y}\frac{\partial\eta}{\partial x}$$

to

$$\frac{\partial\delta}{\partial\hat{x}}\frac{\partial\hat{x}}{\partial x} + \frac{\partial\delta}{\partial y}\frac{\partial\eta}{\partial x} + \frac{\partial\delta}{\partial v}\frac{\partial v}{\partial y}\frac{\partial\eta}{\partial x}$$

- *Thanks to Wen-xiu Ma and his class for catching this and several other errors.*

⋆⋆⋆ *On page 197, line -10, change*

SO(3)–invariant solutions exist.

to

SO(3)–invariant solutions can be constructed by this technique.

⋆⋆⋆ *On page 202, lines 13–14, change*

linear system of ordinary differential equations,

to

linear, constant coefficient system of ordinary differential equations,

⋆⋆⋆ *On page 207, line -2, change*

Example 2.64

to

Example 2.44

⋆⋆⋆ *On page 212, on both line 12 and line 13, change*

M/G

to

M/G^2

⋆⋆⋆ *On page 214, line 13, change*

$\widetilde{F}(x) = F[\pi(x)]$

to

$F(x) = \widetilde{F}[\pi(x)]$

⋆⋆⋆ *On page 215, line -1, change*

$\widetilde{F} = \widetilde{F}(\pi_1, \ldots, \pi_{m-s})$

to

$\widetilde{F} = \widehat{F}(\pi_1, \ldots, \pi_{m-s})$

$\star\star\star$ *On page 217, line 6, change*

$F(R, K) = 0$

to

$\widehat{F}(R, K) = 0$

$\star\star\star$ *On page 238, Exercise 3.7, change the formula to*

$$R = \left(\frac{t^2 E}{p_0}\right)^{1/5} h\left[p_0\left(\frac{t^6}{\rho_0^3\, E^2}\right)^{1/5}\right]$$

- *Thanks to Kameron Decker Harris for spotting this.*

$\star\star\star$ *On page 273, line -7, change*

tecnique

to

technique

$\star\star\star$ *On page 280, in the table, change*

$I_x = x\,D - y\,A + \frac{1}{2}\,x\,u\,u_t + t\,M_x$

to

$I_x = x\,D + y\,A + \frac{1}{2}\,x\,u\,u_t + t\,M_x$

- *Thanks to Gehrt Hartjen for checking through this table and the table on page 340 in his Mathematics Diplomarbeit in Aachen, 2001.*

$\star\star\star$ *On page 285, line 5, change*

4.13. (a)

to

** 4.13. (a)

$\star\star\star$ *On page 285, line 7, change*

Prove that the reduced system Δ/G for the G-invariant solutions of Δ is also the Euler-Lagrange equations for some variational problem on the quotient manifold M/G. Does this generalize to nonvariational symmetry groups?

to

Is the reduced system Δ/G for the G-invariant solutions of Δ necessarily the Euler-Lagrange equations for some variational problem on the quotient manifold M/G? See I.M. Anderson and M. Fels, Symmetry reduction of variational bicomplexes and the principle of symmetric criticality, *Amer. J. Math.* **119** (1997) 609–670, for details.

⋆⋆⋆ *On page 290, line -4, change*

Exercise 2.33

to

Exercise 2.35

⋆⋆⋆ *On page 323, line -8 change*

a third order evolution equation is integrable

to

a third order evolution equation in which u_{xxx} occurs linearly is integrable

⋆⋆⋆ *On page 328, line 10, change*

Bluman and Kumei, [3],

to

Bluman and Kumei, [2],

⋆⋆⋆ *On page 331, insert minus sign after equals sign in second displayed equation:*

$$D_\Delta^* Q = -\, q_t - q_{xx} + u\, q_x$$

⋆⋆⋆ *On page 340, in line 4 of the table, change*

$-\, y\, u_{xxx} + x\, u_{xyy} + u_{xy}$

to

$-\, y\, u_{xxx} + x\, u_{xxy} + u_{xy}$

⋆⋆⋆ *On page 340, in line 5 of the table, change*

$u_{xx}(y\, u_{yt} + \frac{1}{2}\, u_t) - u_{yy}(x\, u_{xt} + \frac{1}{2}\, u_t)$

to

$-\, u_{xx}(y\, u_{yt} + \frac{1}{2}\, u_t) + u_{yy}(x\, u_{xt} + \frac{1}{2}\, u_t)$

⋆⋆⋆ *On page 350, lines -8 to -7, change*

the their Fréchet

to

their Fréchet

⋆⋆⋆ *On page 364, Theorem 5.92, first line, change*

$P[u] \in \mathcal{A}^p$

to

$P[u] \in \mathcal{A}^q$

- *Thanks to Thomas von Schroeter for spotting this.*

$\star\star\star$ *On page 366, line -7, change*

$J \setminus I$

to

$I \setminus J$

- *Thanks to Rob Thompson for catching this and several other errors.*

$\star\star\star$ *On page 381, Exercise 3.16a*:

The system does not, in fact have a recursion operator, although there is a recursive formula for generating the higher order symmetries. On the other hand, the related system

$$u_t = u_{xx} + v^2, \qquad v_t = v_{xx},$$

does admit a recursion operator. Details can be found in: Beukers, F., Sanders, J.A., and Wang, J.P., On integrability of systems of evolution equations, *J. Diff. Eq.* **172** (2001), 396-408.

$\star\star\star$ *On page 381, Exercise 3.16b*:

A proof that the Bakirov system has only one generalized symmetry can now be found in: Beukers, F., Sanders, J.A., and Wang, J.P., One symmetry does not imply integrability, *J. Diff. Eq.* **146** (1998), 251–260.

$\star\star\star$ *On page 397, equation* (6.17)*, change closing* $\}$ *to* $\rangle$:

$$\{F, H\}(x) = \langle\, x\,;[\,\nabla F(x), \nabla H(x)\,]\,\rangle, \qquad x \in \mathfrak{g}^*, \tag{6.17}$$

$\star\star\star$ *On page 420, line -8, change*

rocedure.

to

procedure.

$\star\star\star$ *On page 427, lines 23–24, change*

Recently, Weinstein, [3], proposed the less historically motivated term "Casimir function" for these objects, ...

to

Recently, R. Littlejohn, *AIP Conference Proc.* **88** (1982), 47–66, and A. Weinstein, *AIP Conference Proc.* **88** (1982), 1–11, and [3], have proposed the less historically motivated term "Casimir function" for these objects, ...

- *Thanks to Phil Morrison for alerting me to these earlier references.*

$\star\star\star$ *On page 469, change title of second reference by H. Bateman to*

On dissipative systems and related variational principles

⋆⋆⋆ *On page 480, line 2 change*

Leipz. Berich. **1** (1895)

to

Leipz. Berichte **47** (1895)

⋆⋆⋆ *On page 480, line 5 change*

Leipz. Berich. **3** (1897)

to

Leipz. Berichte **49** (1897)