Canonical Forms for Compatible BiHamiltonian Systems

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In this note, I will review recent results on the canonical forms for compatible biHamiltonian systems of complex-analytic ordinary differential equations based on Turiel's classification, [8], of compatible non-degenerate Hamiltonian pairs. The resulting explicit forms for general biHamiltonian systems in canonical coordinates lead to a complete analysis of their integrability. More details of these results can be found in the author's paper [6].

A system of differential equations is called *biHamiltonian*, [3], [5], if it can be written in Hamiltonian form in two distinct ways:

$$\dot{x} = J_1 \nabla H_1 = J_2 \nabla H_0 . \tag{1}$$

Here $J_1(x), J_2(x)$ are Hamiltonian operators (matrices), not constant multiples of each other, determining Poisson brackets: $\{F, G\}_\nu = \nabla F^T J_\nu(x) \nabla G$. The Hamiltonian pair J_1, J_2 is *compatible* if $J_1 + J_2$ also determines a Poisson bracket, i.e. the Jacobi identity holds. The pair is *nondegenerate* if one of the Poisson structures is symplectic. According to the fundamental theorem of Magri, [3], any biHamiltonian system associated with a nondegenerate Hamiltonian pair induces a hierarchy of commuting Hamiltonians and flows, and, provided enough of these Hamiltonians are functionally independent, is therefore completely integrable.

Theorem 1. Suppose J_1, J_2 form a compatible Hamiltonian pair, with J_1 symplectic. Given a biHamiltonian system (1), there exists a hierarchy of Hamiltonian functions $H_0, H_1, H_2, H_3, \ldots,$ all in involution with respect to either Poisson bracket, $\{H_j, H_k\}_\nu = 0$, and generating mutually commuting biHamiltonian flows

$$\dot{x} = J_1 \nabla H_k = J_2 \nabla H_{k-1} . \tag{2}$$

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We classify Hamiltonian pairs pointwise according to the algebraic invariants of the skew-symmetric matrix pencil $\lambda J_1(x) + \mu J_2(x)$ at each x. According to the Weierstrass theory, cf. [1], the complete algebraic invariants of a non-degenerate matrix pencil are provided by the eigenvalues, elementary divisors and Segre characteristic. (Degenerate pairs of skew-symmetric matrices are handled by the more detailed Kronecker theory.) A pencil is called *elementary* if it has just one complex eigenvalue, and *irreducible* if it has Segre characteristic [(nn)], analogous to a single Jordan block. Every non-degenerate complex matrix pencil is algebraically the direct sum of irreducible matrix pencils. (For simplicity, we restrict our attention to complex-analytic systems in this paper, although the real case offers little additional difficulty.)

The algebraic invariants, i.e. eigenvalues, elementary divisors and Segre characteristic, of a Hamiltonian pair are invariant under the flow of any associated biHamiltonian system. A Hamiltonian pair is *generic* on a domain M if it has constant Segre characteristic, and the number of functionally independent eigenvalues does not change on M. The main classification theorem for nondegenerate biHamiltonian systems is the following:

Theorem. Every generic non-degenerate, compatible Hamiltonian pair can be locally expressed as a Cartesian product of elementary Hamiltonian pairs. Every associated biHamiltonian system decomposes into independent subsystems corresponding to the elementary sub-pairs, each of which consists of an autonomous Hamiltonian system whose dimension is twice the number of irreducible sub-pairs for the given eigenvalue, coupled with a sequence of linear, non-autonomous Hamiltonian systems.

When an eigenvalue is constant, the elementary sub-pair decomposes into a Cartesian product of irreducible sub-pairs; however, this decomposition does *not* hold in the case of non-constant eigenvalues. We will now present the details of the Turiel classification and the structure of associated biHamiltonian systems.

