Conservation Laws in Elasticity

I. General Results

PETER J. OLVER

Communicated by C. DAFERMOS

Abstract

In this paper the basic results involved in the application of NOETHER's theorem relating symmetry groups and conservation laws to the variational problems of homogeneous elastostatics are outlined. General methods and conditions for the existence of variational and generalized symmetries are presented. Applications will be considered in subsequent papers in this series.

1. Introduction

This is the first in a series of papers devoted to applying NOETHER's general theorem relating symmetry groups and conservation laws to the variational problems of linear and nonlinear elasticity. Although NOETHER's theorem has been available for over sixty years, and despite the well-acknowledged importance of group theory in elasticity, this series of papers are, to the best of my knowledge, the first systematic implementation of the full power of NOETHER's theorem in this field. Indeed, not until the work of GÜNTHER, [14], and KNOWLES & STERNBERG, [16], was even a limited variant of NOETHER's theorem applied to elasticity. (This situation is, however, not unique to elasticity, as recent work on new conservation laws in fluid mechanics, [3], has made clear.) The full historical reasons behind the singular delay in adequately applying this powerful theorem to even the most basic systems arising in mathematical physics and engineering are not at all clear, and would make an extremely interesting study in the history of mathematics in this century. To this day, it is fair to say that NOETHER's theorem remains the most quoted, but most under-utilized result in all of the literature of mathematical physics.

This state of affairs becomes even more incredible when one realizes that the basic techniques are completely constructive and amenable to straight-forward computational methods. (These are mechanical enough that one can easily envisage implementing them on a symbol manipulating program.) For any system of differential equations (or even free boundary problems, [3]) the Lie-Ovsiannikov theory provides the general computational framework for the complete classification of all continuous geometrical (or Lie) symmetry groups *i.e.* those realized as physical transformations on the space representing the independent and dependent variables of the system. (For basic references, see [5, 22, 25].) For systems arising from variational principles, NOETHER's theorem provides a means for associating a conservation law to each one-parameter variational symmetry group. These groups, which are included among all the symmetries of the Euler-Lagrange equations, are characterized by their leaving the variational integral itself invariant over arbitrary subdomains. It is this first version of NOETHER's theorem that GÜNTHER and KNOWLES & STERNBERG employed in their analysis.

In 1922, BESSEL-HAGEN, [4], showed how to generalize NOETHER's theorem to give essentially a one-to-one correspondence between generalized variational symmetries and conservation laws. There are two main directions of generalization. First, enlarge the class of symmetries to include transformations depending on the derivatives of the dependent variables, as well as the independent and dependent variables themselves—the generalized symmetries. (These can no longer be realized as geometrical transformations on any finite dimensional space.) More recently, these generalized symmetries have resurfaced in the theory of "soliton" equations, [19, 12], where they also go under the unfortunate misnomer of "Lie-Bäcklund transformations", [1]. BESSEL-HAGEN'S second generalization was to expand the class of variational symmetries to include those (generalized) symmetry groups which infinitesimally leave the Lagrangian in the variational integral invariant only up to a divergence. Besides [4], a complete discussion of this result can be found in [1] and [22].

KNOWLES & STERNBERG did not use either of BESSEL-HAGEN'S generalizations, so their claims of completeness in the classification of conservation laws cannot be correct. This will be borne out in subsequent papers in this series, where new conservation laws will be found. Their methods, however, have persisted in later discussions; *cf.* [6, 10]. In a recent paper EDELEN, [7], makes the same criticism of their work. One of the main purposes of this series is to implement EDELEN'S problem of providing "a detailed classification of all invariance transformations and conservation laws" both for linear and nonlinear elasticity.

In elasticity, besides more general applications to global existence and conservative properties of solutions, conservation laws are of especial interest for problems in propagation of cracks in elastic media, [2, 26], dislocation theory, [9, 17] and scattering of waves in elastic media. Future papers in this series will treat the various applications of the conservation laws found. For the present, we concentrate on the variational problems arising in hyperelastostatics of homogeneous materials, although nonhomogeneities may be readily addressed by the same methods. In this context, ESHELBY'S energy-momentum tensor, [9], corresponding to translational invariance, was the first significant example of a conservation law. Extensions of our results to problems in elastodynamics, along the lines discussed in FLETCHER, [11], will be undertaken in a later paper.

In this first paper, the basics of symmetry group theory and NOETHER's theorem are outlined in a form amenable to applications in elasticity. The basic symmetry conditions for homogeneous variational problems of the form

$$\int W(\partial u)\,dx$$

are discussed, and some simplifications in the steps required to compute symmetries and conservation laws indicated. This material can be regarded as the necessary preliminary analysis required before proceeding to the specific examples to be discussed in subsequent papers. Some of these results appear in the Proceedings of the NATO Advanced Study Institute on Systems of Nonlinear Partial Differential Equations held in Oxford, England, July, 1982. It is a pleasure to thank JOHN BALL for originally sparking my interest in the applications of these general results to elasticity.

2. Noether's Theorem

In this section we outline the general form of NOETHER's theorem, [4], [18] relating symmetry groups of a variational problem to conservation laws of the associated Euler-Lagrange equations. Since the results are in slightly abbreviated form, we refer the reader to [1, 22] for more details.

A. Symmetry Groups of Differential Equations

Let $x=(x^1,\dots,x^p)\in\mathbb{R}^p$ be the independent and $u=(u^1,\dots,u^q)\in\mathbb{R}^q$ the dependent variables. (In three-dimensional elasticity $p=q=3$ and x is the material coordinate, $u=f(x)$ the deformation.) Consider a system of partial differential equations

$$\Delta(x, u, \partial u, \dots, \partial^n u) = 0, \tag{2.1}$$

where $\partial^k u$ represents the k^{th} order partial derivatives of u with respect to x, denoted

$$u^i_J = \partial^k u/\partial x^{j_1}\dots\partial x^{j_k}, \qquad J=(j_1,\dots,j_k), \qquad 1\leqq j_\nu\leqq p.$$

A *geometrical* or *Lie symmetry group* G is a connected local group of point transformations of $\mathbb{R}^p\times\mathbb{R}^q$:

$$g\colon (x,u)\mapsto(\tilde{x},\tilde{u}) = (A(x,u), B(x,u)).$$

The transformations g act on solutions $u=f(x)$ of (2.1) by transforming their graphs, thus

$$g\colon u=f(x)\mapsto \tilde{u}=\tilde{f}(\tilde{x}),$$

which is defined implicitly (locally) by the equation

$$B(x,f(x)) = \tilde{f}(A(x,f(x))).$$

By definition, G is a symmetry group of the system (2.1) if $\tilde{u}=\tilde{f}(\tilde{x})$ is a solution whenever $u=f(x)$ is for all $g\in G$.

