Conservation Laws in Elasticity.
III. Planar Linear Anisotropic Elastostatics

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Abstract

The first order conservation laws for an arbitrary homogeneous linear planar elastic material are completely classified. In all cases, both isotropic and anisotropic, besides the standard Betti reciprocity laws, there are two infinite-dimensional families of conservation laws, each depending on an arbitrary analytic function of two complex variables.

1. Introduction

In the first two papers in this series, [4], [5] (see also [7]), the general structure of the conservation laws for linear isotropic elasticity in both two and three dimensions was completely determined. The purpose of the present paper is to extend those results to linear anisotropic elastic materials in the plane. The main theorem to be proved is that, as in the case of planar isotropic elasticity, for a general linear, homogeneous elastic material, there are three infinite families of conservation laws depending on material coordinate, deformation and deformation gradient. One of these is the familiar Betti reciprocity relation; the other two each depend on an arbitrary analytic function of two complex variables, which are certain linear combinations of the material coordinates and the deformation gradient. The precise structure of these families of conservation laws appears in Theorem 12.

This result is based on a new theory of change of variables of quadratic variational problems, and the resulting canonical elastic moduli in the plane, [9]. The main result of this theory is that every anisotropic planar material is equivalent, under a linear change of variables, to an *orthotropic* material. Thus the problem of determining conservation laws for general anisotropic materials reduces to the simpler problem of determining conservation laws for orthotropic materials, a problem that is solved in Section 4. Section 5 is devoted to an implementation of the explicit change of variables and the resulting structure of the space of conservation laws.

Unfortunately, these computations only work in two dimensions, and the classification of conservation laws for general three-dimensional anisotropic elastic materials remains open. I hope to return to this problem in a future paper in this series.

2. Linear Elasticity

We let $\boldsymbol{x}$ denote the material coordinates, $\boldsymbol{u}$ the deformation, and $\nabla \boldsymbol{u}$ the deformation gradient. In the absence of body forces, the equations of homogeneous linear elasticity are the Euler-Lagrange equations for the variational problem of minimizing

$$\mathscr{W}[\boldsymbol{u}] = \int_{\Omega} W(\nabla \boldsymbol{u})\, d\boldsymbol{x}, \tag{1}$$

where the *stored energy function* $W(\nabla \boldsymbol{u})$ is a symmetric quadratic function of the strain tensor $\boldsymbol{e} = \frac{1}{2}(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T)$,

$$W(\nabla \boldsymbol{u}) = \Sigma\, c_{ijkl} e_{ij} e_{kl}. \tag{2}$$

The constants c_{ijkl} are the *elastic moduli* which describe the physical properties of the elastic material of which the body is composed. The symmetry of the strain tensor implies that we can assume, without loss of generality, that the elastic moduli obey the basic symmetry restrictions

$$c_{ijkl} = c_{jikl} = c_{ijlk}, \qquad c_{ijkl} = c_{klij}.$$

Thus in planar elasticity there are 6 independent elastic moduli. Additional symmetry restrictions stemming from the constitutive properties of the elastic material may place additional constraints on the moduli.

Following [9], we define the *symbol* of the quadratic variational problem (1) to be the biquadratic polynomial

$$Q(\boldsymbol{x}, \boldsymbol{u}) = W(\boldsymbol{x} \otimes \boldsymbol{u})$$

obtained by replacing $\nabla \boldsymbol{u}$ by the rank one tensor $\boldsymbol{x} \otimes \boldsymbol{u}$, *i.e.* we replace e_{ij} in (2) by $\frac{1}{2}(x_i u^j + x_j u^i)$. Note that the symbol Q is *symmetric*, *i.e.*

$$Q(\boldsymbol{x}, \boldsymbol{u}) = Q(\boldsymbol{u}, \boldsymbol{x}).$$

The *Legendre-Hadamard strong ellipticity condition* requires that the symbol Q be *positive definite* in the sense that

$$Q(\boldsymbol{x}, \boldsymbol{u}) > 0 \quad \text{whenever } \boldsymbol{x} \neq \boldsymbol{0} \text{ and } \boldsymbol{u} \neq \boldsymbol{0}. \tag{3}$$

We will assume that our quadratic variational problem (1) satisfies this condition throughout this paper.

3. Reciprocity

As discussed in [8; Proposition 5.45], any linear self-adjoint system of partial differential equations $\Delta[\boldsymbol{u}] = 0$ always possesses a *reciprocity relation*, which is a divergence identity of the general form

$$\boldsymbol{v} \cdot \Delta[\boldsymbol{u}] - \boldsymbol{u} \cdot \Delta[\boldsymbol{v}] = \operatorname{Div} P[\boldsymbol{u}, \boldsymbol{v}], \tag{4}$$

where P is some bilinear expression involving $\boldsymbol{u}$ and $\boldsymbol{v}$. (P is *not* uniquely determined since there are trivial reciprocity relations $\operatorname{Div} P_0 \equiv 0$; see [6].) If $\boldsymbol{v}$ is a solution to the system, then $P[\boldsymbol{u}, \boldsymbol{v}]$ forms a conservation law of the system. For a linearly elastic material (2), one explicit form of the Betti reciprocal theorem is

$$P[\boldsymbol{u}, \boldsymbol{v}] = \boldsymbol{e}[\boldsymbol{v}] \cdot S[\boldsymbol{u}] - \boldsymbol{e}[\boldsymbol{u}] \cdot S[\boldsymbol{v}], \tag{5}$$

where $S[\boldsymbol{u}]$, with components $S_{ij} = \Sigma\, c_{ijkl} e_{kl}$, is the stress tensor associated with the deformation $\boldsymbol{u}$; *cf.* [2; page 98].