Without loss of generality, we may assume that neither 0 nor ∞ is an eigenvalue, so that the Hamiltonian pair is determined by two compatible symplectic Hamiltonian operators. (Otherwise, replace J_1, J_2 by two other linearly independent members of the corresponding pencil.) Darboux' theorem, [5; Theorem 6.22], implies that we can write the first Hamiltonian operator in canonical form

$$J_1 = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \qquad (3)$$

relative to canonically conjugate coordinates $(\mathbf{p}, \mathbf{q})$. Therefore, only the canonical form of the second Hamiltonian operator needs to be explicitly indicated.

Given a Hamiltonian pair J_1, J_2, any associated biHamiltonian systems is a solution to the linear system of partial differential equations

$$\nabla H_1 = M \nabla H_0, \qquad M = J_1^{-1} \cdot J_2, \tag{4}$$

where M is the transpose of the recursion operator, [5]. We remark here that the simple system of differential equations (4), which arises in a surprising number of different contexts, is not well understood, except when the matrix M is constant, in which case the general solution can be found in [2]. In the present case, the solutions all have a similar pattern. On any convex open subdomain, the two Hamiltonians H_0, H_1 are given as a sum of "basic" Hamiltonians $H_0^{(k)}, H_1^{(k)}$, which are individually solutions to (4):

$$H_0(\mathbf{x}) = H_0^{(0)}(\mathbf{x}) + H_0^{(1)}(\mathbf{x}) + \ldots + H_0^{(n)}(\mathbf{x}), \quad H_1(\mathbf{x}) = H_1^{(0)}(\mathbf{x}) + H_1^{(1)}(\mathbf{x}) + \ldots + H_1^{(n)}(\mathbf{x}).$$

Moreover, each basic pair $H_0^{(k)}, H_1^{(k)}$, can be most simply expressed in terms of the derivatives with respect to a parameter s evaluated at $s = 0$ of a single arbitrary analytic function $F(\xi_1(\mathbf{x}, s), \ldots, \xi_m(\mathbf{x}, s))$ depending on certain parameterized variables $\xi_j(\mathbf{x}, s)$. We can therefore summarize the general classification results in this convenient form.

I) Irreducible, Constant Eigenvalue Pairs,

Canonical coordinates: $\quad (\mathbf{p}, \mathbf{q}) = (p_0, p_1, \ldots, p_n, q_0, q_1, \ldots, q_n), \qquad n \geq 0.$

Second Hamiltonian operator:

$$J_2 = \begin{pmatrix} 0 & \lambda I + U \\ -\lambda I - U^T & 0 \end{pmatrix},$$

where $\lambda I + U$ denotes the irreducible $(n+1) \times (n+1)$ Jordan block

$$\lambda I + U = \begin{pmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \lambda & 1 & \\ & & & \cdots & \cdots \end{pmatrix}.$$

Parametrized variables:

$$\pi(s) = p_0 + s\,p_1 + s^2\,p_2 + \dots + s^n\,p_n\,, \qquad \varpi(s) = q_n + s\,q_{n-1} + s^2\,q_{n-2} + \dots + s^n\,q_0\,.$$

Basic Hamiltonians:

$$H_0^{(k)}(\mathbf{x}) = \mu\,\frac{\partial^k}{\partial s^k} F_k(\pi(s), \varpi(s))\Bigg|_{s=0} + k\,\frac{\partial^{k-1}}{\partial s^{k-1}} F_k(\pi(s), \varpi(s))\Bigg|_{s=0}\,,$$

$$H_1^{(k)}(\mathbf{x}) = \frac{\partial^k}{\partial s^k} F_k(\pi(s), \varpi(s))\Bigg|_{s=0}\,, \qquad 0 \le k \le n\,.$$

Note that these Hamiltonians are polynomials in the "minor variables" $p_1,\dots,p_n, q_0,\dots,q_{n-1}$, whose coefficients are certain derivatives of the arbitrary smooth functions $F_k(p_0, q_n)$ of the remaining two "major variables" p_0, q_n. This implies, cf. [6], that any biHamiltonian system corresponding to an irreducible, constant eigenvalue Hamiltonian pair is completely integrable, since it can be reduced to a single two-dimensional (planar) autonomous Hamiltonian system for the major variables, with Hamiltonian $n!\,F_n(p_0, q_n)$. (Curiously, the major variables are *not* canonically conjugate for the standard symplectic structure given by J_1, nor are they conjugate for J_2.) The time evolution of the minor variables is then determined by successively solving a sequence of orced linear planar Hamiltonian systems in the variables p_k, q_{n-k}.