LIE made the fundamental discovery that the symmetry group of a given system of differential equations can be effectively computed by infinitesimal techniques. The infinitesimal generators of a group are vector fields

$$\vec{v} = \sum_{\alpha} \xi^{\alpha}(x, u) \frac{\partial}{\partial x^{\alpha}} + \sum_{i} \varphi_i(x, u) \frac{\partial}{\partial u^i}. \tag{2.2}$$

The one-parameter subgroups can be recovered by integrating $\vec{v}$:

$$\frac{dx^{\alpha}}{d\varepsilon} = \xi^{\alpha}, \quad \frac{du^i}{d\varepsilon} = \varphi_i,$$

where ε is the group parameter.

Since G transforms functions $u = f(x)$, it also simultaneously transforms their derivatives. This defines the prolonged group action $\mathrm{pr}\, G$. Although this action is extremely complicated to write down explicitly (even for fairly simple groups), the infinitesimal generators, which are vector fields of the form

$$\mathrm{pr}\, \vec{v} = \vec{v} + \sum_{i,J} \varphi_i^J \frac{\partial}{\partial u_J^i}, \tag{2.3}$$

have a relatively simple expression:

Theorem 2.1 ([1, 22, 25]). *The coefficients* φ_i^J *of* $\mathrm{pr}\, \vec{v}$ *are given by*

$$\varphi_i^J = D^J \psi_i + \sum_{\alpha} \xi^{\alpha} u_{J,\alpha}^i, \tag{2.4}$$

where

$$\psi_i = \varphi_i - \sum_{\alpha} \xi^{\alpha} u_{\alpha}^i. \tag{2.5}$$

In these formulae $u_{\alpha}^i = \partial u^i / \partial x^{\alpha}$, $u_{J,\alpha}^i = \partial u_J^i / \partial x^{\alpha}$, and D^J denotes the total derivative

$$D^J = D_{j_1} D_{j_2} \ldots D_{j_k},$$

where D_j is the total derivative with respect to x^j.

Theorem 2.2. *The group G is a symmetry group of* (2.1) *if and only if*

$$\mathrm{pr}\, \vec{v}(\Delta) = 0 \quad \text{whenever} \quad \Delta = 0 \tag{2.6}$$

for all infinitesimal generators $\vec{v}$ of G.

(There are two technical conditions on (2.1) for this theorem to be true as stated. One is that the gradient of Δ with respect to all the variables x^{α}, u_J^i never vanishes on the set $\Delta = 0$; this is easy to verify and holds in almost every system of physical interest. The other is an existence result that the only restrictions on derivatives of solutions of the system are those following directly from the system itself and its derivatives. This is harder, but will be assumed in the subsequent exposition. See the appendix in [23] for a detailed discussion.)

In practice, (2.6) yields a large number of elementary differential equations that must be satisfied by the coefficient functions ξ^α, φ_i of any infinitesimal generator $\vec{v}$ of a symmetry group of (2.1). The general solution of this system, which can in most cases be found explicitly, constitutes the most general infinitesimal symmetry of (2.1). See [5, 22, 25] for examples of the calculation of symmetry groups by this technique.

B. Generalized Symmetries

This generalization of the notion of symmetry group arises by permitting the coefficient functions ξ^α, φ_i of the vector field $\vec{v}$, (2.2), to depend also on the derivatives u^i_J of u. The prolongation formulae (2.3–5) and infinitesimal symmetry condition (2.6) remain as before (although now (2.6) holds whenever $\Delta = 0$ *and* all derivatives $D^J \Delta = 0$.) There is no longer a nice geometrical interpretation of the group transformations themselves.

To understand generalized symmetries better, first change $\vec{v}$ into the *standard form*:

$$\tilde{v} = \sum_i \psi_i \frac{\partial}{\partial u^i}, \tag{2.7}$$

where ψ_i is defined by (2.5). Note that by (2.4), the prolongation of $\tilde{v}$ has the simple form

$$\mathrm{pr}\, \tilde{v} = \sum_{i,J} D^J \psi_i \frac{\partial}{\partial u^i_J}. \tag{2.8}$$

Lemma 2.3, [22]. *A vector field $\vec{v}$ is a (generalized) symmetry of* (2.1) *if and only if its standard form $\tilde{v}$ is.*

For this reason, we can work exclusively with vector fields in standard form. Note that even if $\vec{v}$ generates a Lie symmetry group, $\tilde{v}$ is a generalized symmetry since by (2.5) the ψ_i depend on first derivatives u^i_α. However, we immediately have the criterion that a generalized symmetry $\tilde{v}$ is the standard form of a geometrical symmetry if and only if ψ_i have the form (2.5) where φ_i and ξ^α depend only on x and u.

The integration of a generalized symmetry in standard form can be effected by solving the system of evolution equations, [1, 22],

$$\frac{\partial u^i}{\partial \varepsilon} = \psi_i, \quad u(x, 0) = f(x). \tag{2.9}$$

Again ε denotes the group parameter, so that the transformations take the form

$$g_\varepsilon : f(x) = u(x, 0) \mapsto \tilde{f}_\varepsilon(x) = u(x, \varepsilon),$$

where $u(x, \varepsilon)$ is the solution of (2.9). (Here we are assuming that (2.9) is locally uniquely solvable for initial data $u = f(x)$ in some appropriate space of functions.) Thus $\tilde{v}$ is a generalized infinitesimal symmetry of the system (2.1) if $u = \tilde{f}_\varepsilon(x)$ is a solution whenever $u = f(x)$ is. With this interpretation, theorem 2.2 remains in force.

