Multi-Hamiltonian structure of the Born–Infeld equation

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The multi-Hamiltonian structure, conservation laws, and higher order symmetries for the Born–Infeld equation are exhibited. A new transformation of the Born-Infeld equation to the equations of a Chaplygin gas is presented and explored. The Born–Infeld equation is distinguished among two-dimensional hyperbolic systems by its wealth of such multi-Hamiltonian structures.

I. INTRODUCTION

A nonlinear modification of Maxwell's electrodynamics was proposed by Born and Infeld in 1934.[1] The simplest example of this system of nonlinear field equations is the quasilinear second-order equation in $1+1$ dimensions:

$$(1+\varphi_x^2)\varphi_{tt} - 2\varphi_t\varphi_x\varphi_{xt} - (1-\varphi_t^2)\varphi_{xx} = 0, \quad (1.1)$$

which is known as the Born–Infeld equation.[2] The Born–Infeld also governs minimal surfaces in $2+1$-dimensional Minkowski space, which is a special case of the Nambu string.[3] The world sheet of the Nambu string is parametrized by harmonic coordinates, familiar from the theory of minimal surfaces, rather than the light cone gauge.[4] We will also consider the representation of Eq. (1.1) in null coordinates:

$$x' = x+t, \quad t' = x-t,$$

in terms of which the Born–Infeld equation can be rewritten as

$$\varphi_{x'}^2\varphi_{t't'} - 2(2+\varphi_{t'}\varphi_{x'})\varphi_{x't'} + \varphi_{t'}^2\varphi_{x'x'} = 0. \quad (1.2)$$

In this paper we shall discuss the Hamiltonian structure, symmetries, and conservation laws of the Born–Infeld equation. We shall find that it has a remarkably rich structure. The first step is to recast the Born–Infeld equation as a first-order quasilinear Hamiltonian system of hydrodynamic type.[5,6] Remarkably, this can be done in three inequivalent ways, one of which corresponds to a system of isentropic gas dynamics, with the adiabatic index $\gamma = -1$ corresponding to the pressure–density relation $P = -1/\rho$, which is known as a *Chaplygin gas*.[7] Each of these systems is separable; therefore, the extensive results on Hamiltonian structures, symmetries, and conservation laws of Sheftel'[8] and Olver and Nutku[9] can be used. Even among the separable two-dimensional systems, the Born–Infeld system has a much richer algebraic structure than most, in part due to the multiple Hamiltonian reformulations of the equation. We will see that the Born–Infeld equation admits (at least) six independent Hamiltonian structures, in contrast to two Hamiltonian structures for a general separable system and four Hamiltonian structures in the more general polytropic case. Moreover, the diagonalization techniques introduced by Verosky[10] are then applied to show that these systems admit first-order conserved densities depending on arbitrary functions—which is special to these particular systems.

We assume that the reader is familiar with the basics of Hamiltonian systems of evolution equations, symmetries, and conservation laws, as presented, for example, in Olver.[11] In the interests of brevity, we have omitted many of the more complicated computations.

II. HYPERBOLIC FORMS OF THE BORN–INFELD EQUATION

We begin by showing that the Born–Infeld equation can be rewritten in several ways as a first-order system of quasilinear hyperbolic evolution equations. All of these representations have the form[9]

$$u_t = -D_x\frac{\partial H}{\partial v}, \quad v_t = -D_x\frac{\partial H}{\partial u}, \quad (2.1)$$

where $\mathscr{H}[u,v] = \int H(u,v)\,dx$ is the Hamiltonian functional and D_x is the total x derivative. In vector form, if we let

$$\mathbf{u}(x,t) = \begin{pmatrix} u(x,t) \\ v(x,t) \end{pmatrix},$$

then Eqs. (2.1) are in elementary Hamiltonian form[11]:

$$\mathbf{u}_t = \mathscr{D}^* E_{\mathbf{u}}[H], \quad (2.2)$$

where $E_{\mathbf{u}}$ denotes the Euler operator, or variational derivative with respect to $\mathbf{u}$. The Hamiltonian operator in (2.2) is the constant coefficient skew-adjoint differential operator

$$\mathscr{D}^* = -\sigma_1 \cdot D_x = \begin{pmatrix} 0 & -D_x \\ -D_x & 0 \end{pmatrix}, \quad \text{where } \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (2.3)$$