The following simple result is of use in classifying conservation laws. It says that for a general self-adjoint linear system, any conserved density P which depends linearly on $\boldsymbol{u}$ and its derivatives is actually equivalent to a reciprocity relation.

Proposition 1. Suppose $\Delta[\boldsymbol{u}] = 0$ forms a n^{th} order self-adjoint linear system of partial differential equations, which is totally nondegenerate, *cf.* [8; Definition 2.83]. Suppose $\operatorname{Div} Q[\boldsymbol{u}] = 0$ is a conservation law such that the p-tuple of functions $Q[\boldsymbol{u}]$ depends linearly on $\boldsymbol{u}$ and its derivatives. Then Q is equivalent to some version of the standard reciprocity relation (4), i.e.

$$Q[\boldsymbol{u}] = P[\boldsymbol{u}, \boldsymbol{v}] + P_0[\boldsymbol{u}],$$

where $\boldsymbol{v}$ is a solution to the system, and $P_0[\boldsymbol{u}]$ is a trivial conservation law.

(The total nondegeneracy condition is quite mild; indeed a theorem of NIRENBERG, [3; p. 15], implies that any elliptic system of partial differential equations is totally nondegenerate.)

Proof. According to [8; page 270], the given conservation law is equivalent to one in characteristic form

$$\operatorname{Div} Q'[\boldsymbol{u}] = \boldsymbol{v} \cdot \Delta[\boldsymbol{u}],$$

in which, owing to the linearity of Q, and hence Q', the *characteristic* $\boldsymbol{v}$ depends only on the independent variable $\boldsymbol{x}$. Now let

$$R[\boldsymbol{u}] = Q'[\boldsymbol{u}] - P[\boldsymbol{u}, \boldsymbol{v}]$$

be the p-tuple obtained by subtracting off the reciprocity conservation law (4). Then

$$\operatorname{Div} R[\boldsymbol{u}] = \boldsymbol{u} \cdot \Delta[\boldsymbol{v}],$$

so the expression $\boldsymbol{u} \cdot \Delta[\boldsymbol{v}]$ is a total divergence. Moreover, it is easy to see that this is possible if and only if $\boldsymbol{u} \cdot \Delta[\boldsymbol{v}]$ is identically 0. Thus $\Delta[\boldsymbol{v}] = 0$, and R is a trivial conservation law. This proves the proposition.

4. Orthotropic Materials

Definition 2. A stored energy function of the form

$$W(\nabla \boldsymbol{u}) = u_x^2 + \alpha u_y^2 + 2\beta u_x v_y + \alpha v_x^2 + v_y^2, \tag{6}$$

where $\boldsymbol{x} = (x, y)$, $\boldsymbol{u} = (u, v)$ are both in $\mathbb{R}^2$, is called an *orthotropic Lagrangian*, and the parameters α and β the corresponding *canonical elastic moduli* for such an orthotropic elastic medium.

In conventional elasticity, an orthotropic elastic material is one that has three orthogonal planes of reflectional symmetry, *cf.* [1; page 159]. In two dimensions, it is characterized by the conditions

$$c_{1112} = c_{1222} = 0$$

on the elastic moduli. In [9], it was shown how a simple rescaling of both $\boldsymbol{x}$ and $\boldsymbol{u}$ will convert the stored energy functin of a general orthotropic elastic material into one of the form (6). The Euler-Lagrange equations corresponding to the orthotropic Lagrangian (6) are a "generalized" system of Navier's equations

$$E_u = u_{xx} + \alpha u_{yy} + \beta v_{xy} = 0, \qquad E_v = \beta u_{xy} + \alpha v_{xx} + v_{yy} = 0. \tag{7}$$

Strong ellipticity requires that the canonical elastic moduli α and β satisfy the inequalities

$$\alpha > 0, \qquad |\beta| < \alpha + 1. \tag{8}$$

The special case

$$\alpha + \beta = 1, \tag{9}$$

corresponds to a (rescaled) isotropic material; another exceptional case is when

$$\alpha - \beta = 1, \tag{10}$$

which is easily seen to be equivalent to an isotropic material under a simple reflection. The remaining anisotropic cases naturally fall into two classes. The *strongly orthotropic* Lagrangians are those whose elastic moduli satisfy the more restrictive inequalities

$$\alpha > 0, \qquad |\alpha - 1| > \beta. \tag{11}$$

Any other anisotropic, strongly elliptic, orthotropic Lagrangian can be changed into a strongly orthotropic one by a simultaneous rotation of $\boldsymbol{x}$ and $\boldsymbol{u}$ through 45°. The basic result of [9] is that any strongly elliptic planar Lagrangian is equivalent to either a unique strongly orthotropic Lagrangian or to a unique isotropic Lagrangian, both restricted to $0 < \alpha \leqq 1$, $\beta \geqq 0$; see Section 5.