C. Symmetries of Variational Problems

For a variational problem

$$I_\Omega[u] = \int_\Omega W(x, u, \partial u, \ldots, \partial^n u)\, dx \tag{2.10}$$

with Euler-Lagrange equations

$$\frac{\delta I}{\delta u} = 0 \tag{2.11}$$

there are several types of symmetry groups. The most common, and most restrictive, is to require that the integral I_Ω is invariant under all group transformations:

$$I_{\tilde{\Omega}}[\tilde{u}] = I_\Omega[u]$$

for all $u = f(x)$, all subdomains $\Omega \subset \mathbb{R}^p$, all $g \in G$ where $\tilde{u} = \tilde{f}(\tilde{x})$ is the transformed function and $\tilde{\Omega} = g \cdot \Omega$ the transformed domain (which may depend on f itself). The group is necessarily geometrical. The corresponding infinitesimal condition is

$$\operatorname{pr} \vec{v}(W) + W \operatorname{Div} \xi = 0 \tag{2.12}$$

for every infinitesimal generator $\vec{v}$ of G, where $\operatorname{Div} \xi = \Sigma D_\alpha \xi^\alpha$; *cf.* [1, 22, 25].

BESSEL-HAGEN, [4], noted that one can easily generalize these symmetry groups to include *divergence symmetries*, with infinitesimal criterion

$$\operatorname{pr} \vec{v}(W) + W \operatorname{Div} \xi = \operatorname{Div} B \tag{2.13}$$

for some p-tuple $B = (B_1, \ldots, B_p)$. The effect of the corresponding group transformations on I_Ω is less obvious.

Proposition 2.4, [22]. If $\vec{v}$ is an infinitesimal divergence symmetry of (2.10), then $\vec{v}$ is an infinitesimal symmetry of the Euler-Lagrange equations (2.11).

The converse is *not* true, the main source of counterexamples being groups of scale transformations. This proposition provides an effective means for computing divergence symmetries: namely it suffices to check which of the symmetries of (2.11), calculated using theorem 2.2, satisfy the additional criterion (2.13). This avoids the awkward fact that B in (2.13) is not known *a priori*. See [22] for examples.

In particular, $\vec{v}$ is a divergence symmetry if and only if its standard form $\tilde{v}$ is. (This is not true for the restrictive criterion (2.12).) For $\tilde{v}$ in standard form, (2.13) simplifies to

$$\mathrm{pr}\,\tilde{v}(W) = \mathrm{Div}\, B \tag{2.14}$$

for some B (not necessarily the same as above).

Finally we can generalize (2.14) to include generalized symmetry groups. For brevity, given a variational problem *symmetry* will mean infinitesimal generalized symmetry in standard form of the Euler-Lagrange equations, and *variational symmetry* an infinitesimal divergence symmetry in standard form of the variational problem. In particular, every variational symmetry is a symmetry, but not conversely.

D. Conservation Laws and Noether's Theorem

Given a system of partial differential equations (*e.g.* the Euler-Lagrange equations for some variational problem), a *conservation law* is an equation of the form

$$\mathrm{Div}\, A = 0, \tag{2.15}$$

where $A = (A_1, \ldots, A_p)$ can depend on x, u and the derivatives of u, which must be satisfied for all solutions $u = f(x)$ of the given system.

A conservation law is trivial if (2.15) holds identically. This is equivalent, [21], to the statement that

$$A_\alpha = \sum_\beta D_\beta B_{\alpha\beta} \tag{2.16}$$

where $B_{\alpha\beta}$ depend on x, u, and derivatives of u, with

$$B_{\alpha\beta} = -B_{\beta\alpha}.$$

A deeper characterization of trivial conservation laws is discussed in [23]; this will be used in subsequent analysis here.

NOETHER's theorem asserts that to each variational symmetry of a given variational problem, there corresponds a nontrivial conservation law.

Theorem 2.5 [1, 22]. *If $\tilde{v}$ as in* (2.7) *is a standard variational symmetry of* (2.10), *then the expression*

$$\Sigma \psi_i \frac{\delta I}{\delta u^i} = \mathrm{Div}\, A \tag{2.17}$$

constitutes a conservation law for the Euler-Lagrange equations (2.11).

Explicit formulae for the p-tuple A in (2.16) can be given (*cf.* [22]) but in practice it is simpler to reconstruct A directly.

As detailed in [22], there is also a converse to this theorem, so that to every (nontrivial) conservation law there is a corresponding generalized symmetry. The q-tuple $\psi = (\psi_1, \ldots, \psi_p)$ in (2.16) is called the *characteristic* of the conservation law

$$\operatorname{Div} A = 0.$$

Note that trivial conservation laws have zero characteristic.

3. Hyperelasticity and Variational Problems

In three-dimensional elasticity, the independent variables $x = (x^1, x^2, x^3) \in B \subset \mathbb{R}^3$ represent material coordinates, the dependent variables $u = (u^1, u^2, u^3)$ the deformation so that a particle at position x is deformed to position $u(x)$. The deformation gradient is $\partial u(x)$, which is a 3×3 matrix with entries $u^i_\alpha = \partial u^i / \partial x^\alpha$.

The equations of elastostatics for a hyperelastic material arise as the Euler-Lagrange equations for the variational problem

$$\int_B \{W(x, \partial u) + b(x, u)\}\, dx \tag{3.1}$$

where W is the stored energy function and b the body-force potential.

The body is *homogeneous* if W is independent of x. For simplicity we consider the case of a homogeneous elastic body in the absence of body forces, so that (3.1) simplifies to

$$I = \int_B W(\partial u)\, dx. \tag{3.2}$$

(A subsequent paper will detail how the results change in the more general problem (3.1).)

More generally, we can consider variational problems of the type (3.2) with $x = (x^1, \ldots, x^p) \in B \subset \mathbb{R}^p$ and $u = (u^1, \ldots, u^q) \in \mathbb{R}^q$. For n-dimensional elasticity $p = q = n$, but many of our results will remain true even if $p \neq q$.

We shall use summation notation on repeated indices throughout. Latin indices i, j, k, l will run from 1 to q, while Greek indices $\alpha, \beta, \gamma, \delta$ will run from 1 to p. The stored energy or Lagrangian $W(\partial u)$ will be assumed to be at least C^3 in its arguments u^i_α, although this assumption can certainly be weakened in certain results.

