Differential Invariants and Invariant Differential Equations

Peter J. Olver †
School of Mathematics
University of Minnesota
Minneapolis, Minnesota
U.S.A. 55455

e-mail: olver@ima.umn.edu

Abstract. This paper surveys recent results on the classification of differential invariants of transformation groups, and their applications to invariant differential equations and variational problems.

Consider a group of transformations acting on a jet space coordinatized by the independent variables, the dependent variables, and their derivatives. Scalar functions which are not affected by the group transformations are known as differential invariants. Their importance was emphasized by Sophus Lie, [**13**], who showed that every invariant system of differential equations, [**14**], and every invariant variational problem, [**17**], could be directly expressed in terms of the differential invariants. As such they form the basic building blocks of many physical theories, where one begins by postulating the invariance of the equations, or the variational principle, under a prescribed symmetry group. Lie also demonstrated, [**14**], how differential invariants could be used to integrate invariant ordinary differential equations, and succeeded in completely classifying all the differential invariants for all possible finite-dimensional Lie groups of point transformations in the case of one independent and one dependent variable. Lie's results were pursued by Tresse, [**25**], and, much later, Ovsiannikov, [**20**]. In this paper, I will survey some recent results extending the earlier classification theorems, [**19**], and then discuss recent applications to the study of invariant evolution equations, which is of great interest in image processing

April 8, 1994

† *Supported in part by NSF Grants DMS 91–16672 and DMS 92–04192.*

and computer vision, *cf.* [**21**], [**22**]. Space considerations preclude the inclusion of proofs and significant examples here.

I shall assume the reader is familiar with the fundamentals of the Lie theory of symmetry groups of differential equations, as discussed, for instance in my book, [**18**]. I shall employ the same basic notation here. For simplicity, I shall deal with complex-valued functions here, although most of the results are equally in the real case. Let G be an r-dimensional local transformation group acting on the space $M \subset X \times U \simeq \mathbb{C}^p \times \mathbb{C}^q$ coordinatized by p independent and q dependent variables. In the single dependent variable case, $q = 1$, we allow G to be a group of (first order) contact transformations. (Bäcklund's Theorem, [**2**], implies there are no other contact transformation groups.) Let $G^{(n)}$ denote the associated prolonged group action on the jet space J^n, whose coordinates are denoted by $(x, u^{(n)})$. The space of infinitesimal generators of G — its Lie algebra, will be denoted by $\mathfrak{g}$, with associated prolongation $\mathfrak{g}^{(n)}$.

In order to properly study the differential invariants of a transformation group, we must understand the geometry of its prolongations. Let s_n denote the maximal orbit dimension of the prolonged action $G^{(n)}$, so that $G^{(n)}$ acts (semi-)regularly on the open subset $V^n = \{z \in \mathrm{J}^n \mid \dim \mathfrak{g}^{(n)}|_z = s_n\} \subset \mathrm{J}^n$ consisting of all points contained in orbits of maximal dimension. We shall, in fact, assume that $G^{(n)}$ acts regularly on V^n, although all our results, suitably interpreted, are valid in the semi-regular case. The orbit dimensions satisfy the elementary inequalities

$$s_{n-1} \leq s_n \leq s_{n-1} + q\binom{p+n-1}{n}. \tag{1}$$

In particular, they form a nondecreasing sequence

$$s_0 \leq s_1 \leq s_2 \leq \cdots \leq r, \tag{2}$$

that is bounded by the dimension of G and hence eventually stabilizes: $s_m = s$ for all m sufficiently large. We will call s the stable orbit dimension, and the minimal order n for which $s_n = s$ the *order of stabilization* of the group. The following result is due to Ovsiannikov, [**20**].

Theorem 1. *The stable orbit dimension of a transformation group G is equal to the dimension of G if and only if G acts locally effectively.*

Here "locally effectively" means that the only group element in some neighborhood of the identity which acts trivially on M is the identity itself. If G does not act effectively, we can replace it by the quotient group G/G_M, where $G_M = \{g \mid g \cdot x = x \text{ for all } x \in M\}$ is the global isotropy subgroup, which *does* act effectively on M in essentially the same way as G itself. Consequently, there is no loss in generality in assuming that all our group actions are (locally) effective, and hence $s = r = \dim G$ in all cases.

As we shall see, the determination of the precise order of stabilization n is of great significance. A cautionary note: it is possible for the orbit dimension to "pseudo-stabilize", meaning that $s_k = s_{k+1} < s_{k+2}$ for some $k < n$. However, the following result rules out a pseudo-stabilization unless the prolonged orbits have rather high dimension.

Theorem 2. *Suppose that, for some $n \geq 0$, the maximal orbit dimensions of the prolonged group actions satisfy $s_{n-1} < s_n = s_{n+1} \leq q^{(n)}$. Then n is the order of stabilization of G.*

Corollary 3. *Suppose that the maximal orbit dimensions of the prolonged group actions satisfy $s_k = s_{k+1}$ and, also, $s_n = s_{n+1}$ for some $n > k$. Then $s_m = s_n$ for all $m \geq n$.*

Thus, there can be at most one such pseudo-stabilization. Corollary 3 provides a significant strengthening of Ovsiannikov's stabilization theorem, [**20**; p. 313], which states that if $s_n = s_{n+1} = s_{n+2}$, then $s_m = s_n$ for all $m \geq n$.

Example 4. Let $r \geq 3$. Let $x, u \in \mathbb{C}$. Consider the r-dimensional group generated by the vector fields $\partial_x, \partial_u, x\partial_u, \ldots, x^{r-3}\partial_u, x\partial_x + (r-2)u\partial_u$. The maximal orbit dimensions are given by $s_0 = 1$, $s_1 = 2, \ldots, s_{r-3} = s_{r-2} = r-1$, $s_{r-1} = s_r = \cdots = r$. Thus, the orbit dimensions pseudo-stabilize at order $r-3$, and finally stabilize at order $r-1$. In particular, we see that a pseudo-stabilization can occur at arbitrarily high order. On the other hand, as we shall see, this example is effectively the only known example where pseudo-stabilization of orbit dimensions actually occurs.