II. Elementary, Constant Eigenvalue Pairs.

Canonical coordinates: $\quad (\mathbf{p}, \mathbf{q}) = (\mathbf{p}^1, \dots, \mathbf{p}^m, \mathbf{q}^1, \dots, \mathbf{q}^m)$,

$$\mathbf{p}^i = (p_0^i, \dots, p_{n_i}^i), \quad \mathbf{q}^i = (q_0^i, \dots, q_{n_i}^i), \qquad \text{where} \qquad n_1 \ge n_2 \ge \dots \ge n_k \ge 0\,.$$

Second Hamiltonian operator:

$$J_2 = \begin{pmatrix} 0 & & 0 & \lambda I + U_1 & & 0 \\ & \cdots & & & \cdots & \\ 0 & & 0 & 0 & & \lambda I + U_m \\ -\lambda I - U_1^T & & 0 & 0 & & 0 \\ & \cdots & & & \cdots & \\ 0 & & -\lambda I - U_m^T & 0 & & 0 \end{pmatrix},$$

where $\lambda I + U_i$ denotes an irreducible $(n_i + 1) \times (n_i + 1)$ Jordan block as above.

Parametrized variables:

$$\pi^i(s) = p_0^i + s\, p_1^i + s^2 p_2^i + \ldots + s^{n_i} p_{n_i}^i , \qquad \varpi^i(s) = q_{n_i}^i + s\, q_{n_i-1}^i + s^2 q_{n_i-2}^i + \ldots + s^{n_i} q_0^i .$$

Basic Hamiltonians:

$$H_0^{(k)}(\mathbf{x}) = \mu \frac{\partial^k}{\partial s^k} F_k(\pi^1(s), \varpi^1(s), \ldots, \pi^{m_k}(s), \varpi^{m_k}(s)) \Big|_{s=0} +$$

$$+ \; k \frac{\partial^{k-1}}{\partial s^{k-1}} F_k(\pi^1(s), \varpi^1(s), \ldots, \pi^{m_k}(s), \varpi^{m_k}(s)) \Big|_{s=0} ,$$

$$H_1^{(k)}(\mathbf{x}) = \frac{\partial^k}{\partial s^k} F_k(\pi^1(s), \varpi^1(s), \ldots, \pi^{m_k}(s), \varpi^{m_k}(s)) \Big|_{s=0} . \qquad 0 \le k \le n_1 .$$

Here m_k denotes the number of n_i with $n_i \ge k$, i.e. the number of irreducible sub-pairs of dimension greater than $2k + 1$; in particular $m_0 = m$.

As in the irreducible case, the Hamiltonians are polynomials in the minor variables p_j^i, $q_{n_i-j}^i$, $j \ge 1$, whose coefficients are certain derivatives of arbitrary functions of the major variables p_0^i, $q_{n_i}^i$. Thus, such a biHamiltonian system reduces to an autonomous $2m$ – dimensional Hamiltonian system in the major variables, followed by linear non-autonomous Hamiltonian systems in the appropriate minor variables p_k^i, $q_{n_i-k}^i$, $n_i \ge k \ge 1$.

III. Irreducible, Non-constant Eigenvalue Pairs.

Canonical coordinates: $(\mathbf{p}, \mathbf{q}) = (p_0, p_1, \ldots, p_n, q_0, q_1, \ldots, q_n)$, $\qquad n \ge 0$.