The Euler-Lagrange equations for (3.2) take the form $\mathscr{E}^i(W) = 0$, where

$$-\mathscr{E}^i = \frac{\delta}{\delta u^i} = -D_\alpha \frac{\partial}{\partial u^i_\alpha} + \frac{\partial}{\partial u^i} \tag{3.3}$$

is the Euler operator or variational derivative. Let

$$W^i_\alpha = \frac{\partial W}{\partial u^i_\alpha}, \quad W^{ij}_{\alpha\beta} = \frac{\partial^2 W}{\partial u^i_\alpha\, \partial u^j_\beta}, \quad \text{etc.,}$$

so that

$$\mathscr{E}^i(W) = D_\alpha W_\alpha^i = W_{\alpha\beta}^{ij} u_{\alpha\beta}^j = 0. \tag{3.4}$$

Note that $W_{\alpha\beta}^{ij}$ depends on ∂u. We *linearize* (3.4) by fixing ∂u at some value; the resulting equations are just the Euler-Lagrange equations for the quadratic variational problem

$$I = \int C_{\alpha\beta}^{ij} u_\alpha^i u_\beta^j \, dx,$$

where

$$C_{\alpha\beta}^{ij} = W_{\alpha\beta}^{ij}(\partial u_0) \tag{3.5}$$

is the elasticity tensor at the fixed deformation u_0; *cf.* [15]. (More commonly it is denoted $C_{i\alpha j\beta}$.) The Euler-Lagrange equations read

$$C_{\alpha\beta}^{ij} u_{\alpha\beta}^j = 0. \tag{3.6}$$

Fixing ∂u_0, define the symmetric $q \times q$ matrix $Q(\xi) = (q_{ij}(\xi))$ of quadratic polynomials in $\xi \in \mathbb{R}^p$ by

$$q_{ij}(\xi) = C_{\alpha\beta}^{ij} \xi^\alpha \xi^\beta. \tag{3.7}$$

We note that the *Legendre-Hadamard condition* for strong ellipticity of (3.2) can be written in terms of Q as

$$\eta^T Q(\xi) \, \eta \geqq 0 \tag{3.8}$$

for every $0 \neq \eta \in \mathbb{R}^q$, $0 \neq \xi \in \mathbb{R}^p$. The matrix Q will be used in our subsequent analysis.

4. Reduction to x, u-Independent Symmetries

The first step in the discussion of symmetries is to eliminate the x, u-dependence of the coefficient functions. In this section we give the basic method whereby this can be effected, and outline the procedure for finding the general symmetries from knowledge of the x, u-independent ones. Symmetries and variational symmetries/conservation laws must be treated separately, but the basic result is the same in each case.

A. The Symmetry Equations

To analyze the symmetries of the variational problem (3.2), the first step is to write down the symmetry conditions (2.6) for the Euler-Lagrange equations (3.5). Here we exclusively look at symmetries in standard form (2.7), whose coefficient functions $\overset{i}{\psi}$ depend only on x, u, ∂u. These include all Lie symmetries of (2.6) as well as first order generalized symmetries.

From (2.8), (3.4) we see

$$\operatorname{pr} \vec{v}(\mathscr{E}^i(W)) = W^{ij}_{\alpha\beta} D_\alpha D_\beta \overset{j}{\psi} + W^{ijk}_{\alpha\beta\gamma} u^j_{\alpha\beta} D_\gamma \overset{k}{\psi} \tag{4.1}$$

where

$$W^{ijk}_{\alpha\beta\gamma} = \frac{\partial^3 W}{\partial u^i_\alpha \, \partial u^j_\beta \, \partial u^k_\gamma},$$

and

$$\vec{v} = \overset{i}{\psi} \frac{\partial}{\partial u^i}. \tag{4.2}$$

Note that (4.1) involves derivatives of u of at most third order; hence for (2.6) to be satisfied

$$\operatorname{pr} \vec{v}(\mathscr{E}^i(W)) = \lambda^{ij}_\alpha D_\alpha \mathscr{E}^j(W) + \mu^{ij} \mathscr{E}^j(W), \tag{4.3}$$

for functions λ^{ij}_α, μ^{ij} depending on derivatives of u of order $\leqq 3$.

The terms on the left-hand side of (4.3) depending on $\partial^3 u$ or quadratic in $\partial^2 u$ are

$$W^{ij}_{\alpha\beta}\left[\overset{j}{\psi}{}^k_\gamma u^k_{\alpha\beta\delta} + \overset{j}{\psi}{}^{kl}_{\gamma\delta} u^k_{\alpha\gamma} u^l_{\beta\delta}\right] + W^{ijk}_{\alpha\beta\gamma} \overset{k}{\psi}{}^l_\delta u^j_{\alpha\beta} u^l_{\gamma\delta}$$

where $\overset{j}{\psi}{}^k_\gamma = \partial \overset{j}{\psi} / \partial u^k_\gamma$, *etc.* On the other hand

$$D_\alpha \mathscr{E}^j(W) = W^{jk}_{\beta\gamma} u^k_{\alpha\beta\gamma} + W^{jkl}_{\beta\gamma\delta} u^k_{\beta\gamma} u^l_{\alpha\delta}.$$

Substituting into (4.3), the coefficient of $u^k_{\alpha\beta\gamma}$ yields

$$W^{ij}_{\alpha\beta} \overset{j}{\psi}{}^k_\gamma = \lambda^{ij}_\alpha W^{jk}_{\beta\gamma}, \qquad \text{sym } [\alpha\beta\gamma]; \tag{4.4}$$

hence λ^{ij}_α depends only on x, u, ∂u. In (4.4) sym $[\alpha\beta\gamma]$ indicates that each side of the equation must be summed over all permutations of the indices $\alpha\beta\gamma$. Similarly, the coefficient of $u^k_{\alpha\gamma} u^l_{\alpha\delta}$ is, after use of (4.4),

$$\frac{\partial\left(\overset{j}{\psi}, W^{ij}_{\alpha\beta}\right)}{\partial(u^k_\gamma, u^l_\delta)} \equiv W^{ijl}_{\alpha\beta\delta} \overset{j}{\psi}{}^k_\gamma - W^{ijk}_{\alpha\beta\gamma} \overset{j}{\psi}{}^l_\delta = \left(\mu^{ijl}_{\alpha\delta} - \frac{\partial \gamma^{ij}_\alpha}{\partial u^l_\delta}\right) W^{jk}_{\beta\gamma}, \quad \text{sym } [k[\beta\gamma], l[\alpha\,\delta]], \tag{4.5}$$