In the scalar case, $p = q = 1$, one can obtain very detailed information owing to Lie's complete classification of all possible Lie groups of point and contact transformations acting on a two-dimensional space, [**12**], [**16**]; see also [**24**] for a modern treatment, and [**6**] for recent applications to quantum mechanics. The basic result is that *any* finite-dimensional transformation group G acting on a two-dimensional complex manifold without fixed points (0 dimensional orbits) is locally equivalent, under a point (or contact) transformation, to one of the groups appearing in Tables 1–4 at the end of the paper. The groups of point transformations naturally fall into three classes — the primitive groups, for which there is no invariant foliation, the imprimitive, transitive groups, and, finally, the intransitive groups. In addition, there are just three finite-dimensional groups of contact transformations not contact-equivalent to any point transformation group; in Table 4, I have listed the characteristics $Q(x, u, u_x)$ of the infinitesimal generators, the first order generators themselves being recovered by the standard formula

$$\mathbf{v}^{(1)} = -\frac{\partial Q}{\partial p}\frac{\partial}{\partial x} + \left(Q - p\frac{\partial Q}{\partial p}\right)\frac{\partial}{\partial u} + \left(\frac{\partial Q}{\partial x} + p\frac{\partial Q}{\partial u}\right)\frac{\partial}{\partial p}. \tag{3}$$

The complete classification allows us to determine the stabilization order for every possible transformation group in the plane, and, in addition, the complete system of differential invariants. One remarkable consequence is that, in the scalar case, Example 4 provides the *only* examples of transformation groups that pseudo-stabilize. (Indeed, I do not know of any multi-dimensional examples of pseudo-stabilization which are not straightforward generalizations of this example!)

Theorem 5. *Let $M \subset X \times U \simeq \mathbb{C} \times \mathbb{C}$. Let $G \neq \{e\}$ be an r-dimensional locally effective group of point or contact transformations. Then the prolonged orbit dimensions are given by one of the following three mutually exclusive possibilities:*

i. The Regular Case: $s_k = k+2$ *for* $k \le r-2$*, while* $s_m = r$ *for* $m \ge r-2$*.*

ii. The Intransitive Case: $s_k = k+1$ *for* $k \le r-1$*, while* $s_m = r$ *for* $m \ge r-1$*.*

iii. The Pseudo-stabilization Case: $s_k = k+1$ *for* $k \le r-3$*,* $s_{r-2} = r-1$*,* $s_m = r$ *for* $m \ge r-1$*. In this case, G is necessarily equivalent to the group action in Example 4, which is Case 2.7 for* $k = \alpha$ *in the Tables.*

A *differential invariant* is a scalar function $I: \mathrm{J}^n \to \mathbb{C}$ which satisfies $I(g^{(n)} \cdot (x, u^{(n)})) = I(x, u^{(n)})$ for all $g^{(n)} \in G^{(n)}$, and all $(x, u^{(n)}) \in \mathrm{J}^n$ where the prolonged transformation $g^{(n)} \cdot (x, u^{(n)})$ is defined. Differential invariants (of connected groups) are most easily determined using infinitesimal methods.

Proposition 6. *A function* $I: \mathrm{J}^n \to \mathbb{C}$ *is a differential invariant for a connected transformation group* G *if and only if* $\mathbf{v}^{(n)}(I) = 0$ *for every prolonged infinitesimal generator* $\mathbf{v}^{(n)} \in \mathfrak{g}^{(n)}$.

According to Frobenius' Theorem, there are, in general,

$$i_n = \dim \mathrm{J}^n - s_n = p + q\binom{p+n}{n} - s_n \tag{4}$$

functionally independent differential invariants of order at most n near any point $z \in V^n$. Since each differential invariant of order less than n is included in this count, the integers i_n form a non-decreasing sequence: $i_0 \le i_1 \le i_2 \le \cdots$. The difference $j_n = i_n - i_{n-1}$ will count the number of strictly independent n^{th} order differential invariants.

The basic method for constructing a complete system of differential invariants is to use invariant differential operators. A differential operator is said to be G-invariant if it maps differential invariants to higher order differential invariants, and thus, by iteration, produces hierarchies of differential invariants of arbitrarily large order. For n sufficiently large, we can guarantee the existence of sufficiently many such differential operators so as to completely generate all the higher order independent differential invariants of the group by successively differentiating lower order "fundamental" differential invariants.

The most direct way to construct the required invariant differential operators utilizes differential forms and the contact structure on the jet space J^n. Recall first that a differential form Θ on J^n is called a *contact form* if it is annihilated by all prolonged functions. Every contact form on J^n is a linear combination of the basic contact one-forms

$$\theta_J^\alpha = du_J^\alpha - \sum_{i=1}^{p} u_{J,i}^\alpha \, dx^i, \qquad \alpha = 1, \ldots, q, \quad 0 \le \#J < n, \tag{5}$$

where $u_J^\alpha = D_J u^\alpha$ denotes the J-th derivative of u^α. Contact transformations (including prolonged point transformations) are distinguished by the fact that they map contact forms to contact forms. A one-form ω on J^n is called *horizontal* if it annihilates all vertical tangent directions; equivalently it can be written as $\omega = \sum_{i=1}^{p} P_i(x, u^{(n)}) \, dx^i$.

Definition 7. Let G be a transformation group. A differential form ω on J^n is called *contact-invariant* if and only if, for every $g \in G$, we have $(g^{(n)})^* \omega = \omega + \theta$ for some contact form $\theta = \theta_g$. A *contact-invariant coframe* is a set of p linearly independent horizontal contact-invariant one-forms $\{\omega^1, \ldots, \omega^p\}$ on J^n.

Contact-invariant coframes are the jet space counterparts of the differential geometric coframes that form the foundation of the Cartan equivalence method, [**3**], [**5**]. Note that if I is any differential invariant, its *total differential* $\mathrm{D}I = \sum_{i=1}^p D_i I \, dx^i$, which is just the horizontal component of its ordinary differential dI, is a contact-invariant one-form. Thus, knowledge of sufficiently many independent differential invariants allows us to construct a contact-invariant coframe. (However, in almost every case, such a coframe is not the simplest or lowest order one.) If $F(x, u^{(m)})$ is any differential function, we can rewrite its total differential in terms of the coframe,

$$\mathrm{D}F = \sum_{k=1}^{p} \mathcal{D}_k F \, \omega^k. \tag{6}$$

The resulting *coframe differential operators* $\mathcal{D}_k$ provide the desired invariant differential operators.