Second Hamiltonian operator:

$$J_2 = \begin{pmatrix} 0 & P(\mathbf{p}) \\ -P(\mathbf{p})^T & 0 \end{pmatrix},$$

where $P(\mathbf{p})$ denotes the $(n + 1) \times (n + 1)$ banded upper triangular matrix

$$P_n(\mathbf{p}) = P(\mathbf{p}) = \begin{pmatrix} p_0 & p_1 & p_2 & p_3 & \cdots & p_n \\ & p_0 & p_1 & p_2 & \cdots & p_{n-1} \\ & & p_0 & p_1 & \cdots & \cdots \\ & & & p_0 & \cdots & \cdots \\ & & & & \cdots & \cdots \\ & & & & & p_0 \end{pmatrix}. \qquad (5)$$

(Interestingly, both $P(\mathbf{p})$ and its inverse determine isomorphic Hamiltonian operators!)

Parametrized variables:

$$\pi(s) = p_0 + s\,p_1 + s^2\,p_2 + \ldots + s^n\,p_n\,, \qquad \varpi(s) = q_n + s\,q_{n-1} + s^2\,q_{n-2} + \ldots + s^n\,q_0\,.$$

Basic Hamiltonians:

$$H_0^{(-1)}(\mathbf{x}) = \tilde{h}(p_0)\,, \qquad H_1^{(-1)}(\mathbf{x}) = h(p_0)\,, \qquad \text{where} \qquad \tilde{h}'(\xi) = \xi\,h'(\xi)\,,$$

$$H_0^{(k)}(\mathbf{x}) = \frac{\partial^k}{\partial s^k}\left\{ \pi(s)\,\pi'(s)\,F_k(\pi(s), \varpi(s)) \right\}\Big|_{s=0}\,,$$

$$H_1^{(k)}(\mathbf{x}) = \frac{\partial^k}{\partial s^k}\left\{ \pi'(s)\,F_k(\pi(s), \varpi(s)) \right\}\Big|_{s=0}\,, \qquad 0 \le k \le n-1\,.$$

Here $\pi'(s)$ is the derivative of π with respect to s.

In this case, the eigenvalue is a constant, hence p_0 is a first integral. Once its value is fixed, the other minor variable q_n is determined by solving a single autonomous ordinary differential equation. The remaining minor variables $p_1,\ldots,p_n, q_0,\ldots,q_{n-1}$ satisfy a sequence of forced, linear planar Hamiltonian systems.

IV. Elementary, Non-constant Eigenvalue Pairs.

Canonical coordinates: $(\mathbf{p}, \mathbf{q}) = (p_0, \mathbf{p}^1, \ldots, \mathbf{p}^m, q_0, \mathbf{q}^1, \ldots, \mathbf{q}^m)\,, \qquad m \ge 2\,,$

$$\mathbf{p}^i = (p_1^i, \ldots, p_{n_i}^i), \quad \mathbf{q}^i = (q_1^i, \ldots, q_{n_i}^i), \qquad \text{where} \qquad n_1 \ge n_2 \ge \ldots \ge n_k \ge 1\,.$$

Second Hamiltonian operator:

$$J_2 = \begin{pmatrix} 0 & \mathbf{P}^*(\mathbf{p}) \\ -\mathbf{P}^*(\mathbf{p})^T & 0 \end{pmatrix},$$

where

$$\mathbf{P}^*(\mathbf{p}) = \begin{pmatrix} p_0 & \mathbf{p}^1 & \mathbf{p}^2 & \cdots & \mathbf{p}^m \\ & P_{n_1-1}(\hat{\mathbf{p}}^1) & 0 & \cdots & 0 \\ & & P_{n_2-1}(\hat{\mathbf{p}}^2) & & \\ & & & \cdots & \\ & & & & P_{n_m-1}(\hat{\mathbf{p}}^m) \end{pmatrix}.$$

Here $\hat{\mathbf{p}}^i = (p_0, p^i_1, \ldots, p^i_{n_i-1})$, and the P_{n_i-1}'s are as given in (5). Note that this particular pair is algebraically reducible, but cannot be decoupled using canonical transformations.