where

$$\mu^{ij} = \mu^{ijl}_{\alpha\delta} u^l_{\alpha\delta} + \mu^{ij}_0,$$

$\mu^{ijl}_{\alpha\delta}$ and μ^{ij}_0 depending only on x, u, ∂u. Here (4.5) must be summed over permutations of $\beta\gamma$ and $\alpha\delta$ together with permutations of $k[\beta\gamma]$, $l[\alpha\delta]$. (In other words the subgroup of the groups of permutations of kl and $\alpha\beta\gamma\delta$ which leaves the monomial $u^k_{\beta\gamma} u^l_{\alpha\delta}$ unchanged.)

As a consequence of (4.4), (4.5) we find the important result.

Proposition 4.1. If $\vec{v} = \overset{i}{\psi}(x, u, \partial u)\, \partial/\partial u^i$ is a symmetry of (3.5), then for each fixed x_0, u_0, the vector field $\vec{v}_0 = \overset{i}{\psi}(x_0, u_0, \partial u)\, \partial/\partial u^i$ is also a symmetry.

This simplifies the computation of symmetries as follows: If $\vec{v}_1, \ldots, \vec{v}_N$ form a basis for the space of x, u-independent symmetries, *i.e.* those of the form $\overset{i}{\psi}(\partial u)\, \partial/\partial u^i$, then the proposition implies that all remaining symmetries depending on x, u, ∂u take the form

$$\vec{v} = \sum_{\varkappa=1}^{N} \chi^{\varkappa}(x, u)\, \vec{v}_{\varkappa} \tag{4.6}$$

for suitable functions $\chi^{\varkappa}$. Substituting (4.6) into the general symmetry equations (2.6) leads to a more manageable system for the coefficient functions $\chi^{\varkappa}$. It is thus good strategy to concentrate first on the computation of x, u-independent symmetries before proceeding to the general case. Note that since W is independent of x, u, the vector fields

$$\vec{k}_i = \frac{\partial}{\partial u^i}, \qquad \vec{p}_\alpha = u^i_\alpha \frac{\partial}{\partial u^i}, \tag{4.7}$$

representing translation in the u^i direction, and translation in the x^α direction, respectively, are always symmetries. If no other x, u-independent symmetries exist, we conclude that all symmetries of (3.4) depending on at most first order derivatives of u are geometrical; otherwise there exist generalized symmetries and, indeed, x, u-independent ones. Note further that the conservation laws corresponding to (4.7) are, respectively, the Euler-Lagrange equations themselves and ESHELBY'S energy-momentum tensor, [9].

B. Variational Symmetries

According to theorem 2.5, a vector field $\vec{v} = \overset{i}{\psi}\, \partial/\partial u^i$ is a variational symmetry of (3.2) if and only if

$$\overset{i}{\psi} \mathscr{E}^i(W) = \operatorname{Div} A = D_\alpha \overset{\alpha}{A}, \tag{4.8}$$

where A is the corresponding conservation law. We first note an intrinsic characterization of variational symmetries based on the symmetry equations (4.3); the proof will be deferred until subsection C.

Lemma 4.2. *A vector field $\vec{v}$ is a variational symmetry of* (3.2) *if and only if in* (4.3)

$$\lambda^{ij}_\alpha = \overset{j}{\psi}{}^i_\alpha \quad \text{and} \quad \mu^{ij} = \mathscr{E}^i\big(\overset{j}{\psi}\big) \tag{4.9}$$

(*cf.* (3.3)).

Thus if (4.9) hold, the existence of a conservation law A satisfying (4.8) is assured. A second useful fact is that A can, without loss of generality, be taken to depend on x, u, ∂u.

Lemma 4.3. *If* $\overset{i}{\psi}$ *depend only on* x, u, ∂u *and satisfy* (4.8), *then* $A = \tilde{A} + B$ *where* B *is a trivial null divergence, and* $\tilde{A}$ *depends on* x, u, ∂u.

This is a direct consequence of theorem 5.1 in [23] since the left-hand side of (4.8) is linear in $\partial^2 u$. Replacing A by $\tilde{A}$ in (4.8), the coefficient of $u^j_{\alpha\beta}$ is

$$\overset{i}{\psi} W^{ij}_{\alpha\beta} = \overset{\alpha}{A}{}^j_\beta, \quad \text{sym } [\alpha\beta] \tag{4.10}$$

where $\overset{\alpha}{A}{}^j_\beta = \partial \overset{\alpha}{A} / \partial u^j_\beta$. For $q = 1$, equations (4.10) form the conformal equations for a Riemannian manifold with metric $g_{\alpha\beta} = W^{11}_{\alpha\beta} + W^{11}_{\beta\alpha}$; *cf.* [8]. For this reason we name (4.10) *vector conformal equations.*

If ψ and A are independent of x and u, then (4.10) is equivalent to (4.8). Thus we have the analogue of proposition 4.1 for variational symmetries.

Proposition 4.4. If $\vec{v} = \overset{i}{\psi}(x, u, \partial u)\, \partial/\partial u^i$ is a variational symmetry of (3.2) with conservation law $A(x, u, \partial u)$, then for each fixed x_0, u_0, the vector field $\vec{v}_0 = \overset{i}{\psi}(x_0, u_0, \partial u)\, \partial/\partial u^i$ is a variational symmetry with conservation law $A(x_0, u_0, \partial u)$.

Thus a representation similar to (4.6) for variational symmetries exists. In the case of conservation laws, the simplification is more striking. If $A_1, \ldots, A_N$ form a basis for the x, u-independent conservation laws, (corresponding to variational symmetries $\vec{v}_1, \ldots, \vec{v}_N$), and $B_1, \ldots, B_M$ a basis for the null divergences depending only on ∂u (these are suitable combinations of Jacobian determinants; *cf.* [23]) then all conservation laws depending on x, u, ∂u are of the form

$$A = \sum_{\varkappa=1}^{N} \omega_\varkappa(x, u)\, A_\varkappa + \sum_{\iota=1}^{M} \theta_\iota(x, u)\, B_\iota \tag{4.11}$$

for appropriate scalar function ω, θ.