In [5], a complete analysis of the conservation laws depending on $\boldsymbol{x}$, $\boldsymbol{u}$, and the deformation gradient $\nabla \boldsymbol{u}$ of a planar isotropic Lagrangian was carried out. The goal of this section is to provide the corresponding analysis of the conservation laws of a strongly orthotropic Lagrangian. In accordance with the general procedures of [4], we begin by looking for $\boldsymbol{x}$, $\boldsymbol{u}$-independent conservation laws

$$D_x A + D_y B = 0, \tag{12}$$

where A and B depend only on the deformation gradient $\nabla \boldsymbol{u} = (u_x, u_y, v_x, v_y)$. For simplicity, we denote the derivatives (u_x, u_y, v_x, v_y) by (p, q, r, s) respectively. Now if (12) is to hold on all solutions of the generalized system of Navier's equations (7), and the lefthand side depends on at most second order derivatives of $\boldsymbol{u}$, we must have the identity

$$D_x A + D_y B = \varphi E_u + \psi E_v, \tag{13}$$

holding for all $\boldsymbol{x}$, $\boldsymbol{u}$. Here φ, ψ are certain coefficient functions, which themselves can only depend on $\nabla \boldsymbol{u}$. Equating the coefficients of the various second order derivatives of $\boldsymbol{u}$ in (13), we find that A, B must satisfy the following version of the "vector conformal equations", *cf.* [4; (4.10)],

$$\begin{aligned} A_p &= \varphi, & A_r &= \alpha\psi, \\ A_q + B_p &= \beta\psi, & A_s + B_r &= \beta\varphi, \\ B_q &= \alpha\varphi, & B_s &= \psi. \end{aligned}$$

Let $\nabla A = (A_p, A_q, A_r, A_s)$ denote the "gradient" of A with respect to the deformation gradient variables. Eliminating φ, ψ from the vector conformal equations, we find that A and B satisfy the system of differential equations

$$\nabla A = M \cdot \nabla B, \tag{14}$$

where M is the following matrix:

$$M = \begin{bmatrix} 0 & \dfrac{1}{\alpha} & 0 & 0 \\ -1 & 0 & 0 & \beta \\ 0 & 0 & 0 & \alpha \\ 0 & \dfrac{\beta}{\alpha} & -1 & 0 \end{bmatrix}. \tag{15}$$

The analysis of the gradient equation (14) is based on the next lemma, which is an elementary exercise in linear algebra.

Lemma 3. *Let α, β be the canonical elastic moduli of a strongly orthotropic material. Define the constants*

$$\sigma = \frac{\alpha^2 + 1 - \beta^2}{2\alpha} > 1,$$

and

$$\tau = \sqrt{\sigma + \sqrt{\sigma^2 - 1}} > 1.$$

Then the matrix M has four simple purely imaginary eigenvalues at $\pm\tau i, \pm\tau^{-1} i$, with corresponding complex eigenvectors

$$\boldsymbol{a} = \boldsymbol{a}_1 \pm i\boldsymbol{a}_2, \qquad \boldsymbol{b} = \boldsymbol{b}_1 \pm i\boldsymbol{b}_2, \tag{16}$$

where

$$\boldsymbol{a}_1 = \begin{bmatrix} \beta\tau \\ 0 \\ 0 \\ \tau - \alpha\tau^3 \end{bmatrix}, \quad \boldsymbol{a}_2 = \begin{bmatrix} 0 \\ \alpha\beta\tau^2 \\ \alpha^2\tau^2 - \alpha \\ 0 \end{bmatrix}, \quad \boldsymbol{b}_1 = \begin{bmatrix} \beta\tau^2 \\ 0 \\ 0 \\ \tau^2 - \alpha \end{bmatrix}, \quad \boldsymbol{b}_2 = \begin{bmatrix} 0 \\ \alpha\beta\tau \\ \alpha^2\tau - \alpha\tau^3 \\ 0 \end{bmatrix}.$$

Interestingly, the characteristic equation for M is

$$\lambda^4 + 2\sigma\lambda^2 + 1 = 0,$$

which is exactly the same as the discriminant (see Definition 10 below) of the orthotropic Lagrangian (6). In the isotropic cases (9), (10), M has a pair of double eigenvalues at $\pm i$, while in the remaining strongly elliptic cases not satisfying (11), the eigenvalues lie on the unit circle $|\lambda| = 1$. (The cases when (6) is not strongly elliptic correspond to the cases when M has real eigenvalues.)