Parametrized variables:

$$\pi^i(s) = p_0 + s\,p^i_1 + s^2 p^i_2 + \ldots + s^{n_i} p^i_{n_i}, \qquad \omega^i(s) = q^i_{n_i} + s\,q^i_{n_i-1} + \ldots + s^{n_i-1} q^i_1, \quad i \geq 1,$$

$$\mu^j(s) = \frac{z^j(s)}{s}, \qquad \sigma^j(s) = \varpi(z^j(s)), \qquad \text{where } z^j(s) \text{ solves } \pi^j(z) = \pi^1(s), \qquad j \geq 2.$$

Using the Lagrange inversion formula, [4], the latter two parametrized variables have the alternative expansions

$$\mu^j(s) = \sum_{n=0}^{n_j-1} \frac{s^n (\zeta^1(s))^{n+1}}{(n+1)!} \frac{d^n}{dt^n} \frac{1}{(\zeta^j(t))^{n+1}} \Bigg|_{t=0},$$

$$\sigma^j(s) = q^j_{n_j} + \sum_{n=0}^{n_j-1} \frac{s^n (\zeta^1(s))^n}{n!} \frac{d^n}{dt^n} \frac{1}{(\zeta^j(t))^n} \frac{d\omega^j(t)}{dt} \Bigg|_{t=0},$$

where $\zeta^i(s) = (\pi^i(s) - p_0)/s$. These expansions can be expressed in terms of the remarkable nonlinear series differential operator

$$\mathcal{D} = D^{-1} : e^{sDu} : D = 1 + \sum_{n=1}^{\infty} \frac{s^n}{n!} D^{n-1} u^n D, \qquad D = \frac{d}{dt}, \quad u = u(t),$$

where the colons denote *normal ordering* of the non-commuting operators D and u, which is analogous to the so-called "Wick ordering" in quantum mechanics. This operator has the surprising property that it commutes with *any* analytic function $\Phi(u)$, i.e. $\mathcal{D}\,\Phi(u) = \Phi(\mathcal{D}\,u)$! See [7] for details and applications of this operator in combinatorics, orthogonal polynomials and new higher order derivative identities.

Basic Hamiltonians:

$$H_0^{(-1)}(\mathbf{x}) = \tilde{h}(p_0), \qquad H_1^{(-1)}(\mathbf{x}) = h(p_0), \qquad \text{where} \qquad \tilde{h}'(\xi) = \xi\, h'(\xi),$$

$$H_0^{(k)}(\mathbf{x}) = \frac{\partial^k}{\partial s^k} \left\{ s\, \zeta^1(s)\, \frac{d\pi^1}{ds}\, F_k(\pi^1(s),\mu^2(s),\ldots,\mu^{m_k}(s),\omega^1(s),\sigma^2(s),\ldots,\sigma^{m_k}(s)) \right\} \Bigg|_{s=0},$$

$$H_1^{(k)}(\mathbf{x}) = \frac{\partial^k}{\partial s^k} \left\{ \frac{d\pi^1}{ds}\, F_k(\pi^1(s), \mu^2(s), \ldots, \mu^{m_k}(s), \omega^1(s), \sigma^2(s), \ldots, \sigma^{m_k}(s)) \right\} \Bigg|_{s=0},$$

for $0 \le k \le n_1 - 1$, where $m_k = \#\,\{\, n_i \ge k \,\}$.

In general, such biHamiltonian systems reduce to the integration of a $2m-2$ dimensional autonomous Hamiltonian system for the coordinates $p_1^i, q_{n_i}^i$, $i = 1, \ldots, m$, followed by a sequence of forced linear Hamiltonian systems. The final coordinate q_0 is determined by quadrature. Actually, the initial Hamiltonian system can be reduced in order to $2m-3$ since it only involves the homogeneous ratios of momenta $r^i = p_1^i / p_1^1$, $i \ge 2$, as can be seen from the second formula for μ^j.

Further work: The key outstanding problem in this area is to determine similar canonical forms in degenerate compatible biHamiltonian systems. Unfortunately, Turiel's approach, which is fundamentally tied to the covariant differential form framework for symplectic structures, does not appear to readily generalize, since degenerate Poisson structures can only be readily expressed in the contravariant language of bi-vector fields, [5].

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