Theorem 4.5. *If* $A_1, \ldots, A_N$, $B_1, \ldots, B_M$, A *are as above, then* A *is a conservation law if and only if*

$$\Sigma(D_\alpha \omega_\varkappa)\, A^\alpha_\varkappa + \Sigma(D_\alpha \theta_\iota)\, B^\alpha_\iota = 0 \tag{4.12}$$

identically in x, u, ∂u. ($A_\varkappa = (A^1_\varkappa, \ldots, A^p_\varkappa)$, etc.)

Proof. Substitute (4.11) into (4.8) and note that $A_\varkappa$, B_ι are already conserved, so the only remaining terms on the right-hand side are given by (4.12). Moreover these only depend on x, u, ∂u, hence must vanish.

C. Proof of Lemma 4.2

The result is equivalent to the formula

$$\text{pr}\, \vec{v}\, [\mathscr{E}^i(W)] = \overset{j}{\psi}{}^i_\alpha D_\alpha \mathscr{E}^j(W) + \mathscr{E}^i\left(\overset{j}{\psi}\right) \mathscr{E}^j(W) \tag{4.13}$$

being necessary and sufficient for $\vec{v}$ to be variational. There are a number of ways to establish this formula. One method is to apply the Euler operator $\mathscr{E}^i$ directly to (4.8) and use the basic fact, [13, 21], that

$$\mathscr{E}^i \operatorname{Div} A = 0$$

for any A. The computations are rather lengthy in general. They can be simplified using formulae in chapter 5 of [22], specifically those required in the proof of proposition 5.13 there, which is the same as proposition 2.4 here. Alternatively, the analogue of proposition 2.6 from [20] for partial differential operators will work. In all cases, however, the converse is considerable more tricky.

Computationally, the easiest method is to utilize the theory of differential forms in the formal calculus of variations developed in [21] (see also [13]), whose notation we use here. Since

$$d_* W = -\mathscr{E}^i(W)\, du^i$$

(since the Euler operator (3.3) here is the negative of the usual one), if $\vec{v}$ is a variational symmetry,

$$\begin{aligned} 0 &= -d_*[\operatorname{pr} \vec{v}(W)], \\ &= -\operatorname{pr} \vec{v}(d_* W), \\ &= \operatorname{pr} \vec{v}[\mathscr{E}^i(W)\, du^i], \\ &= \operatorname{pr} \vec{v}[\mathscr{E}^i(W)]\, du^i + \mathscr{E}^i(W)\, d_* \overset{i}{\psi}. \end{aligned}$$

Moreover, integration by parts shows

$$\begin{aligned} \mathscr{E}^i(W)\, d_* \overset{i}{\psi} &= \mathscr{E}^i(W)\, \overset{i}{\psi}{}^j_\alpha\, du^j_\alpha + \mathscr{E}^i(W)\, \partial \overset{i}{\psi} / \partial u^j .\, du^j, \\ &= -D_\alpha[\mathscr{E}^i(W)\, \overset{i}{\psi}{}^j_\alpha]\, du^j + \mathscr{E}^i(W)\, \partial \overset{i}{\psi} / \partial u^j .\, du^j, \\ &= -\left\{\overset{i}{\psi}{}^j_\alpha D_\alpha \mathscr{E}^i(W) + \mathscr{E}^i(W)\, \mathscr{E}^j\big(\overset{i}{\psi}\big)\right\} du^j. \end{aligned}$$

Therefore, changing indices, we have

$$0 = \left\{\operatorname{pr} \vec{v}[\mathscr{E}^i(W)] - \overset{j}{\psi}{}^i_\alpha D_\alpha \mathscr{E}^j(W) - \mathscr{E}^i\big(\overset{j}{\psi}\big)\, \mathscr{E}^j(W)\right\} du^j,$$

from which (4.13) follows immediately.

To prove the converse, suppose (4.13) holds. Then by the above computation

$$d_*[\operatorname{pr} \vec{v}(W)] = 0.$$

But this implies

$$\operatorname{pr} \vec{v}(W) = \operatorname{Div} A,$$

hence the converse.

(This will be the only place we will utilize this differential form theory, but the above proof should give the reader some idea of its power and efficacy for proving complicated variational formulae.)

5. Linearization and Symmetries

From now on we restrict attention to x, u-independent symmetries. The main result of this section is that if the variational problem (3.2) admits an x, u independent symmetry, then for any fixed ∂u_0 the corresponding linearized variational problem admits a linearized version of the same symmetry. Of course, truly nonlinear conditions also arise, so the above condition is not sufficient. We subsequently analyze the linear symmetry conditions, which reduce to questions about matrices of quadratic polynomials, but only partial results have been determined so far, leaving many open questions.

A. Linearized Symmetries

Given a vector field $\vec{v} = \overset{i}{\psi}\,\partial/\partial u^i$, and a fixed deformation gradient ∂u_0, defines the *linearized vector field*

$$\vec{v}_0 = \overset{i}{\psi}{}^j_\alpha(\partial u_0)\, u^j_\alpha \frac{\partial}{\partial u^i}.$$

Theorem 5.1. *If the nonlinear variational problem* (3.2) *admits an* x, u*-independent symmetry* $\vec{v}$, *then for each fixed* ∂u_0, *the linearized vector field* $\vec{v}_0$ *is a symmetry of the corresponding linearized problem* (3.6). *The same result holds for variational symmetries, although the forms of the corresponding conservation laws necessarily differ.*

Proof. It suffices to note that (x, u-independent) $\vec{v} = \overset{i}{\psi}\,\partial/\partial u^i$ is a symmetry of (3.6) if and only if

$$C^{ij}_{\alpha\beta}\overset{j}{\psi}{}^k_\gamma = \lambda^{ij}_\alpha C^{jk}_{\beta\gamma}, \qquad \text{sym}\,[\alpha\beta\gamma] \tag{5.1}$$

holds; indeed this is just (4.4), and the quadratic terms in $\partial^2 u$ leading to (4.5) do not appear in the case of a linear symmetry. But (5.1) is just (4.4) at fixed ∂u_0, so the result holds. The statement for variational symmetries follows analogously from lemma 4.2.