If $\boldsymbol{a} = (a_1, a_2, a_3, a_4)$ is any vector in $\mathbb{C}^4$, we define

$$\boldsymbol{a} \cdot \nabla \boldsymbol{u} = a_1 u_x + a_2 u_y + a_3 v_x + a_4 v_y = a_1 p + a_2 q + a_3 r + a_4 s.$$

Theorem 4. *Let α, β be strongly orthotropic elastic moduli. Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be the eigenvectors of the matrix M defined by* (16). *Define the complex deformation gradients*

$$\xi = \xi_1 + i\xi_2 = \boldsymbol{a} \cdot \nabla \boldsymbol{u}, \qquad \eta = \eta_1 + i\eta_2 = \boldsymbol{b} \cdot \nabla \boldsymbol{u}. \tag{17}$$

Then the pair of functions A, B form the components of an $\boldsymbol{x}$, $\boldsymbol{u}$-independent conservation law for the Euler-Lagrange equations (7) *if and only if*

$$A = F_1 + G_1, \qquad B = \tau^{-1} F_2 + \tau G_2, \tag{18}$$

where

$$F_1 + iF_2 = F(\xi) \quad \text{and} \quad G_1 + iG_2 = G(\eta)$$

are arbitrary analytic functions of their complex argument.

Proof. Define the matrix

$$C = [\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{b}_1, \boldsymbol{b}_2],$$

which places M into real canonical form

$$C \cdot M \cdot C^{-1} = \begin{bmatrix} 0 & \tau & 0 & 0 \\ -\tau & 0 & 0 & 0 \\ 0 & 0 & 0 & \dfrac{1}{\tau} \\ 0 & 0 & -\dfrac{1}{\tau} & 0 \end{bmatrix}.$$

If we perform a linear change of variables in (14), replacing (p, q, r, s) by $(\xi_1, \xi_2, \eta_1, \eta_2)$, then we have the simpler system

$$A_{\xi_1} = \tau B_{\xi_2}, \qquad A_{\xi_2} = -\tau B_{\xi_1},$$
$$A_{\eta_1} = \frac{1}{\tau} B_{\eta_2}, \quad A_{\eta_2} = -\frac{1}{\tau} B_{\eta_1}. \tag{19}$$

Cross-differentiation shows that all the mixed ξ, η partial derivatives of A and B vanish; for example

$$A_{\xi_1\eta_1} = \tau B_{\xi_2\eta_1} = -\tau^2 A_{\xi_2\eta_2} = \tau^3 B_{\xi_1\eta_2} = \tau^4 A_{\xi_1\eta_1},$$

which must therefore vanish since $\tau \neq \pm 1$ in the strongly orthotropic case. Therefore,

$$A = A_1(\xi_1, \xi_2) + A_2(\eta_1, \eta_2), \qquad B = B_1(\xi_1, \xi_2) + B_2(\eta_1, \eta_2),$$

and (19) decouples into a pair of Cauchy-Riemann equations for the complex functions

$$F(\xi) = A_1 + i\tau B_1, \qquad G(\eta) = A_2 + \frac{i}{\tau} B_2.$$

From this the theorem easily follows.

Define the complex coordinates

$$z = x + i\tau y, \qquad w = x + \frac{i}{\tau} y \tag{20}$$

with corresponding complex total derivatives

$$D_z = \tfrac{1}{2}\left(D_x - \frac{i}{\tau} D_y\right), \qquad D_w = \tfrac{1}{2}(D_x - i\tau\, D_y). \tag{21}$$

We note that the strongly orthotropic Navier equations have two alternative simple expressions in terms of the complex coordinates and corresponding complex deformation gradients; either

$$D_z \xi = 0, \qquad \text{or} \qquad D_w \eta = 0, \tag{22}$$

are both equivalent to the full system (7). We also note that the two types of conservation laws (18) can be written in the compact forms

$$\operatorname{Re}\{D_z F\} = 0, \qquad \text{when} \qquad A = F_1, B = \tau^{-1} F_2, \tag{23a}$$

and

$$\operatorname{Re}\{D_w G\} = 0, \qquad \text{when} \qquad A = G_1, B = \tau\, G_2. \tag{23b}$$

From these observations, we easily find the most general conservation law depending on the material coordinate $\boldsymbol{x}$ and the deformation gradient $\nabla \boldsymbol{u}$:

Lemma 5. *Let $A(\boldsymbol{x}, \nabla \boldsymbol{u})$, $B(\boldsymbol{x}, \nabla \boldsymbol{u})$ form the components of a conservation law* (12) *which does not explicitly depend on $\boldsymbol{u}$. Then*

$$A = F_1 + G_1 + A_0, \qquad B = \tau^{-1} F_2 + \tau G_2 + B_0, \tag{24}$$

where

$$F_1 + iF_2 = F(\bar{z}, \xi), \quad \text{and} \quad G_1 + iG_2 = G(\bar{w}, \eta)$$

are arbitrary analytic functions of their complex arguments, and A_0, B_0 depend only on $\boldsymbol{x}$, and form the components of a trivial conservation law.

The easy proof is omitted.

It remains to investigate those conservation laws which depend explicitly on $\boldsymbol{u} = (u, v)$. Before doing this, we must finally deal with the trivial conservation laws, meaning those in which

$$A = D_y Q, \quad B = -D_x Q,$$

where $Q(\boldsymbol{x}, \boldsymbol{u})$ is an arbitrary smooth function of position and deformation. (See [5] for a proof that these are all the trivial conservation laws in this situation.)