The induced Poisson bracket on the space of densities is given by the standard formula

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$$\{\mathscr{F},\mathscr{H}\} = \frac{1}{2}\int E_u[F]\cdot\mathscr{D}^* E_u[H]dx$$
$$= \frac{1}{2}\int \{E_v[F]D_xE_u[H] - E_u[F]D_xE_v[H]\}dx.$$

We begin by looking at Eq. (1.1) in the physical variables. Since (1.1) can be derived from a variational principal where the Lagrangian depends only on the gradient of φ, we know that it can be expressed as the integrability condition of a first-order system.[6] To effect this change, we introduce a new potential ψ given by

$$\psi_x = \varphi_x/\sqrt{1+\varphi_x^2-\varphi_t^2}, \quad \psi_t = \varphi_t/\sqrt{1+\varphi_x^2-\varphi_t^2}. \tag{2.4}$$

Inverting Eqs. (2.4) for the first derivatives of φ we find the same expressions, with the roles of φ and ψ interchanged. Equation (1.1) is then realized as the integrability condition for system (2.4): Moreover, its companion equation expressing the integrability conditions for φ is again (1.1), with ψ replacing φ. We shall now formulate these equations in terms of a pair of conservation laws. For this purpose, we introduce the variables

$$r = \varphi_x, \quad s = \psi_x.$$

Solving (2.4) for φ_t, ψ_t we deduce that the one-forms

$$\alpha = r\,dx + s\sqrt{(1+r^2)/(1+s^2)}\,dt = d\varphi,$$
$$\omega = s\,dx + r\sqrt{(1+s^2)/(1+r^2)}\,dt = d\psi,$$

are exact; the implication that they are closed gives rise to the following pair of quasilinear evolution equations:

$$\begin{aligned} r_t &= \left[rs/\sqrt{(1+r^2)(1+s^2)}\right]r_x + \sqrt{(1+r^2)/(1+s^2)^3}\,s_x, \\ s_t &= \sqrt{(1+s^2)/(1+r^2)^3}\,r_x + \left[rs/\sqrt{(1+r^2)(1+s^2)}\right]s_x. \end{aligned} \tag{2.5}$$

We will call the quasilinear system (2.5) the *physical version* of the Born–Infeld equation. It is easy to see that (2.5) is in the standard Hamiltonian form (2.2), where

$$\widetilde{H}^*(r,s) = \sqrt{(1+r^2)(1+s^2)} \tag{2.6}$$

is the Hamiltonian density. We note that there are alternative ways of reexpressing (1.1) as the integrability condition of a first-order system such as (2.2), but there is a unique choice of ψ which will result in a Hamiltonian system of equations. (An alternative first-order form of the Born–Infeld equation that is not Hamiltonian can be found in Whitham.[2])

A similar reasoning applies to the Born–Infeld equation, rewritten in the null coordinates (1.2). Dropping the primes on x, t, we similarly introduce a new potential χ by

$$\chi_x = -\varphi_x/\sqrt{1+\varphi_x\varphi_t}, \quad \chi_t = \varphi_t/\sqrt{1+\varphi_x\varphi_t}. \tag{2.7}$$

As in (2.4), the companion equation for χ is identical to (1.2). Define

$$z = \varphi_x, \quad w = \chi_x.$$

Note that the one-forms

$$\alpha = z\,dx - (1/z - z/w^2)dt = d\varphi,$$
$$\omega = w\,dx - (1/w - w/z^2)dt = d\chi$$

are exact, leading to an alternative system of quasilinear evolution equations:

$$\begin{aligned} z_t &= (1/z^2 + 1/w^2)z_x - (2z/w^3)w_x, \\ w_t &= -(2w/z^3)z_x + (1/z^2+1/w^2)w_x, \end{aligned} \tag{2.8}$$

which will be called the *null coordinate version* of the Born–Infeld equation. Again, (2.8) are in Hamiltonian form (2.2), with the Hamiltonian density

$$\widehat{H}^*(z,w) = z/w + w/z. \tag{2.9}$$

Although the two versions of the Born–Infeld equation can be obtained by a transformation between physical and null coordinates, it is rather remarkable that there is also a transformation of the dependent variables which maps one to the other, as shown in the following theorem.