Analyzing (5.1) further, recall that $Q(\xi)$ is the matrix with entries

$$q_{ij}(\xi) = C^{ij}_{\alpha\beta}\xi^\alpha\xi^\beta.$$

Define $L(\xi)$ to be the matrix with

$$l_{ij}(\xi) = \overset{i}{\psi}{}^j_\alpha\xi^\alpha,$$

and $M(\xi)$ to have entries

$$m_{ij}(\xi) = \lambda^{ij}_\alpha\xi^\alpha.$$

Then (5.1) is just the matrix equation

$$Q(\xi)\,L(\xi) = M(\xi)\,Q(\xi). \tag{5.2}$$

Proposition 5.2. Let $Q(\xi)$ be the matrix of quadratic polynomials corresponding to the quadratic variational problem (3.6). Then (3.6) admits a nonvariational symmetry if and only if there exists a non-zero matrix $N(\xi)$ of linear polynomials is ξ such that

$$Q(\xi)\, N(\xi) = -N^T(\xi)\, Q(\xi). \tag{5.3}$$

Similarly, (3.6) admits an x, u-independent variational symmetry not of the form (4.7) if and only if there exists a matrix $L(\xi)$ of linear polynomials in ξ, with $L(\xi) \neq l(\xi)\, \mathrm{I}$, such that

$$Q(\xi)\, L(\xi) = L^T(\xi)\, Q(\xi). \tag{5.4}$$

Proof. The second statement is obvious from lemma 4.2. Thus there exist nonvariational symmetries if and only if (5.2) holds for some $M \neq L^T$. Set $N = M - L^T$, and it easily follows that N satisfies (5.3).

It remains to determine what (5.3) or (5.4) imply for the form of Q. The second condition seems particularly difficult, and I have been unable to make any progress in discerning its general meaning. In three dimensions (5.3) can be fully analyzed, as will be seen.

B. Nonvariational Symmetries in three Dimensions

In this section we restrict attention to quadratic variational problems satisfying the Legendre-Hadamard condition with $x \in \mathbb{R}^3$, $u \in \mathbb{R}^3$.

Theorem 5.3. *Suppose $W(\partial u)$ is quadratic, satisfies the Legendre-Hadamard condition. The linear Euler-Lagrange equations admit nonvariational symmetries if and only if there is a linear change of variables $\tilde{u} = Au$ such that either*

a) $$W(\partial\tilde{u}) = F_1(\partial\tilde{u}^1) + F_1(\partial\tilde{u}^2) + F_3(\partial\tilde{u}^3),$$

so the Euler-Lagrange equations decouple with at least two being identical, or

b) $$W(\partial\tilde{u}) = F(\partial\tilde{u}^1) + F(\partial\tilde{u}^2) + F(\partial\tilde{u}^3) + [g(\partial\tilde{u})]^2,$$

where F is quadratic and g linear in their arguments.

Note that the case of linear isotropic elasticity falls into case b), which we therefore name *quasi-isotropic*. For most g, a further linear change in the x variables will convert this to the isotropic case. The Euler-Lagrange equation for a quasi-isotropic W take the form

$$Lu + (\delta \otimes \delta)\, u = 0,$$

where L is a scalar second order and δ a first order differential operator.

For a nonlinear variational problem, the question of whether every linearized version is equivalent, under a change of variables, to a decoupled or quasi-isotropic quadratic problem seems to be rather difficult. The problem is that the linear charge of variables can depend on the point ∂u_0 at which the linearization is taken. More work is needed on this problem. One further cautionary note should be added. It is not true that for (3.2) to admit nonvariational symmetries every linearized problem admit nonvariational symmetries. Indeed, in (4.9) if the λ_α^{ij} are correct, the corresponding linearized symmetry must be variational, whereas it does not necessarily follow that the μ^{ij} will be of the right form.

Lemma 5.4. *If Q is symmetric, positive definite, QN skew-symmetric, then*

$$Q = C^T Q_1 C,$$

C independent of ξ, with either

a)

$$Q_1 = \Delta(\xi) + \boldsymbol{\lambda}(\xi) \otimes \boldsymbol{\lambda}(\xi), \tag{5.5}$$

or

b)

$$Q_1 = \Delta(\xi) + \mu(\xi)\, M(\xi), \tag{5.6}$$

where Δ is a diagonal matrix of quadratic functions δ_1, δ_2, δ_3, and μ, $\boldsymbol{\lambda}$, M are, respectively, a scalar, vector, matrix of linear functions.

Proof. The (1, 1) entry of QN is

$$\sum_j q_{1j}(\xi)\, n_{j1}(\xi) = 0. \tag{5.7}$$

Since q_{11} is positive definite,

$$\{\xi \mid n_{11} = 0\} \subset \{\xi \mid n_{21} = n_{31} = 0\},$$

hence there are constants a, b with

$$n_{11} + an_{21} + bn_{31} = 0.$$

Then (5.7) reads

$$n_{21}(q_{12} - aq_{11}) + n_{31}(q_{13} - bq_{11}) = 0;$$

hence there is a linear $\mu_1(\xi)$ with $q_{12} - aq_{11} = n_{31}\mu_1$, $q_{13} - bq_{11} = -n_{21}\mu_1$. Let

$$C_1 = \begin{pmatrix} 1 & -a & -b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix};$$

so that the matrix

$$\tilde{Q} = C_1^T Q C_1$$

has first row

$$(q_{11}, \mu_1 n_{31}, -\mu_1 n_{21}).$$

This changes N to

$$\tilde{N} = C^{-1}NC_1,$$

with first column

$$(0, n_{21}, n_{11})^T.$$

Similar analysis of the (2, 2) entry of $\tilde{Q}\tilde{N}$ shows that

$$\tilde{q}_{21} - a'\tilde{q}_{22} = \tilde{n}_{32}\mu_2, \tilde{q}_{23} - b'\tilde{q}_{22} = \tilde{n}_{12}\mu_2$$

for constants a', b' and linear μ_2. Moreover, since $\tilde{Q}$ is symmetric, $a' = 0$, as otherwise $\tilde{q}_{22}$ would vanish whenever $\mu_1 = \mu_2 = 0$. Now set

$$C_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -b' \\ 0 & 0 & 1 \end{pmatrix},$$

$$Q^* = C_2^T\tilde{Q}C_2,$$

$$N^* = C_2^{-1}\tilde{N}C_2.$$

A similar analysis of the (3, 3) entry shows that the off-diagonal entries of Q^* are all products of linear functions of ξ. Further, the symmetry of Q^* shows easily that there is a diagonal matrix C_3 with

$$Q_1 = C_3^T Q^* C_3$$

and with Q_1 of one of the two forms in the lemma.