Proposition 8. *Let $\mathcal{D}_1, \ldots, \mathcal{D}_p$ be the coframe differential operators associated with a contact-invariant coframe on J^n. If $I(x, u^{(m)})$ is any differential invariant of order m, then $\mathcal{D}_k I$ is a differential invariant of order $\leq \max\{n, m+1\}$.*

Theorem 9. *Suppose that G is a transformation group, and let n be its order of stabilization. Then there exists a contact-invariant coframe $\omega^1, \ldots, \omega^p$ on J^n, with corresponding invariant differential operators $\mathcal{D}_1, \ldots, \mathcal{D}_p$, and a system of* fundamental differential invariants *$J_1, \ldots, J_m$, of order at most $n+2$, such that, locally, every differential invariant can be written as a function of these differential invariants and their derivatives $\mathcal{D}_{j_1} \cdots \mathcal{D}_{j_\kappa} J_\nu$, $\kappa \geq 0$, $1 \leq j_\mu \leq p$, $\nu = 1, \ldots, m$.*

Except in the case of one independent variable, the precise number m of fundamental differential invariants required to construct the complete system of differential invariants is not known. If $p = 1$, then it can be proved, [**19**], that the number of fundamental differential invariants one needs is exactly q, the number of dependent variables. Let us now specialize even further, to the scalar case $p = q = 1$. Assuming $G \neq \{e\}$ acts locally effectively, Theorem 5 implies that there are precisely two fundamental differential invariants, $I(x, u^{(s)})$ and $J(x, u^{(r)})$, having orders $0 \leq s < r = \dim G$ respectively. Moreover, there exists a contact-invariant horizontal form $\omega = L(x, u^{(t)}) \, dx$ having order $t \leq n$, the stabilization order of G; the corresponding invariant differential operator is $\mathcal{D} = (1/L) D_x$. In the regular case, the lowest order differential invariant $I(x, u^{(r-1)})$ has order exactly $r - 1$, and every other differential invariant (including J) can be written in terms of I and its derivatives $\mathcal{D}^m I$. In the intransitive case, the lowest order differential invariant $I(x, u)$ has order 0, whereas in the pseudo-stabilization case, the lowest order differential invariant $I(x, u^{(r-2)})$ has order $r - 2$. In these two cases, the differential $\mathrm{D}I = D_x I\, dx$ provides a contact-invariant one-form, with corresponding differential operator $\mathcal{D} = d/dI$; every other differential invariant can be written in terms of I, J, and the differentiated invariants $\mathcal{D}^m J = d^m J/dI^m$.

In [**14**], Lie determined the differential invariants for each of the point transformation groups appearing in Tables 1–3. Apparently, he did not publish the formulas for the differential invariants of the three contact transformation groups. A complete list of differential invariants and invariant one-forms for the point and contact transformation groups appears in Table 5. In this table, and below, we use the notation $u_n = D_x^n u$ for the higher order derivatives of the scalar function u. Of particular interest are certain subgroups of the projective group SL(3) — Case 1.3 in Table 1. The Euclidean group E(2), which is a real form of the complex transformation group of Type 2.7 for $k = 1$, $\alpha = 0$, the special affine group $\mathrm{SA}(2) = \mathrm{SL}(2) \ltimes \mathbb{C}^2$, Case 1.1, and the full affine group $\mathrm{A}(2) = \mathrm{GL}(2) \ltimes \mathbb{C}^2$, Case 1.2, play a key role in differential geometric applications, discussed in detail in Guggenheimer, [**10**]. In these cases, the lowest order invariant one-form is the group-invariant arc length element ds, and the lowest order differential invariant is the group-invariant curvature κ. In particular, in the Euclidean case, $ds = \sqrt{1 + u_x^2}\, dx$ and $\kappa = u_{xx}/(1 + u_x^2)^{3/2}$. Thus, for the above metnioned groups, a complete system of differential invariants is provided by the curvature and its derivatives with respect to arc length, $d^m \kappa/ds^m$. Indeed, one is tempted to define, for any regular group of point or contact transformations in the plane, the group-invariant arc length to be the lowest order contact-invariant one-form, which is unique up to constant multiple, and the group-invariant curvature to be the lowest order fundamental differential invariant, which is unique up to a function thereof.

We recall next how differential invariants are used to characterize systems of differential equations and variational problems which admit a prescribed symmetry group; see [**13**], [**17**], [**18**], [**20**]. The basic result holds for arbitrary numbers of independent and dependent variables.

Theorem 10. *Let G be a transformation group, and let $I_1, \ldots, I_k$, $k = i_n$, be a complete system of functionally independent n^{th} order differential invariants on an open subset $V^n \subset \mathrm{J}^n$. A system of differential equations admits G as a symmetry group if and only if, when restricted to the subset V^n, it can be rewritten in terms of the differential invariants:*

$$\Delta_\nu(x, u^{(n)}) = F_\nu(I_1(x, u^{(n)}), \ldots, I_k(x, u^{(n)})) = 0, \qquad \nu = 1, \ldots, l. \tag{7}$$

Thus, the only invariant systems of differential equations of order n which are not described by differential invariants are those contained in the singular subvariety

$$\mathcal{S}^n = \mathrm{J}^n \setminus V^n = \left\{ z \in \mathrm{J}^n \;\middle|\; \dim \mathfrak{g}^{(n)}|_z < s_n \right\},$$

where the orbits of the prolonged action are not maximal. In particular, if G acts locally effectively and n is at least the stabilization order of G, then $\mathcal{S}^n$ is just the subset of J^n

where the prolonged infinitesimal generators of G are linearly dependent. Any G-invariant system of differential equations can then be decomposed into the union of a regular component, which is a subset of V^n and can be (locally) characterized by the vanishing of a system of differential invariants, as in Theorem 10, and a singular component, which is a subset of $\mathcal{S}^n$ and hence is characterized by the linear dependence of the infinitesimal generators, together with (possibly) additional conditions.