Theorem 1: Given r, s with $rs > 1$, define the transformation

$$\begin{aligned} z &= (1+r^2)^{1/4}(1+s^2)^{1/4}[(rs+1)^{1/2} + (rs-1)^{1/2}], \\ w &= (1+r^2)^{1/4}(1+s^2)^{1/4}[(rs+1)^{1/2} - (rs-1)^{1/2}]. \end{aligned} \tag{2.10}$$

If (r,s) satisfy the physical version of the Born–Infeld equation (2.5), then (z,w) satisfy the null coordinate version (2.8).

The proof is a straightforward, but lengthy calculation. In Sec. III we shall see how the transformation (2.10) can be systematically deduced by referring to the second Hamiltonian structure of (2.5).

We now turn to a remarkable transformation from the Born–Infeld system to a system of quasilinear equations arising in polytropic gas dynamics.

Theorem 2: Define the variables

$$u = -(1/z^2 + 1/w^2), \quad v = zw/2. \tag{2.11}$$

Then z, w satisfy the Born–Infeld system (2.8) if and only if u,v satisfy the gas dynamics system

$$u_t + uu_x + v^{-3}v_x = 0, \quad v_t + (uv)_x = 0. \tag{2.12}$$

The proof is again a straightforward calculation. The system (2.12) corresponds to the equations of isentropic, polytropic gas dynamics with the adiabatic index $\gamma = -1$, known as a Chaplygin gas.[7] The system (2.12) is distinguished from such quasilinear hyperbolic systems by the fact that shocks do not form[12,2]: This system is also in the elementary Hamiltonian form (2.2), with the Hamiltonian density

$$H^*(u,v) = u^2v/2 + 1/2v. \tag{2.13}$$

We remark that the reduction of a gas dynamics system to a single second-order hyperbolic equation, which includes the reduction of a Chaplygin gas to the Born–Infeld equation (1.2), can be found in Garabedian.[13] Note, also, that the physical version (2.5) can be transformed directly to the gas dynamics version (2.12) by composing the transformations (2.10) and (2.11):

$$u = rs/\sqrt{(1+r^2)(1+s^2)}, \quad v = \sqrt{(1+r^2)(1+s^2)}. \tag{2.14}$$

We thus have three distinct ways of reformulating the

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Born–Infeld equation as a Hamiltonian system of quasilinear evolution equations of the type (2.2). To keep track of various functions and operators in the different coordinate systems, we will adopt the following conventions: In the physical (r,s) version, these quantities will have an overtilde, e.g., $\widetilde{H}$; in the null (z,w) version, they will have a caret, e.g., $\widehat{H}$; and the gas dynamics (u,v) coordinates will not have any distinguishing mark, e.g., H.

III. FIRST-ORDER HAMILTONIAN OPERATORS

We now investigate other first-order Hamiltonian structures for the Born–Infeld equation using the methods found in Refs. 6 and 9. First, we recall that the most general skew-adjoint first-order matrix differential operator has the form

$$\mathscr{D} = M\cdot D_x + D_x\cdot M + Q_x = \begin{pmatrix} 2mD_x + m_x & 2pD_x + p_x + q_x \\ 2pD_x + p_x - q_x & 2nD_x + n_x \end{pmatrix}, \quad (3.1)$$

where

$$M = \begin{pmatrix} m & p \\ p & n \end{pmatrix}$$

is a general symmetric matrix,

$$Q = \begin{pmatrix} 0 & q \\ -q & 0 \end{pmatrix}$$

is a general skew-symmetric matrix, and where the coefficients m, n, p, and q are allowed to depend on the dependent variables. The particular Hamiltonian operator (2.3) corresponds to the choice

$$\mathscr{D}^*:\ m^* = n^* = q^* = 0, \quad p^* = -\tfrac{1}{2}. \quad (3.2)$$

In order that the Poisson bracket associated with the operator (3.1) satisfies the Jacobi identity, the coefficients m, n, p, and q must satisfy additional first-order partial differential equations.[6]

Besides the standard Hamiltonian form (2.2), any polytropic gas dynamics system can be written in two additional, alternative Hamiltonian forms involving first-order Hamiltonian operators[6] and making it a tri-Hamiltonian system:

$$\mathbf{u}_t = \mathscr{D}_0 E_{\mathbf{u}}(H_2) = \mathscr{D}_1 E_{\mathbf{u}}(H_1) = \mathscr{D}_2 E_{\mathbf{u}}(H_0). \quad (3.3)$$