We now prove theorem 5.3 for Q_1 of the form (5.5). (For Q_1 of the form (5.6) it can be shown by similar methods that Q is similar to a diagonal matrix.) Assume no two entries of $\boldsymbol{\mu}$ are multiples of the same linear function, otherwise we are back in case (5.6). The above calculations show that $N_1 = CNC^{-1}$ takes the form

$$N_1 = \begin{pmatrix} 0 & a_2\mu_3 & -a_3\mu_2 \\ -a_1\mu_3 & 0 & a_3\mu_1 \\ a_1\mu_2 & -a_2\mu_1 & 0 \end{pmatrix}$$

for a_i independent of ξ. Comparing the off-diagonal entries of Q_1N_1 (which must be skew-symmetric) we find that

$$a_i(\delta_j - \mu_j^2) = a_j(\delta_i - \mu_i^2)$$

for all i, j, hence $a_i \neq 0$ for all i, and

$$\delta_i = a_i p + \mu_i^2$$

for some quadratic p. Thus

$$Q_1 = pA + \boldsymbol{\mu} \otimes \boldsymbol{\mu}$$

with A diagonal with entries a_i. Finally choose B with $B^T AB = I$, and let

$$Q_0 = B^T Q_1 B, \ \lambda = B\mu,$$

proving the theorem in this case.

The research reported here was supported in part by the U. S. National Science Foundation, Grant NSF MCS 81-00786.

References

1. ANDERSON, R. L., & IBRAGIMOV, N. H., *Lie-Bäcklund Transformations in Applications*, SIAM Studies in Applied Math. Vol. 1, Philadelphia, 1979.
2. ATKINSON, C., "Stress singularities and fracture mechanics," Appl. Mech. Rev. **32** (1979) 123–135.
3. BENJAMIN, T. B., & OLVER, P. J., "Hamiltonian structure, symmetries and conservation laws for water waves," J. Fluid Mech. **125** (1982) 137–185.
4. BESSEL-HAGEN, E., "Über die Erhaltungssätze der Elektrodynamik", Math. Ann. **84** (1921) 258–276.
5. BLUMAN, G. W., & COLE, J. D.. *Similarity Methods for Differential Equations*, Applied Math. Sci. No. 13, Springer-Verlag, Berlin-Heidelberg-New York, 1974.
6. CHEN, F. H. K., & SHIELD, R. T., "Conservation laws in elasticity of the J-integral type", J. Appl. Math. Phys. (ZAMP) **28** (1977) 1–22.
7. EDELEN, D. G. B., "Aspects of variational arguments in the theory of elasticity: fact and folklore", Int. J. Solids Structures **17** (1981) 729–740.
8. EISENHART, L. P., *Riemannian Geometry*, Princeton University Press, Princeton, 1949.
9. ESHELBY, J. D., "The continuum theory of lattice defects", in *Solid State Physics*, (ed. F. SEITZ & D. TURNBULL) vol. 3, Academic Press, New York, 1956.
10. ESHELBY, J. D., "The elastic energy-momentum tensor", J. Elasticity **5** (1975) 321–335.
11. FLETCHER, D. C., "Conservation laws in linear elastodynamics", Arch. Rational Mech. Anal. **60** (1976) 329–353.
12. FOKAS, A. S., "A symmetry approach to exactly solvable evolution equations," J. Math. Phys. **21** (1980) 1318–1325.
13. GEL'FAND, I. M., & DIKII, L. A., "Asymptotic series for Sturm-Liouville equations and the algebra of the Korteweg-de Vries equation," Russian Math. Surveys **30** (1975). 77–113.
14. GÜNTHER, W., "Über einige Randintegrale der Elastomechanik," Abh. Braunschw. wiss. Ges. **14** (1962) 53–72.
15. GURTIN, M. E., *An Introduction to Continuum Mechanics*, Academic Press, New York, 1981.
16. KNOWLES, J. K., & STERNBERG, E., "On a class of conservation laws in linearized and finite elastostatics," Arch. Rational Mech. Anal. **44** (1972) 187–211.
17. LARDNER, R. W., *Mathematical Theory of Dislocations and Fracture*, University of Toronto Press, Toronto, 1974.
18. NOETHER, E., "Invariante Variationsprobleme," Kgl. Ges. Wiss. Nachr. Göttingen, Math.-Physik, Kl. **2** (1918) 235–257.
19. OLVER, P. J., "Evolution equations possessing infinitely many symmetries," J. Math. Phys. **18** (1977) 1212–1215.
20. OLVER, P. J., "Euler operators and conservation laws of the BBM equation," Math. Proc. Camb. Phil. Soc. **85** (1979) 143–160.

21. OLVER, P. J., "On the Hamiltonian structure of evolution equations," Math. Proc. Camb. Phil. Soc. **88** (1980) 71–88.
22. OLVER, P. J., "Applications of Lie groups to differential equations," Lecture Notes, University of Oxford, 1980; Graduate Texts in Mathematics, Springer-Verlag. Berlin-Heidelberg-New York (to appear).
23. OLVER, P. J., "Conservation laws and null divergences." Math. Proc. Camb. Phil. Soc. (to appear).
24. OLVER, P. J., "Conservation laws in elasticity II. Linear homogeneous isotropic elastostatics," Arch. Rational Mech. Anal. **85** (1984) 131–160.
25. OVSIANNIKOV, L. V., *Group Analysis of Differential Equations*, Academic Press, New York, 1982.
26. RICE, J. R., "A path-independent integral and the approximate analysis of strain concentration by notches and cracks," J. Appl. Mech. **35** (1968) 379–386.

School of Mathematics
University of Minnesota
Minneapolis

(Received July 6, 1983)