In the scalar case, the singular subvariety can be characterized using the method of Lie determinants, *cf.* [**14**]. Let

$$\mathbf{v}_\mu^{(r-2)} = \xi_\mu \frac{\partial}{\partial x} + \sum_{k=0}^{r-2} \varphi_\mu^k \frac{\partial}{\partial u_k}, \qquad \mu = 1, \ldots, r,$$

where $u_k = D_x^k u$, be the prolonged infinitesimal generators of the r-dimensional transformation group G. According to Theorem 5, in the regular case the stabilization order is $r-2$, hence the singular subvariety is given by the vanishing of the *Lie determinant*

$$\det \begin{vmatrix} \xi_1 & \varphi_1 & \varphi_1^1 & \cdots & \varphi_1^{r-1} \\ \xi_2 & \varphi_2 & \varphi_2^1 & \cdots & \varphi_2^{r-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \xi_r & \varphi_r & \varphi_r^1 & \cdots & \varphi_r^{r-1} \end{vmatrix} = 0, \tag{8}$$

which defines a single ordinary differential equation for u as a function of x. Equation (8) is the *only* G-invariant differential equation of order $\leq r-2$. In the anomalous cases (intransitive or pseudo-stabilization), the Lie determinant (8) vanishes identically. Since the stabilization order is $r-1$, the singular subvariety is given as the subset of J^{r-1} where the prolonged infinitesimal generators $\mathbf{v}_1^{(r-1)}, \ldots, \mathbf{v}_r^{(r-1)}$ are linearly dependent, which can be checked by forming an $r \times (r+1)$ matrix having the form (8) but whose columns go up to order $r-1$, and computing the determinant of a suitable maximal $r \times r$ minor. By a slight abuse of terminology, we shall call this maximal minor the Lie determinant in this case.

Theorem 11. *Suppose G is an r-dimensional transformation group acting on $M \subset X \times U \simeq \mathbb{C} \times \mathbb{C}$. Then every invariant differential equation can either be written in terms of the fundamental differential invariants or by the vanishing of the associated Lie determinant.*

Example 12. Consider the four parameter group generated by $\partial_x, x\partial_x, \partial_u, u\partial_u$, which is Case 2.9 for $k=1$. The second prolongations of these vector fields are ∂_x, $x\partial_x - u_x\partial_{u_x} - 2u_{xx}\partial_{u_{xx}}$, ∂_u, and $u\partial_u + u_x\partial_{u_x} + u_{xx}\partial_{u_{xx}}$, hence the Lie determinant is

$$\det \begin{vmatrix} 1 & 0 & 0 & 0 \\ x & 0 & -u_x & -2u_{xx} \\ 0 & 1 & 0 & 0 \\ 0 & u & u_x & u_{xx} \end{vmatrix} = -u_x u_{xx}.$$

Therefore, the singular invariant differential equations are $u_x = 0$ and $u_{xx} = 0$. Every other invariant differential equation can be written in terms of the fundamental differential invariant $I = u_x u_{xxx}/u_{xx}^2$ and its invariant derivatives $\mathcal{D}^m I$, where $\mathcal{D} = (u_x/u_{xx})D_x$. For example, the invariant third order equations are all of the form $u_x u_{xxx} = c u_{xx}^2$.

The full list of Lie determinants for all Lie groups of point and contact transformations in the complex plane can also be found in Table 5. In this table, we have omitted any inessential constant factors.

Detailed results on the symmetry classification of ordinary differential equations, and their integration, follows immediately from the results in Table 5. See Lie, [**14**], for a full range of applications. As an example, we deduce a general result on the characterization of differential equations having maximal and submaximal symmetry groups. First, we need to know when an ordinary differential equation admits a finite-dimensional symmetry group. The following theorem is due to Lie; see [**15**], [**7**], for a proof.

Theorem 13. *The point transformation symmetry group of a normal system of ordinary differential equations of order $n \geq 2$ is finite-dimensional. The contact transformation symmetry group of a normal ordinary differential equation of order $n \geq 3$ is a finite-dimensional.*

Theorem 14. *Let $\Delta(x, u^{(n)}) = 0$ be a n^{th} order scalar ordinary differential equation.*

i. *If $n = 2$, then $\Delta = 0$ admits at most an eight-parameter symmetry group of point transformations. Moreover, the symmetry group is eight-dimensional if and only if $\Delta = 0$ is equivalent to the linear equation $u_{xx} = 0$, with symmetry group of type 1.3.*

ii. *If $n \geq 3$, then $\Delta = 0$ admits at most an $(n+4)$-parameter symmetry group of point transformations. Moreover, the symmetry group is $(n+4)$-dimensional if and only if $\Delta = 0$ is equivalent to the linear equation $u_n = 0$, with symmetry group of type 2.11, for $k = n$.*

iii. *If $n = 3$, then $\Delta = 0$ admits at most a ten-parameter symmetry group of contact transformations. Moreover, the symmetry group is ten-dimensional if and only if $\Delta = 0$ is equivalent to the linear equation $u_{xxx} = 0$, with symmetry group of type 4.3.*

iv. *If $n \geq 4$, then $\Delta = 0$ admits at most an $(n+4)$-parameter symmetry group of contact transformations. Moreover, the symmetry group is $(n+4)$-dimensional if and only if $\Delta = 0$ is equivalent to the linear equation $u_n = 0$.*

The ordinary differential equations with submaximal symmetry groups, meaning those whose dimension is as large as possible without being maximal, are also of interest. In the second order case, the submaximal point symmetry group has dimension at most 3; the invariant ordinary differential equations are

$$u_{xx} = \frac{3u_x^2}{2u} + cu^3, \qquad u_{xx} = 6uu_x - 4u^3 + c(u_x - u^2)^{3/2},$$
$$u_{xx} = cu_x^{(\alpha-2)/(\alpha-1)}, \qquad u_{xx} = ce^{-u_x},$$

where c is a constant; the associated symmetry groups are, respectively, of types 2.1, 2.2, 2.7, with $k = 1$, $\alpha \neq 0, \frac{1}{2}, 1, 2$, and 2.8, with $k = 1$, *cf.* [**11**]. For $n = 3$, the submaximal point symmetry group has dimension 6; the invariant differential equation is

$$2u_x u_{xxx} - 3u_{xx}^2 = 0, \tag{9}$$

which has symmetry group $\mathrm{SL}(2) \times \mathrm{SL}(2)$ of type 2.4. For $n = 5$, the submaximal point or contact symmetry group has dimension 6; the invariant differential equation is

$$9u_{xx}^2 u_{xxxxx} - 45u_{xx}u_{xxx}u_{xxxx} + 40u_{xxx}^3 = 0, \tag{10}$$

which has symmetry group $\mathrm{SL}(3)$ of type 1.3. The solutions $u = f(x)$ of (10) are all graphs of conic sections. For $n = 7$, the submaximal contact symmetry group has dimension 10; the invariant differential equation is

$$10u_3^3 u_7 - 70u_3^2 u_4 u_6 - 49u_3^2 u_5^2 + 280u_3 u_4^2 u_5 - 175u_4^4 = 0. \tag{11}$$

which has contact symmetry group $\mathrm{SO}(5)$ of type 4.3. In all other cases, the submaximal symmetry group has dimension $n+2$. The equation is equivalent to either a linear equation (which is not equivalent to $u_n = 0$), or

$$3u_{xx}u_{xxxx} - 5u_{xxx}^2 = 0, \qquad \text{or} \qquad (n-1)u_{n-2}u_n - nu_{n-1}^2,$$

having respective symmetry groups of type 2.6, 1.2, or 2.11, for $k = n - 2$.