Lemma 6. *Suppose*

$$A = F_1 + G_1, \quad B = \tau^{-1} F_2 + \tau G_2, \tag{25}$$

where $F(\boldsymbol{x}, \boldsymbol{u}, \xi)$ and $G(\boldsymbol{x}, \boldsymbol{u}, \eta)$, are analytic in ξ, η respectively, form the components of a trivial conservation law. Then

$$\begin{aligned} F &= [i\tau(\tau^2 - \alpha) \cdot Q_u + \beta\tau^2 Q_v]\,\xi + Q_x - i\tau Q_y + H_1 + i\tau H_2, \\ G &= [i(\alpha\tau^2 - 1) \cdot Q_u - \beta\tau Q_v]\,\eta - H_1 - i\tau^{-1} H_2, \end{aligned} \tag{26}$$

where Q, H_1, H_2 are arbitrary smooth functions of x, $\boldsymbol{u}$. (Here the functions H_1, H_2 reflect the slight ambiguity in the "definition" (25) of F and G, and contribute nothing to the conservation law itself.)

Theorem 7. *Let $A(\boldsymbol{x}, \boldsymbol{u}, \nabla \boldsymbol{u})$, $B(\boldsymbol{x}, \boldsymbol{u}, \nabla \boldsymbol{u})$ form the components of a conservation law for a strongly orthotropic elastic material. Then*

$$A = F_1 + G_1 + A_R + A_0, \quad B = \tau^{-1} F_2 + \tau G_2 + B_R + B_0, \tag{27}$$

where

$$F_1 + iF_2 = F(\bar{z}, \xi), \quad \text{and} \quad G_1 + iG_2 = G(\bar{w}, \eta)$$

are arbitrary analytic functions of their complex arguments, A_R, B_R form the components of the reciprocity law (5), and A_0, B_0 form the components of a trivial conservation law.

Proof. From the basic Theorem 4.5 of [4], we know that any conservation law of the required form has to be written as in (25), with

$$D_x A + D_y B = \operatorname{Re}\{D_z F + D_w G\} = 0$$

now holding as an identity in $\boldsymbol{x}$, $\boldsymbol{u}$, ξ, η. Expanding and using the definition (17) of ξ, η, we find that F and G must satisfy the identity

$$\operatorname{Re}\{F_z + u_z F_u + v_z F_v + G_w + u_w G_u + v_w G_v\} = 0, \tag{28}$$

where

$$u_z = \tfrac{1}{2}\left(u_x - \frac{i}{\tau} u_y\right) = \frac{2\tau(\tau^2-\alpha)\bar{\xi} + (\tau^2-1)(\alpha\tau^2-1)\eta + (\tau^2+1)(\alpha\tau^2-1)\bar{\eta}}{2\alpha\beta\tau^2(\tau^4-1)},$$

$$v_z = \tfrac{1}{2}\left(v_x - \frac{i}{\tau} v_y\right) = i\,\frac{2\tau\bar{\xi} + (\tau^2-1)\eta - (\tau^2+1)\bar{\eta}}{2\alpha\tau(\tau^4-1)},$$

$$u_w = \tfrac{1}{2}(u_x - i\tau u_y) = \frac{(\tau^2-\alpha)(\tau^2-1)\xi + (\tau^2-\alpha)(\tau^2+1)\bar{\xi} + 2\tau(\alpha\tau^2-1)\bar{\eta}}{2\alpha\beta\tau(\tau^4-1)},$$

$$v_w = \tfrac{1}{2}(v_x - i\tau v_y) = i\,\frac{(\tau^2-1)\xi + (\tau^2+1)\bar{\xi} - 2\tau\bar{\eta}}{2\alpha(\tau^4-1)}.$$

The following elementary lemma is now of use:

Lemma 8. *Let A, B, C, D, E, F, G, H be complex analytic functions of a single complex variable, and let ξ, η be independent complex variables. Then*

$$\operatorname{Re}\{A(\xi) + \bar{\xi}B(\xi) + \eta C(\xi) + \bar{\eta}D(\xi) + E(\eta) + \bar{\eta}F(\eta) + \xi G(\eta) + \bar{\xi}H(\eta)\} = 0$$

if and only if

$$\begin{aligned}
A(\xi) &= \gamma - (\bar{\delta} + \psi + \bar{\chi})\,\xi, & E(\eta) &= -\bar{\gamma} - (\varepsilon + \bar{\theta} + \bar{\varphi})\,\eta,\\
B(\xi) &= ia\xi + \delta, & F(\eta) &= ib\eta + \varphi,\\
C(\xi) &= \lambda\xi + \varepsilon, & G(\eta) &= -\lambda\eta + \psi,\\
D(\xi) &= \mu\xi + \theta, & H(\eta) &= -\bar{\mu}\eta + \chi,
\end{aligned}$$

where γ, δ, ε, θ, φ, ψ, χ are complex constants, and a, b are real constants.