For the case of the adiabatic index $\gamma = -1$, the Hamiltonian operators in (3.3) have the form

$$\mathscr{D}_0 = \mathscr{D}^*:\ m_0 = 0, \quad n_0 = 0, \quad p_0 = -\tfrac{1}{2}, \quad q_0 = 0, \quad (3.4)$$

$$\mathscr{D}_1:\ m_1 = 1/v^3, \quad n_1 = v, \quad p_1 = -u, \quad q_1 = 2u, \quad (3.5)$$

$$\mathscr{D}_2:\ m_2 = u/v^3, \quad n_2 = uv, \quad p_2 = -u^2/2 - 1/2v^2, \quad q_2 = u^2 \quad (3.6)$$

and are mutually compatible.[11] We note that (3.4)–(3.6) are genuinely distinct Hamiltonian operators, meaning that $\mathscr{D}_2$ is not related to $\mathscr{D}_0$ and $\mathscr{D}_1$ according to a well-known recurrence formula[14] which generates higher order Hamiltonian operators from any bi-Hamiltonian system. The corresponding Hamiltonian densities placing (2.12) in the tri-Hamiltonian form (3.3) are

$$H_0 = v, \quad H_1 = uv, \quad H_2 = u^2v/2 + 1/2v, \quad (3.7)$$

which appear in the well-known hierarchy of conserved densities for gas dynamics.[9] (See Sec. IV.)

Before proceeding to the tri-Hamiltonian structure of the null coordinate and physical versions of the Born–Infeld equation, it helps to recall how Hamiltonian operators transform under a change of variables.

Lemma 3:[14,15] Let $\mathbf{u} = \varphi(\mathbf{z})$ be a change of variables and let $\mathbf{J}$ denote the Jacobian matrix of φ. Let $\mathscr{D}$ denote a Hamiltonian operator in the $\mathbf{u}$ coordinates and $\widehat{\mathscr{D}}$ the corresponding Hamiltonian operator in the $\mathbf{z}$ coordinates; then these two operators are related by the change of variables formula

$$\mathscr{D} = \mathbf{J}\cdot\widehat{\mathscr{D}}\cdot\mathbf{J}^T. \quad (3.8)$$

Thus for Hamiltonian operators of the form (3.1), we find the corresponding coefficient matrices have the change of variables formula

$$M = \mathbf{J}\cdot\widehat{M}\cdot\mathbf{J}^T, \quad Q_x = \mathbf{J}\cdot\widehat{Q}_x\cdot\mathbf{J}^T + \mathbf{J}\cdot\widehat{M}\cdot\mathbf{J}_x^T - \mathbf{J}_x\cdot\widehat{M}\cdot\mathbf{J}^T. \quad (3.9)$$

Dubrovin and Novikov[5] have pointed out that the Poisson brackets defined by Hamiltonian operators for equations of hydrodynamic type give rise to Riemannian metrics with vanishing torsion and curvature. The metric corresponding to an operator of the form (3.1) is given by

$$ds^2 = (n\,du^2 - 2p\,du\,dv + m\,dv^2)/(mn - p^2). \quad (3.10)$$

Since the metric (3.10) is flat we know that a (possibly complex) change of variables $\mathbf{u} = \varphi(\mathbf{z})$ will bring it to the canonical form $d\hat{s}^2 = 2\,dz\,dw$, determining the maximal analytic extension of the metric and corresponding to the elementary Hamiltonian operator (2.3). Remarkably, the transformations (2.11) and (2.14) are precisely the ones needed to place the metrics determined by the Hamiltonian operators $\mathscr{D}_1$, $\mathscr{D}_2$ in canonical form.

Proposition 4: Under the transformations (2.11) and (2.14) the Hamiltonian operators and densities for the gas dynamics system (2.12) are mapped to the following Hamiltonian operators and densities for the null and physical versions of the Born–Infeld equation:

Null coordinate version—Hamiltonian operators:

$$\widehat{\mathscr{D}}_0:\ \hat{m}_0 = zw^{-3}(z^{-2} - w^{-2})^{-2}, \quad \hat{n}_0 = wz^{-3}(z^{-2} - w^{-2})^{-2},$$

$$\hat{p}_0 = -\tfrac{1}{2}(z^{-2} - w^{-2})^{-1}, \quad \hat{q}_0 = (z^{-2} - w^{-2})^{-1},$$

$$\widehat{\mathscr{D}}_1 = -2\mathscr{D}^*:\ \hat{m}_1 = 0, \quad \hat{n}_1 = 0, \quad \hat{p}_1 = 1, \quad \hat{q}_1 = 0,$$

$$\widehat{\mathscr{D}}_2:\ \hat{m}_2 = -\frac{z}{w^3}, \quad \hat{n}_2 = -\frac{w}{z^3}, \quad \hat{p}_2 = -\frac{1}{2z^2} + \frac{1}{2w^2}, \quad \hat{q}_2 = \frac{1}{z^2} - \frac{1}{w^2}.$$

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Hamiltonian densities:

$$\hat{H}_0(z,w) = zw/2,$$
$$\hat{H}_1(z,w) = -z/2w - w/2z,$$
$$\hat{H}_2(z,w) = w/4z^3 + 3/2zw + z/4w^3.$$

Physical version—Hamiltonian operators:

$$\tilde{\mathscr{D}}_0\colon\ \tilde{m}_0 = \frac{-2rs(1+r^2)^2}{(r^2-s^2)^2},\quad \tilde{n}_0 = \frac{-2rs(1+s^2)^2}{(r^2-s^2)^2},$$
$$\tilde{p}_0 = \frac{(r^2+s^2)(1+r^2)(1+s^2)}{(r^2-s^2)^2},$$
$$\tilde{q}_0 = \frac{(r^2+s^2+2r^2s^2)}{(r^2-s^2)},$$
$$\tilde{\mathscr{D}}_1\colon\ \tilde{m}_1 = \frac{(1+r^2)^{3/2}(r^2+s^2+2r^2s^2)}{(1+s^2)^{1/2}(r^2-s^2)^2},$$
$$\tilde{n}_1 = \frac{(1+s^2)^{3/2}(r^2+s^2+2r^2s^2)}{(1+r^2)^{1/2}(r^2-s^2)^2},$$
$$\tilde{p}_1 = \frac{(1+r^2)^{1/2}(1+s^2)^{1/2}}{(r^2-s^2)^2}rs(2+r^2+s^2),$$
$$\tilde{q}_1 = [2rs/(r^2-s^2)](1+r^2)^{1/2}(1+s^2)^{1/2},$$
$$\tilde{\mathscr{D}}_2 = \mathscr{D}^*\colon\ \tilde{m}_2 = 0,\quad \tilde{n}_2 = 0,\quad \tilde{p}_2 = -\tfrac{1}{2},\quad \tilde{q}_2 = 0.$$

Hamiltonian densities:

$$\tilde{H}_0(r,s) = \sqrt{(1+r^2)(1+s^2)},$$
$$\tilde{H}_1(r,s) = rs,$$
$$\tilde{H}_2(r,s) = (r^2s^2+1)/2\sqrt{(1+r^2)(1+s^2)}.$$

IV. RECURSION OPERATORS AND CONSERVED DENSITIES

According to Magri's theorem,[16] any compatible bi-Hamiltonian system has an associated recursion operator. The Hamiltonian operators $\mathscr{D}_0$, $\mathscr{D}_1$, and $\mathscr{D}_2$ are mutually compatible[6]; thus there are three recursion operators for the gas dynamics system,

$$\mathscr{R}_1 = \mathscr{D}_1\cdot\mathscr{D}_0^{-1},\quad \mathscr{R}_2 = \mathscr{D}_2\cdot\mathscr{D}_0^{-1},\quad \mathscr{R}_3 = \mathscr{D}_2\cdot\mathscr{D}_1^{-1}, \tag{4.1}$$

although there is a trivial relation between them:

$$\mathscr{R}_2 = \mathscr{R}_3\cdot\mathscr{R}_1.$$

Similar recursion operators can be constructed for the null coordinate and physical versions of the Born–Infeld equation. Now, a curious phenomenon occurs when we apply the recursion operator to the hierarchy where the Born–Infeld Hamiltonian lies. We find that the hierarchy of Hamiltonian flows $\mathscr{R}_1$ terminates after just two steps:

$$\mathscr{R}_1\colon\ H_0 \to H_1 \to H_2 \to 0$$

because the second Hamiltonian H_2 is a distinguished functional (Casimir) for the Hamiltonian structure determined by $\mathscr{D}_1$. Therefore, the hierarchy guaranteed by Magri's theorem[16] degenerates into just three independent Hamiltonians; we have a nontrivial example of a bi-Hamiltonian system which does not satisfy one of the technical hypotheses of Magri's theorem, which states that the hierarchy of Hamiltonians be independent functionals.[11,16] However, the second recursion operator $\mathscr{R}_2$ does generate further members of the gas dynamics hierarchy of conserved densities.[9] (We remark that in Ref. 9 we failed to show that this property of the hierarchy of flows generated by one of the recursion operators can occasionally degenerate. The equations following (4.3) of Ref. 9 should read as

$$\mathscr{R}_1(\mathbf{Q}_n) = (n\gamma - n - 1)\mathbf{Q}_{n+1},$$
$$\mathscr{R}_2(\mathbf{Q}_n) = (n/2)(n\gamma + \gamma - n - 3)\mathbf{Q}_{n+2},$$
$$\mathscr{R}_1(\tilde{\mathbf{Q}}_n) = (n\gamma - n + 1)\tilde{\mathbf{Q}}_{n+1},$$
$$\mathscr{R}_2(\tilde{\mathbf{Q}}_n) = [(n+1)/2](n\gamma - n + 1)\tilde{\mathbf{Q}}_{n+2},$$

leading to degeneracies if γ has one of the forms $1 \pm 1/n$, $1 \pm 2/n$ for some integer n.)

Another interesting anomaly occurs for the physical version of the Born–Infeld equation. Here, from the point of view of Ref. 6, the most natural recursion operator would be

$$\mathscr{R}^* = \tilde{\mathscr{D}}_1\cdot\tilde{\mathscr{D}}_2^{-1} = \tilde{\mathscr{D}}_1\cdot\mathscr{D}^{*-1}.$$

Again, this recursion operator does not produce a hierarchy of symmetries and conserved Hamiltonian densities. In fact, as the reader can check, the recursion operator repeats after two steps:

$$\mathscr{R}^*\colon\ H_0 \to H_1 \to H_0 \to H_1 \to H_0 \to \cdots,$$

resulting in an infinite loop; again the functionals produced by Magri's theorem[16] are not independent. [At first glance, this result does not seem reconciled with the gas dynamics version under the transformation (2.14). However, we note that since the recursion operator involves the inverse of the Hamiltonian operator $\mathscr{D}^*$, we can add in any element of its kernel at each step. Thus the explanation is that we have just chosen different elements of ker $\tilde{\mathscr{D}}_2$ to add in.]

The gas dynamics, null coordinate, and physical versions of the Born–Infeld equation are examples of *separable systems*,[8,9] meaning that the Hamiltonian density H in the representation (2.2) satisfies

$$H_{uu}/H_{vv} = \lambda(u)/\mu(v). \tag{4.2}$$

For the three versions, the separation coefficients are given by

gas dynamics [(2.12)]:

$$\lambda(u) = 1,\quad \mu(v) = v^{-4},$$

null version [(2.8)]:

$$\hat{\lambda}(z) = z^{-4},\quad \hat{\mu}(w) = w^{-4}, \tag{4.3}$$

physical version [(2.5)]:

$$\tilde{\lambda}(r) = (1+r^2)^{-2},\quad \tilde{\mu}(s) = (1+s^2)^{-2},$$

It is standard that the zeroth-order conserved densities for such a system can be found by solving a separable linear wave equation.[8,9]

Proposition 5: A function $F(u,v)$ is a conserved density of a separable Hamiltonian system (2.2) and (4.2) if and only if it is a solution to the linear wave equation

$$F_{uu}/\lambda(u) = F_{vv}/\mu(v). \tag{4.4}$$

Any Hamiltonian system (2.2) admits the conserved densities 1, u, v, and uv. In the separable case, there are four

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fundamental hierarchies of solutions to the wave equation (4.4), each of the form

$$H_n(u,v) = \sum_{i=0}^{n} F_i(u)\cdot G_{n-i}(v), \tag{4.5}$$

where the functions F_i and G_i are generated by the recursion relations

$$\frac{\partial^2 F_i}{\partial u^2} = \lambda(u) F_{i-1}, \quad F_i(0) = F_i'(0) = 0,$$

$$\frac{\partial^2 G_i}{\partial v^2} = \mu(v) G_{i-1}, \quad G_i(0) = G_i'(0) = 0.$$