As mentioned in the introduction, differential invariants are also used to characterize all invariant variational problems associated with a given transformation group. Here, by symmetry of a variational problem, we shall mean a standard variational symmetry, without any divergence terms, *cf.* [**18**]. The following result originally appears in Lie, [**17**].

Theorem 15. *Let G be a transformation group, and assume that there exists a contact-invariant horizontal p-form $\Omega_0 = L_0(x, u^{(n)})\, dx$ on J^n. A variational problem admits G as a variational symmetry group if and only if it is of the form $\int I\Omega = \int I L_0\, dx$, where I is an arbitrary differential invariant of G.*

Any contact-invariant coframe $\omega^1, \ldots, \omega^p$ produces a contact-invariant p-form $\Omega = \omega^1 \wedge \cdots \wedge \omega^p$. Thus every G-invariant variational problem has the form

$$\mathcal{L}[u] = \int L(x, u^{(n)})\, dx = \int F\big(I_1(x, u^{(n)}), \ldots, I_k(x, u^{(n)})\big)\ \omega^1 \wedge \cdots \wedge \omega^p, \tag{12}$$

where $I_1, \ldots, I_k$ are a complete set of functionally independent differential invariants. In the scalar case, then, the most general invariant variational problem has the form $\int I\omega = \int IL\, dx$, in which I is an arbitrary differential invariant, and $\omega = L\, dx$ is the invariant one-form. In our geometric interpretation, then, $\omega = ds$ is the G-invariant element of arc length, and I is and arbitrary function of the curvature and its derivatives with respect to arc length.

Table 5 immediately provides a symmetry classification of the scalar variational problems, generalizing results of González–López, [**8**], for point transformation groups. Recall first that, in the scalar case, a n^{th} order Lagrangian $L(x, u^{(n)})$ is called *nonsingular* if it satisfies the nondegeneracy condition $\partial^2 L(\partial u_n)^2 \neq 0$.

Theorem 16. *A nonsingular first order Lagrangian admits a symmetry group of dimension* ≤ 3. *A nonsingular* n^{th} *order Lagrangian,* $n \geq 2$, *admits a symmetry group of dimension* $\leq n+3$.

The maximally symmetric first order Lagrangians are all equivalent, under a complex-valued transformation, to a constant multiple of one of the following,

$$u_x^\alpha, \qquad \sqrt{u_x - u^2}, \qquad e^{-u_x}, \tag{13}$$

having respective symmetry group of type 2.7, with $k = 1$, 2.2, and 2.8. For $n \geq 2$, one family of maximally symmetric Lagrangians is given by $L = u_n^{2/(n+1)}$, having a symmetry group of type 2.10, with $k = n$. There are, in addition, five "anomalous" maximally symmetric Lagrangians:

$$\begin{gathered} \sqrt[3]{u_{xx}}, \qquad \frac{\sqrt{2u_x u_{xxx} - 3u_{xx}^2}}{u_x}, \qquad \sqrt[3]{u_{xxx}}, \\ \frac{\sqrt[3]{9u_{xx}^2 u_{xxxxx} - 45u_{xx}u_{xxx}u_{xxxx} + 40u_{xxx}^3}}{u_{xx}}, \\ \frac{\sqrt[4]{10u_3^3u_7 - 70u_3^2u_4u_6 - 49u_3^2u_5^2 + 280u_3u_4^2u_5 - 175u_4^4}}{u_3}. \end{gathered} \tag{14}$$

The symmetry groups are Cases 1.1, 2.4, 4.1, 1.3, 4.3 — the third and fifth being maximally symmetric only for contact symmetry groups. Each of the anomalous Lagrangians defines the invariant arc length functional for a particular geometric group. Interestingly, the simple quadratic Lagrangian $L = u_n^2$, which has linear Euler–Lagrange equation, is *not* maximally symmetric for $n \geq 2$ — it has only an $(n+2)$-dimensional symmetry group, also of type 2.7. Thus, quadratic Lagrangians do *not* have the most symmetry, providing an explicit counterexample to the "meta-theorem" that linear objects are the ones with the highest degree of symmetry. (It should be remarked, however, the quadratic Lagrangians *are* maximally symmetric for divergence symmetries, [**8**].)

These remarks motivate an interesting unsolved problem. Any symmetry of a variational problem is also a symmetry of its Euler-Lagrange equations (although the converse is not necessarily true). Thus the Euler-Lagrange equation for each G-invariant Lagrangian can be rewritten in terms of the differential invariants of G. However, I do not know a general formula for calculating the invariant formulation of the Euler-Lagrange equations directly from the invariant formula for the Lagrangian, although special cases do appear in [**1**]. In the scalar case, the Euler-Lagrange equation for the G-invariant functional is expressed in terms of the G-invariant curvature and its derivatives with respect to arc length. In simple cases (Euclidean or special affine) the Euler-Lagrange equation is a multiple of the curvature, and so the arc-length minimizing curves are those having zero curvature. For more general cases (including the full affine and projective groups) this is not true — the curvature of the arc-length minimizing curves satisfies a certain interesting differential equation. I do not understand what this implies about the underlying geometry.