Applying Lemma 8 to (28), we see that F and G must satisfy the following differential equations:

$$\begin{aligned}
\alpha\beta\tau^2(\tau^4-1)\,F_z &= \gamma - (\bar{\delta} + \psi + \bar{\chi})\,\xi,\\
\alpha\beta\tau^2(\tau^4-1)\,G_w &= -\bar{\gamma} - (\varepsilon + \bar{\theta} + \bar{\varphi})\,\eta,\\
2\tau(\tau^2-\alpha)\,F_u + 2i\beta\tau^2 F_v &= ia\xi + \delta,\\
2\tau^2(\alpha\tau^2-1)\,G_u - 2i\beta\tau^3 G_v &= ib\eta + \varphi,\\
(\tau^2-1)\{(\alpha\tau^2-1)\,F_u + i\beta\tau F_v\} &= \lambda\xi + \varepsilon,\\
(\tau^2-1)\{\tau(\tau^2-\alpha)\,G_u + i\beta\tau^2 G_v\} &= -\lambda\eta + \psi,\\
(\tau^2+1)\{(\alpha\tau^2-1)\,F_u - i\beta\tau F_v\} &= \mu\xi + \theta,\\
(\tau^2+1)\{\tau(\tau^2-\alpha)\,G_u + i\beta\tau^2 G_v\} &= -\bar{\mu}\eta + \chi.
\end{aligned}\tag{29}$$

The first conclusion is that F and G must be of the form

$$F = F'(\bar{z}, \xi) + f(\boldsymbol{x}, \boldsymbol{u})\,\xi + f_0(\boldsymbol{x}, \boldsymbol{u}), \qquad G = G'(\bar{w}, \eta) + g(\boldsymbol{x}, \boldsymbol{u})\,\eta + g_0(\boldsymbol{x}, \boldsymbol{u})$$

i.e. all the $\boldsymbol{u}$-dependence is in the linear and constant ξ and η terms. Since by Lemma 5, F' and G' already constitute a conservation law, we can ignore them, and concentrate on the functions f, f_0, g, g_0.

The third and fourth equations in (29) show that the combinations

$$(\tau^2 - \alpha) f_u + i\beta\tau f_v \quad \text{and} \quad (\alpha\tau^2 - 1) g_u - i\beta\tau g_v$$

must both be purely imaginary functions. A straightforward computation shows that this requires f and g to be of the forms

$$f = i\tau(\tau^2 - \alpha) \cdot Q_u + \beta\tau^2 Q_v \qquad g = i(\alpha\tau^2 - 1) \cdot R_u - \beta\tau R_v, \tag{30}$$

for real-valued functions $Q(\boldsymbol{x}, \boldsymbol{u})$ and $R(\boldsymbol{x}, \boldsymbol{u})$. However, the fifth through eighth equations in (29) show that, except for terms that do not depend on $\boldsymbol{u}$, we can take $Q = R$. But this implies that the ξ, η terms in the representation (30) are the same as the ξ, η terms in the form (26) of a trivial conservation law. Subtracting off the trivial conservation law, we are left with F, G of the special form

$$F = f_1(\boldsymbol{x})\, \xi + f_0'(\boldsymbol{x}, \boldsymbol{u}), \qquad G = g_1(\boldsymbol{x})\, \eta + g_0'(\boldsymbol{x}, \boldsymbol{u}),$$

to be analyzed. However, (29) easily shows that f_0' and g_0' are at most linear in $\boldsymbol{u}$. Thus we are left with a conservation law which is linear in $\boldsymbol{u}$ and $\nabla \boldsymbol{u}$. According to the general result in Proposition 1, this latter conservation law must be equivalent to the Betti reciprocity relation (4). This completes the proof of Theorem 7, and hence the classification of conservation laws of orthotropic elastic materials.

5. Change of Variables

Before discussing changes of variables, we note that in any variational problem, one can always add any *null Lagrangian* or total divergence to the integrand without affecting the Euler-Lagrange equations, *cf.* [8; Chapter 4]. For example, in the planar quadratic case we can add in any constant multiple of the Jacobian determinant

$$u_x v_y - u_y v_x = D_x(u v_y) + D_y(-u v_x)$$

to the stored energy $W(u_x, u_y, v_x, v_y)$ without affecting the Euler-Lagrange equations. Thus, the Lagrangians $u_x v_y$ and $u_y v_x$ and $\frac{1}{2}[u_x v_y + u_y v_x]$ all have exactly the same Euler-Lagrange equations, and are considered to be *equivalent Lagrangians.*

The basic method employed to determine conservation laws for a general linearly elastic material is to try to simplify the stored energy function as much as possible through the use of specially "adapted" coordinates. Since we are restricting our attention to symmetric quadratic variational problems, we will only consider linear changes of variables of the special form

$$\boldsymbol{x} = A\boldsymbol{x}', \qquad \boldsymbol{u} = A^{-1}\boldsymbol{u}', \tag{31}$$

in which A is a nonsingular $p \times p$ matrix; this ensures that the new stored energy remains a function of the new strain tensor $\boldsymbol{e}'$. In terms of the new variables

$\boldsymbol{x}'$, $\boldsymbol{u}'$, the variational problem has an analogous form

$$\mathcal{W}'[\boldsymbol{u}'] = \int_{\Omega'} W'(\nabla \boldsymbol{u}')\, d\boldsymbol{x}',$$

where the new stored energy function W' has the same form (2), but with new elastic moduli c'_{ijkl}. We will call two such functions W and W' *equivalent* if there exists a nonsingular matrix A and a null Lagrangian N such that

$$W'(\nabla \boldsymbol{u}') = \{W(\nabla \boldsymbol{u}) + N(\nabla \boldsymbol{u})\}\, |\det A|,$$

under the transformation (31).