The hierarchies depend on the initial selection of $H_0 = F_0 \cdot G_0$:

$$H_0^{(1)} = 1, \quad F_0^{(1)} = G_0^{(1)} = 1,$$

$$H_0^{(2)} = u, \quad F_0^{(2)} = u, \quad G_0^{(2)} = 1,$$

$$H_0^{(3)} = v, \quad F_0^{(3)} = 1, \quad G_0^{(3)} = v,$$

$$H_0^{(4)} = uv, \quad F_0^{(4)} = u, \quad G_0^{(4)} = v.$$

Our transformations do not respect this hierarchical structure of the conserved densities. For example, (2.11) maps the first and fourth null Born–Infeld hierarchies to combinations of all four gas dynamics hierarchies, so that up to a multiple,

$$\widehat{H}_{2j}^{(1)} \to H_j^{(1)}, \quad \widehat{H}_{2j}^{(4)} \to H_j^{(3)},$$

$$\widehat{H}_{2j+1}^{(1)} \to H_j^{(2)}, \quad \widehat{H}_{2j+1}^{(4)} \to H_j^{(4)}.$$

On the other hand, the second and third hierarchies are mapped to algebraic conserved densities for the gas dynamics version (2.12). For example, the conserved density $\widehat{H}_0^{(2)} = z$ is mapped to the conserved density

$$\sqrt{v - uv^2} + \sqrt{-v - uv^2},$$

which does not show up in any of the standard gas dynamics hierarchies. The hierarchies in the physical r, s variables are no longer rational functions and we shall not write them explicitly: They do not correspond to any of the hierarchies in the other variables (with isolated exceptions) and provide yet other nonpolynomial conserved densities for gas dynamics system (2.12).

V. HIGHER ORDER HAMILTONIAN STRUCTURES

In Olver and Nutku[9] it was shown that any separable Hamiltonian system has a second Hamiltonian structure involving a complicated third-order matrix differential operator. The resulting recursion operator recovers results on symmetries and conservation laws due to Sheftel'.[8] For the Born–Infeld equations, each of the gas dynamics, null coordinate, and physical versions is separable, and so we are led to three distinct third-order Hamiltonian structures. This is probably quite special to these particular systems, but we have no proof of this fact. In particular, it would be interesting to see whether any of the other polytropic gas dynamics systems have additional Hamiltonian structures.

Theorem 6: Consider a separable Hamiltonian system (2.2), where the Hamiltonian density satisfies (4.2). Define the matrix variables

$$U_x = \begin{pmatrix} u_x & \mu(v)v_x \\ v_x & \lambda(u)u_x \end{pmatrix}, \quad V_x = \begin{pmatrix} \lambda(u)u_x & \mu(v)v_x \\ v_x & u_x \end{pmatrix}. \tag{5.1}$$

Then the system can be written in the bi-Hamiltonian form

$$\mathbf{u}_t = \mathscr{D}^* E_{\mathbf{u}}(H) = \mathscr{E} E_{\mathbf{u}}(H^*) \tag{5.2}$$

using the third-order matrix differential operator

$$\mathscr{E} = D_x \cdot V_x^{-1} \cdot D_x \cdot U_x^{-1} \cdot \sigma_1 \cdot D_x$$

$$= D_x \cdot V_x^{-1} \cdot D_x \cdot \sigma_1 \cdot V_x^{-T} \cdot D_x. \tag{5.3}$$

In particular, $\mathscr{E}$ is Hamiltonian and compatible with $\mathscr{D}^*$.

In the case of gas dynamics the matrix variables coincide:

$$U_x = V_x = \begin{pmatrix} u_x & v^{-4}v_x \\ v_x & u_x \end{pmatrix}$$

and the corresponding Hamiltonian operator (5.3) is

$$\mathscr{E}_0 = D_x \cdot U_x^{-1} \cdot D_x \cdot U_x^{-1} \cdot \sigma_1 \cdot D_x, \tag{5.4}$$

which is compatible with $\mathscr{D}_0 = \mathscr{D}^*$. The second Hamiltonian in (5.2) turns out to be

$$H^* = H_2^{(3)} = u^4 v/24 + u^2/2v + 1/24v^3,$$

which appears in the third hierarchy (4.5) of conserved densities. The corresponding recursion operator is the square of the simple recursion operator

$$\mathscr{R} = D_x \cdot U_x^{-1}, \tag{5.5}$$

so that

$$\mathscr{E} \cdot \mathscr{D}_0^{-1} = D_x \cdot U_x^{-1} \cdot D_x \cdot U_x^{-1} = \mathscr{R}^2.$$