Finally, I shall describe some recent applications to the symmetry classification of evolution equations. This task was begun by Sokolov, [**24**], in his study of integrability and solitons. The classification has received added impetus from recent work, done in collaboration with G. Sapiro and A. Tannenbaum, on applications to computer vision and image processing, including connections with geometric curve-shortening flows; see [**23**], [**21**], [**22**]. Let G be a transformation group acting on $M \subset X \times U$, and consider a scalar evolution equation

$$u_t = K(x, u^{(n)}), \tag{15}$$

in which t is an additional independent variable (the time), and the right hand side depends only on the spatial (x) derivatives of u. The group action is extended to $M \times \mathbb{C}$, with G acting trivially on $t \in \mathbb{C}$. An interesting remark is that the evolution equation (15) admits G as a symmetry group if and only if the contact form

$$\frac{\theta}{K} = \frac{1}{K(x, u^{(n)})} \left\{ du - \sum_{i=1}^{p} u_i \, dx^i \right\}, \tag{16}$$

is a G-invariant one-form on J^n. The following result provides a complete characterization of all the G-invariant evolution equations, *cf.* [**22**].

Theorem 17. *Suppose $L(x, u^{(n)})$ is a G-invariant Lagrangian with nonvanishing Euler-Lagrange expression, $E(L) \neq 0$. Then every G-invariant evolution equation has the form*

$$u_t = \frac{L}{E(L)} \, I, \tag{17}$$

where I is an arbitrary differential invariant of G.

Remark: Theorem 17 can be extended to several dependent variables: one requires q, the number of dependent variables, distinct G-invariant Lagrangians $L_1, \ldots, L_q$, with the property that their "Euler–Lagrange matrix" $\mathbf{E} = \big(\mathrm{E}_\alpha(L_\beta)\big)$ is invertible. The most general G-invariant evolution equation then has the form

$$u_t = L_1 \, \mathbf{E}^{-1} \, \mathbf{I}, \tag{18}$$

where $\mathbf{I}$ is a column vector of differential invariants. (Note that each $L_\beta = I_\beta L_1$ for some differential invariant I_β, so the L_1 in (18) can be replaced by any other L_β by suitably modifying the invariant vector $\mathbf{I}$.)

The evolution equation (17) provides the most general invariant evolution equation; however, choosing $I =$ const does not necessarily yield the simplest such invariant equation. For symmetry groups of importance in image processing, there is an alternative way of characterizing invariant evolution equations.

Theorem 18. *Let G be a subgroup of the projective group* SL(3). *Let $ds = L\,dx$ denote the G-invariant one-form of lowest order and $\kappa = I$ its fundamental differential invariant. Then every G-invariant evolution equation has the form*

$$u_t = \frac{u_{xx}}{L^2}\,J, \tag{19}$$

where J is an arbitrary differential invariant for G, and thus a function of κ and its arc-length derivatives $d^k\kappa/ds^k$.

Note that, as a corollary of Theorems 17 and 18 we find that, for subgroups G of the projective group, the Euler-Lagrange expression associated with any G-invariant Lagrangian, including the G-invariant arc-length functional, has the form $\mathrm{E}(L) = JL^3/u_{xx}$ for some differential invariant J. For the similarity, special affine, affine, and full projective groups, (19) with J constant is distinguished as the *unique* G-invariant evolution equation of *lowest* order. For the Euclidean group, the simplest nontrivial invariant evolution equation is given by $u_t = c\sqrt{1+u_x^2}$ since, in this case, the curvature invariant κ has order 2, so we can take $J = c/\kappa$. In the Euclidean case, the flow (19) defines the fundamental *curve shortening flow*, in which ones moves the curve in its normal direction by an amount proportional to the curvature κ. This flow has been of great interest in geometry; see particularly the foundational work of Gage and Hamilton, [4], and Grayson, [**9**]. The special affine version is also a second order diffusion equation, in which one moves in the normal direction to the curve in proportion to $\kappa^{1/3}$ — see [**23**], [**21**], [**22**], for applications of these flows to image processing. Finally, if the group G is "volume-preserving", meaning that it leaves the $(p+1)$-form $dx^1 \wedge \cdots \wedge dx^p \wedge du$ invariant, then $\mathrm{E}(L)$ itself is a differential invariant, and hence the simplest invariant evolution equation is $u_t = P$, where $P\,dx$ is the lowest order invariant p-form. This case includes the Euclidean and special affine groups.

Acknowledgements. It is a pleasure to thank V. Sokolov and R. Heredero for their help in finding the differential invariants of the contact transformation groups, and I. Anderson, G. Sapiro and A. Tannenbaum for important remarks.

References

[**1**] Anderson, I.M., *The Variational Bicomplex*, Academic Press, to appear.

[**2**] Bäcklund, A.V., Ueber Flachentransformationen, *Math. Ann.* **9** (1876), 297–320.

[**3**] Cartan, É., Les problèmes d'équivalence, *in: Oeuvres Complètes*, part. II, vol. 2, Gauthiers–Villars, Paris, 1952, pp. 1311–1334.

[**4**] Gage, M., and Hamilton, R.S., The heat equation shrinking convex plane curves, *J. Diff. Geom.* **23** (1986), 69–96 .

[**5**] Gardner, R.B., *The Method of Equivalence and Its Applications*, SIAM, Philadelphia, 1989.

[**6**] González–López, A., Kamran, N., and Olver, P.J., Quasi–exact solvability, *Contemp. Math.*, to appear.

[**7**] González–Gascón, F., and González–López, A., Symmetries of differential equations. IV, *J. Math. Phys.* **24** (1983), 2006–2021.

[8] González–López, A., Symmetry bounds of variational problems, *J. Phys. A* **27** (1994), 1205–1232.

[9] Grayson, M., The heat equation shrinks embedded plane curves to round points, *J. Diff. Geom.* **26** (1987), 285–314.

[10] Guggenheimer, H.W., *Differential Geometry*, McGraw–Hill, New York, 1963.

[11] Ibragimov, N.H., Group analysis of ordinary differential equations and the invariance principle in mathematical physics (for the 150th anniversary of Sophus Lie), *Russian Math. Surveys* **47**:4 (1992), 89–156.

[12] Lie, S., Theorie der Transformationsgruppen I, *Math. Ann.* **16** (1880), 441–528; also *Gesammelte Abhandlungen*, vol. 6, B.G. Teubner, Leipzig, 1927, pp. 1–94.

[13] Lie, S., Über Differentialinvarianten, *Math. Ann.* **24** (1884), 537–578; also *Gesammelte Abhandlungen*, vol. 6, B.G. Teubner, Leipzig, 1927, pp. 95–138.