The fundamental theorem of [9] states that every planar linear elastic medium is equivalent under a linear change of variables to an orthotropic elastic medium. Specifically, we have the following:

Theorem 9. *Let $W(\nabla \boldsymbol{u})$ be a first order planar quadratic Lagrangian which satisfies the Legendre-Hadamard strong ellipticity condition. Then W is equivalent either to an isotropic material, or to a strongly orthotropic Lagrangian* (6), *where the canonical elastic moduli α and β satisfy the inequalities* (11).

In other words, for planar elasticity, while the general planar elastic problem in a general coordinate system has 6 independent elastic moduli c_{ijkl}, Theorem 9 shows that if we choose a special adapted coordinate system, there are in reality only *two* independent moduli.

An important feature of this result is that the construction of the linear transformation (31) which places a general elastic stored energy function into canonical form is *completely explicit*. To describe this, we introduce an important quartic polynomial associated with any quadratic planar Lagrangian.

Definition 10. Let $W(\nabla \boldsymbol{u})$ be a quadratic planar Lagrangian, and let $Q(\boldsymbol{x}, \boldsymbol{u})$ be its symbol. We write Q as a homogeneous quadratic polynomial in $\boldsymbol{x}$,

$$Q(\boldsymbol{x}, \boldsymbol{u}) = A(\boldsymbol{u})\, x^2 + B(\boldsymbol{u})\, xy + C(\boldsymbol{u})\, y^2,$$

where A, B, and C are homogeneous quadratic polynomials in $\boldsymbol{u}$. Set $a(z) = A(z, 1)$, $b(z) = B(z, 1)$, $c(z) = C(z, 1)$. Then the *discriminant* of W is the quartic polynomial

$$\Delta(z) = b(z)^2 - 4a(z)\, c(z). \tag{31}$$

The structure of the roots of the discriminant Δ provides the key to the construction of the required linear transformation. According to the Fundamental Theorem of Algebra, there are, counting multiplicities, precisely four complex roots, which we denote by z_1, z_2, z_3, z_4, so $\Delta(z_i) = 0$, $i = 1, \ldots, 4$. The Legendre-Hadamard condition (3) implies that the roots cannot be real, and hence z_1, z_2 and z_3, z_4 are complex conjugate roots. There are then two distinct cases:

1. The Isotropic Case. If there is a single complex conjugate pair of double roots, so $z_1 = z_3, z_2 = z_4$, then the material is equivalent to a unique isotropic material satisfying $0 < \alpha \leqq 1, \beta = 1 - \alpha$.

2. The Anisotropic Case. If the roots are simple, then the material is equivalent to a unique strongly orthotropic material satisfying $0 < \alpha < 1, 0 \leqq \beta < 1 - \alpha$.

The explicit form of the change of variables (31) is found from the roots z_j as follows, *cf.* [9]: Define the matrix

$$B = \begin{bmatrix} \tau \cos \varphi & -\sin \varphi \\ r\tau \cos(\theta - \varphi) & r \sin(\theta - \varphi) \end{bmatrix}. \tag{33}$$

Here (r, θ) are the polar coordinates of the first root z_1. The quantity -2φ is the angle between the vertical line through the first pair of complex conjugate roots z_1 and z_2 and the circle passing through the four roots z_1, z_2, z_3 and z_4. (In the isotropic case, $\varphi = 0$.) See Figure 1.

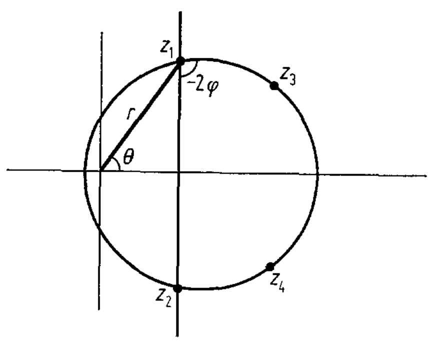

Fig. 1

Also, the quantity τ is given by

$$\tau = \frac{s_1 + s_2}{\sqrt{s_3 \cdot s_4}},$$

where s_1, s_2, s_3, s_4 are the lengths of the four indicated line segments in Figure 2.

