Homework Assignment # 4

**Exercise 4.1.** Draw a picture of the complex plane with the complex solutions to \( z^6 = 1 \) marked. What is the exact formula (no trigonometric functions allowed) for the primitive sixth root of unity \( \zeta = \zeta_6 \)? Verify explicitly that \( 1 + \zeta + \zeta^2 + \zeta^3 + \zeta^4 + \zeta^5 = 0 \). Explain this identity using vector addition for complex numbers.

**Exercise 4.2.** Explain why all the discrete Fourier coefficients in Example 11.22 are real. Can you state a general theorem?

**Exercise 4.3.** The discrete Fourier transform defines a linear function from the sample vector \( \mathbf{f} \) to its Fourier coefficients \( \mathbf{c} \) and so can be described by matrix multiplication. The reconstruction formula for is given by matrix inversion \( \mathbf{f} = \mathbf{F}^{-1} \mathbf{c} \). Find a formula for the Fourier matrix \( \mathbf{F} = \mathbf{F}_n \) such that \( \mathbf{c} = \mathbf{F} \mathbf{f} \). Write down the particular cases for \( \mathbf{F}_4 \) and \( \mathbf{F}_8 \) explicitly. Prove that

\[
\mathbf{F}^{-1} = n \mathbf{F}^\dagger,
\]

where \( \mathbf{F}^\dagger = \mathbf{F}^T \) is the Hermitian transpose of \( \mathbf{F} \), obtained by transposing \( \mathbf{F} \) and then applying complex conjugation to all entries.

**Exercise 4.4.** Each step of the Fast Fourier transform defines a linear transformation, and so is also given by matrix multiplication. For \( n = 2^r \) with \( r = 2, 3 \) write the Fourier transform matrix \( \mathbf{F}_n \) as a product of \( r \) “simpler ” matrices that represent the individual steps of the Fast Fourier transform.

**Example 4.5.** Let

\[
f(x) = \begin{cases} 
-x, & 0 \leq x \leq \frac{1}{3}\pi, \\
 x - \frac{2}{3}\pi, & \frac{1}{3}\pi \leq x \leq \frac{4}{3}\pi, \\
 -x + 2\pi, & \frac{4}{3}\pi \leq x \leq 2\pi. 
\end{cases}
\]

(a) Use MATLAB to construct the discrete Fourier coefficients for \( f(x) \) based on \( n = 128 \) data points.

(b) Graph the reconstructed function when using the data compression algorithm that retains only the 10 and 20 lowest frequency modes. Discuss what you observe.

**Due:** Friday, April 21

**Second Midterm:** Friday, May 5.

You will be allowed to use one 8” \( \times \) 11” sheet of notes.