[14] Lie, S., Klassifikation und Integration von gewohnlichen Differential- gleichungen zwischen x, y, die eine Gruppe von Transformationen gestatten I, II, *Math. Ann.* **32** (1888), 213–281; also *Gesammelte Abhandlungen*, vol. 5, B.G. Teubner, Leipzig, 1924, pp. 240–310.

[15] Lie, S., *Vorlesungen über Differentialgleichungen mit Bekannten Infinitesimalen Transformationen*, B.G. Teubner, Leipzig, 1891..

[16] Lie, S., Gruppenregister, *in: Gesammelte Abhandlungen*, vol. 5, B.G. Teubner, Leipzig, 1924, pp. 767–773.

[17] Lie, S., Über Integralinvarianten und ihre Verwertung für die Theorie der Differentialgleichungen, *Leipz. Berichte* **4** (1897), 369–410; also *Gesammelte Abhandlungen*, vol. 6, B.G. Teubner, Leipzig, 1927, pp. 664–701.

[18] Olver, P.J., *Applications of Lie Groups to Differential Equations*, Second Edition, Graduate Texts in Mathematics, vol. 107, Springer–Verlag, New York, 1993.

[19] Olver, P.J., Differential invariants, *in: Algebraic and Geometric Structures in Differential Equations*, P.H.M. Kersten and I.S. Krasil'shchik, eds., Proceedings, University of Twente, 1993, to appear .

[20] Ovsiannikov, L.V., *Group Analysis of Differential Equations*, Academic Press, New York, 1982.

[21] Olver, P.J., Sapiro, G., and Tannenbaum, A., Differential invariant signatures and flows in computer vision: a symmetry group approach, *in: Geometry–Driven Diffusion in Computer Vision*, B. M. Ter Haar Romeny, ed., Kluwer Acad. Publ., Dordrecht, the Netherlands, 1994.

[22] Olver, P.J., Sapiro, G., and Tannenbaum, A., Invariant geometric evolutions of surfaces and volumetric smoothing, preprint, University of Minnesota, 1994.

[23] Sapiro, G., and Tannenbaum, A., On affine plane curve evolution, *J. of Func. Anal.* **119** (1994), 79–120.

[24] Sokolov, V.V., On the symmetries of evolution equations, *Russ. Math. Surveys* **43:5** (1988), 165–204.

[25] Tresse, A., Sur les invariants différentiels des groupes continus de transformations, *Acta Math.* **18** (1894), 1–88.

Table 1

Primitive Lie algebras of vector fields in $\mathbb{C}^2$

	Generators	Dim	Structure
1.1.	$\partial_x, \partial_u, x\partial_x - u\partial_u, u\partial_x, x\partial_u$	5	$\mathfrak{sa}(2)$
1.2.	$\partial_x, \partial_u, x\partial_x, u\partial_x, x\partial_u, u\partial_u$	6	$\mathfrak{a}(2)$
1.3.	$\partial_x, \partial_u, x\partial_x, u\partial_x, x\partial_u, u\partial_u, x^2\partial_x + xu\partial_u, xu\partial_x + u^2\partial_u$	8	$\mathfrak{sl}(3)$

Table 2

Transitive, imprimitive Lie algebras of vector fields in $\mathbb{C}^2$

	Generators	Dim	Structure
2.1.	$\partial_x, x\partial_x - u\partial_u, x^2\partial_x - 2xu\partial_u$	3	$\mathfrak{sl}(2)$
2.2.	$\partial_x, x\partial_x - u\partial_u, x^2\partial_x - (2xu+1)\partial_u$	3	$\mathfrak{sl}(2)$
2.3.	$\partial_x, x\partial_x, u\partial_u, x^2\partial_x - xu\partial_u$	4	$\mathfrak{gl}(2)$
2.4.	$\partial_x, x\partial_x, x^2\partial_x, \partial_u, u\partial_u, u^2\partial_u$	6	$\mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$
2.5.	$\partial_x, \eta_1(x)\partial_u, \ldots, \eta_k(x)\partial_u$	$k+1$	$\mathbb{C} \ltimes \mathbb{C}^k$
2.6.	$\partial_x, u\partial_u, \eta_1(x)\partial_u, \ldots, \eta_k(x)\partial_u$	$k+2$	$\mathbb{C}^2 \ltimes \mathbb{C}^k$
2.7.	$\partial_x, x\partial_x + \alpha u\partial_u, \partial_u, x\partial_u, \ldots, x^{k-1}\partial_u$	$k+2$	$\mathfrak{a}(1) \ltimes \mathbb{C}^k$
2.8.	$\partial_x, x\partial_x + (ku + x^k)\partial_u, \partial_u, x\partial_u, \ldots, x^{k-1}\partial_u$	$k+2$	$\mathfrak{a}(1) \ltimes \mathbb{C}^k$
2.9.	$\partial_x, x\partial_x, u\partial_u, \partial_u, x\partial_u, x^2\partial_u, \ldots, x^{k-1}\partial_u$	$k+3$	$(\mathfrak{a}(1) \oplus \mathbb{C}) \ltimes \mathbb{C}^k$
2.10.	$\partial_x, 2x\partial_x + (k-1)u\partial_u, x^2\partial_x + (k-1)xu\partial_u,$ $\partial_u, x\partial_u, x^2\partial_u, \ldots, x^{k-1}\partial_u$	$k+3$	$\mathfrak{sl}(2) \ltimes \mathbb{C}^k$
2.11.	$\partial_x, x\partial_x, x^2\partial_x + (k-1)xu\partial_u, u\partial_u,$ $\partial_u, x\partial_u, x^2\partial_u, \ldots, x^{k-1}\partial_u$	$k+4$	$\mathfrak{gl}(2) \ltimes \mathbb{C}^k$

In Cases 2.5 and 2.6, the functions $\eta_1(x), \ldots, \eta_k(x)$ satisfy a k^{th} order constant coefficient homogeneous linear ordinary differential equation $\mathcal{D}[u] = 0$.

In Cases 2.5 – 2.11 we require $k \geq 1$. Note, though, that if we set $k = 0$ in Case 2.10, and replace u by u^2, we obtain Case 2.1. Similarly, if we set $k = 0$ in Case 2.11, we obtain Case 2.3. Cases 2.7 and 2.8 for $k = 0$ are equivalent to the Lie algebra $\{\partial_x, e^x\partial_u\}$ of type 2.5. Case 2.9 for $k = 0$ is equivalent to the Lie algebra $\{\partial_x, \partial_u, u\partial_u\}$ of type 2.6.