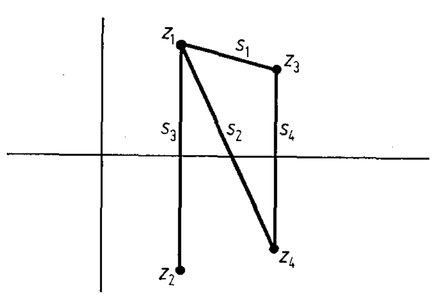

Fig. 2

Theorem 11. *Let $W(\nabla \boldsymbol{u})$ be the stored energy function for a linearly elastic material. Let z_1, z_2, z_3, z_4 be the complex roots of the discriminant $\Delta(\boldsymbol{u})$. Let B be the matrix determined by* (33). *Then the linear transformation*

$$\boldsymbol{x} = B^{-1} \cdot \boldsymbol{x}', \quad \boldsymbol{u} = B \cdot \boldsymbol{u}',$$

will convert W into a scalar multiple of either an isotropic or a strongly orthotropic stored energy function. (The scalar multiple can be eliminated by a simple rescaling of $\boldsymbol{x}$ and/or $\boldsymbol{u}$.)

Although isotropic and more general orthotropic materials have similar looking Lagrangians, the structure of their associated conservation laws is quite dissimilar.

Theorem 12. *Let $\mathscr{W}[u]$ be a strongly elliptic quadratic planar variational problem, with corresponding Euler-Lagrange equations $E(W) = 0$.*

*1. **The Isotropic Case.** If W is equivalent to an isotropic material, then there exists a complex linear combination z of the variables (x, y), a complex linear combination ω of the variables (u, v), and two complex linear combinations ξ, η of the components of the deformation gradient (u_x, u_y, v_x, v_y) having the properties:*

a) The two Euler-Lagrange equations can be written as a single complex differential equation in the form

$$D_z \eta = 0.$$

b) Any first order conservation law is equivalent to a real linear combination of
i) the Betti reciprocity relations,
ii) the two families of complex convservations laws

$$\mathrm{Re}\,[D_z F] = 0,$$

and

$$\mathrm{Re}\,\{D_z[(\xi + z)\, G_\eta + \overline{G}]\} = 0,$$

where $F(z, \eta)$ and $G(z, \eta)$ are arbitrary complex analytic functions of their two arguments,
iii) the extra conservation law

$$\mathrm{Re}\,[D_z\{\omega\eta - iz\eta^2\}] = 0.$$

*2. **The Anisotropic Case.** If W is equivalent to a strongly orthotropic material, then there exist two complex linear combinations z, w of the variables (x, y), and two corresponding complex linear combinations ξ, η of the components of the deformation gradient (u_x, u_y, v_x, v_y) with the properties:*

a) The two Euler-Lagrange equations can be written as a single complex differential equation in either of the two forms

$$D_z \xi = 0, \quad \text{or} \quad D_w \eta = 0.$$

b) Any first order conservation law is equivalent to a real linear combination of
i) the Betti reciprocity relations, and
ii) the two families of complex conservation laws

$$\operatorname{Re}[D_z F] = 0, \quad \text{and} \quad \operatorname{Re}[D_w G] = 0,$$

where $F(z, \xi)$ and $G(w, \eta)$ are arbitrary complex analytic functions of their two arguments.

Thus the striking result is that in *both* isotropic and anisotropic planar elasticity, there are three infinite families of conservation laws. One family is the well-known Betti reciprocity relation. The other two are determined by two arbitrary analytic functions of two complex variables. However, the detailed structure of these latter two families is markedly different depending upon whether we are in the isotropic or truly anisotropic (orthotropic) case. The two orthotropic families degenerate to a single isotropic family, but a second family makes its appearance in the isotropic case. In addition, the isotropic case is distinguished by the existence of one extra anomalous conservation law, the significance of which is not at all clear.

The details of the proof of this theorem in the isotropic case have appeared in [5; Theorem 4.2] (although there is a misprint, corrected in [6]). The strongly orthotropic case follows immediately from Theorems 7 and 11. Indeed, using the matrix $\boldsymbol{B}$ determined by (33) and the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ determined by (16), we set

$$z = (1, i\tau) \cdot \boldsymbol{B} \cdot \boldsymbol{x}, \qquad w = (1, i\tau^{-1}) \cdot \boldsymbol{B} \cdot \boldsymbol{x},$$

and

$$\xi = \boldsymbol{a} \cdot (B^{-1} \nabla \boldsymbol{u} B^{-1}), \quad \eta = \boldsymbol{b} \cdot (B^{-1} \nabla \boldsymbol{u} B^{-1}).$$

In the isotropic case, we have

$$z = (1, i) \cdot \boldsymbol{B} \cdot \boldsymbol{x}, \quad \omega = (1, i) \cdot B^{-1} \cdot \boldsymbol{u},$$

and

$$\xi = \boldsymbol{a} \cdot (B^{-1} \nabla \boldsymbol{u} B^{-1}), \quad \eta = \boldsymbol{b} \cdot (B^{-1} \nabla \boldsymbol{u} B^{-1}),$$

where $\boldsymbol{a} = (1, i\alpha, -i\alpha, 1)$, $\boldsymbol{b} = (1, i, i, -1)$, *cf.* [5; § 4]. Then the general anisotropic or isotropic conservation law takes the form prescribed in Theorem 12.

We hope to return to the application of these families of conservation laws in a future publication.

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