Table 3

Intransitive Lie algebras of vector fields in $\mathbb{C}^2$

	Generators	Dim	Structure
3.1.	$\eta_1(x)\partial_u, \ldots, \eta_k(x)\partial_u$	k	$\mathbb{C}^k$
3.2.	$\eta_1(x)\partial_u, \ldots, \eta_k(x), u\partial_u$	$k+1$	$\mathbb{C} \ltimes \mathbb{C}^k$
3.3.	$\partial_u, u\partial_u, u^2\partial_u$	3	$\mathfrak{sl}(2)$

Table 4

Lie algebras of contact transformations in $\mathbb{C}^2$

	Generators	Dim	Structure
4.1.	$1, x, x^2, u_x, xu_x, u_x^2$	6	$\mathfrak{sa}(2) \ltimes \mathbb{C}$
4.2.	$1, x, x^2, u, u_x, xu_x, u_x^2$	7	$\mathfrak{a}(2) \ltimes \mathbb{C}$
4.3.	$1, x, x^2, u, u_x, xu_x, x^2u_x - 2xu,$ $u_x^2, 2uu_x - xu_x^2, 4xuu_x - 4u^2 - x^2u_x^2$	10	$\mathfrak{so}(5) \simeq \mathfrak{sp}(4)$

Table 5

Differential invariants of transformation groups in $\mathbb{C}^2$

	Fundamental differential invariant(s)	Invariant one-form	Lie determinant
1.1.	$u_2^{-8/3} R_4$	$u_2^{1/3}\, dx$	u_2^3
1.2.	$R_4^{-3/2} S_5$	$u_2^{-1}\sqrt{R_4}\, dx$	$u_2^2 R_4$
1.3.	$S_5^{-8/3} V_7$	$u_2^{-1} S_5^{1/3}\, dx$	$u_2 S_5^2$
2.1.	$u^{-4}(2uu_2 - 3u_1^2)$	$u\, dx$	u^2
2.2.	$(u_1 - u^2)^{-3/2}(u_2 - 6uu_1 + 4u^3)$	$\sqrt{u_1 - u^2}\, dx$	$u_1 - u^2$
2.3.	$Q_2^{-3/2} S_3$	$u^{-1}\sqrt{Q_2}\, dx$	$u\, Q_2$
2.4.	$Q_3^{-3} U_5$	$u_1^{-1}\sqrt{Q_3}\, dx$	$u_1 Q_3^2$
2.5.	$W(x)^{-1}\mathcal{D}[u]$	dx	$W(x)$
2.6.	$D_x \log \mathcal{D}[u]$	dx	$W(x)\mathcal{D}[u]$
2.7a.	$u_k^{(\alpha-k)^{-1}-1} u_{k+1} \qquad k \neq \alpha$	$u_k^{-(\alpha-k)^{-1}}\, dx$	u_k
2.7b.	$u_k, \quad u_{k+1}^{-2} u_{k+2} \qquad k = \alpha$	$u_{k+1}\, dx$	u_{k+1}
2.8.	$u_{k+1} e^{u_k/k!}$	$e^{-u_k/k!}\, dx$	1
2.9.	$u_{k+1}^{-2} u_k u_{k+2}$	$u_k^{-1} u_{k+1}\, dx$	$u_k u_{k+1}$
2.10.	$u_k^{-2(k+3)/(k+1)} Q_{k+2}$	$u_k^{2/(k+1)}\, dx$	u_k^2
2.11.	$Q_{k+2}^{-3/2} S_{k+3}$	$u_k^{-1}\sqrt{Q_{k+2}}\, dx$	$u_k Q_{k+2}$
3.1.	$x, \quad \mathcal{D}[u]$	dx	$W(x)$
3.2.	$x, \quad D_x \log \mathcal{D}[u]$	dx	$W(x)\mathcal{D}[u]$
3.3.	$x, \quad u_1^{-2} Q_3$	dx	u_1^3
4.1	$u_3^{-8/3} \widetilde{R}_5$	$u_3^{1/3}\, dx$	u_3^3
4.2	$\widetilde{R}_5^{-3/2} \widetilde{S}_6$	$u_3^{-1}\sqrt{\widetilde{R}_5}\, dx$	$u_3^2 \widetilde{R}_5$
4.3	$T_7^{-5/2} Z_9$	$u_3^{-1} T_7^{1/4}\, dx$	$u_3 T_7^2$

$W(x)$ denotes the Wronskian determinant of $\eta_1(x), \ldots, \eta_k(x)$, and $\mathcal{D}$ is a k^{th} order linear ordinary differential operator whose kernel is spanned by $\eta_1(x), \ldots, \eta_k(x)$. Furthermore,

$$Q_{k+2} = (k+1)u_k u_{k+2} - (k+2)u_{k+1}^2, \quad R_4 = 3u_2 u_4 - 5u_3^2,$$
$$S_{k+3} = (k+1)^2 u_k^2 u_{k+3} - 3(k+1)(k+3)u_k u_{k+1} u_{k+2} + 2(k+2)(k+3)u_{k+1}^3,$$
$$\widetilde{R}_5 = 3u_3 u_5 - 5u_4^2, \quad \widetilde{S}_6 = 9u_3^2 u_6 - 45u_3 u_4 u_5 + 40u_4^3,$$
$$T_7 = 10u_3^3 u_7 - 70u_3^2 u_4 u_6 - 49u_3^2 u_5^2 + 280u_3 u_4^2 u_5 - 175u_4^4,$$
$$U_5 = u_1^2\left[Q_3 D_x^2 Q_3 - \tfrac{5}{4}(D_x Q_3)^2\right] + u_1 u_2 Q_3 D_x Q_3 - (2u_1 u_3 - u_2^2)Q_3^2,$$
$$V_7 = u_2^2\left[S_5 D_x^2 S_5 - -\tfrac{7}{6}(D_x S_5)^2\right] + u_2 u_3 S_5 D_x S_5 - \tfrac{1}{2}(9u_2 u_4 - 7u_3^2)S_5^2,$$
$$Z_9 = u_3^2\left[T_7 D_x^2 T_7 - \tfrac{9}{8}(D_x T_7)^2\right] + u_3 u_4 T_7 D_x T_7 - \tfrac{4}{5}(7u_3 u_5 - 5u_4^2)T_7^2.$$