Similarly, we have a third-order recursion operator in the null variables (z,w). We define the matrix variables

$$Z_x = \begin{pmatrix} z_x & w^{-4}w_x \\ w_x & z^{-4}z_x \end{pmatrix}, \quad W_x = \begin{pmatrix} z^{-4}z_x & w^{-4}w_x \\ w_x & z_x \end{pmatrix}$$

and the operator

$$\widehat{\mathscr{E}}_1 = D_x \cdot W_x^{-1} \cdot D_x \cdot Z_x^{-1} \cdot \sigma_1 \cdot D_x$$

$$= D_x \cdot W_x^{-1} \cdot D_x \cdot \sigma_1 \cdot W_x^{-T} \cdot D_x$$

is Hamiltonian. Moreover, the Hamiltonian operators $\widehat{\mathscr{E}}_1$ and $\widehat{\mathscr{D}}_1 = -2\mathscr{D}^*$ are compatible; therefore, they form a Hamiltonian pair. The null Born–Infeld equation (2.8) can be written as a bi-Hamiltonian system

$$\mathbf{z}_t = \widehat{\mathscr{D}}_1 E_{\mathbf{z}}[\widehat{H}] = \widehat{\mathscr{E}}_1 E_{\mathbf{z}}[\widehat{H}^*], \tag{5.6}$$

where the Hamiltonian is a multiple of the Hamiltonian $\widehat{H}_4^{(2)}$ in the fourth hierarchy (4.5):

$$\widehat{H}^*(z,w) = 2\widehat{H}_2^{(4)}(z,w) = w/12z^3 + 1/2zw + z/12w^3.$$

Note that the transformation (2.11) cannot map the above two higher order Hamiltonian operators to each other since the corresponding bi-Hamiltonian structures do not match, nor are the compatibility relations preserved. Indeed, a long calculation proves that the gas dynamics recursion operator arising from the bi-Hamiltonian Form (5.6) under the transformation (2.11) is the operator

$$\widehat{\mathscr{R}}_1 = \widehat{\mathscr{E}}_1 \widehat{\mathscr{D}}_1^{-1} \to -2\mathscr{R}_1 \mathscr{R}^2,$$

where $\mathscr{R}$ is the gas dynamics recursion operator given by (5.5) and $\mathscr{R}_1$ is the recursion operator (4.1) arising from Nutku's[6] Hamiltonian structures for gas dynamics. Therefore the operator

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$$\mathscr{E}_1 = -2\mathscr{R}_1\mathscr{E}_0\mathscr{R}_1$$

in another third-order Hamiltonian operator for Eqs. (2.12) which is compatible with the first-order Hamiltonian operator $\mathscr{D}_1$, but not with either $\mathscr{D}_0$ or $\mathscr{D}_2$.

Finally there is yet another third-order Hamiltonian operator arising from the physical version of the Born–Infeld equation. The operator takes the form

$$\begin{aligned}\widetilde{\mathscr{E}}_2 &= D_x \cdot S_x^{-1} \cdot D_x \cdot R_x^{-1} \cdot \sigma_1 \cdot D_x \\ &= D_x \cdot S_x^{-1} \cdot D_x \cdot \sigma_1 \cdot S_x^{-T} \cdot D_x,\end{aligned}$$

where

$$R_x = \begin{pmatrix} r_x & (1+s^2)^{-2}s_x \\ s_x & (1+r^2)^{-2}r_x \end{pmatrix},$$

$$S_x = \begin{pmatrix} (1+r^2)^{-2}r_x & (1+s^2)^{-2}s_x \\ s_x & r_x \end{pmatrix}.$$

This Hamiltonian operator is compatible with $\widetilde{\mathscr{D}}_2 = \mathscr{D}^*$ and so, when transformed back to the other coordinate systems, it provides yet another Hamiltonian structure for the Born–Infeld equation.

In summary, then, we have found that the Born–Infeld equation in any of its evolutionary forms (2.5), (2.8), or (2.12) possesses six distinct Hamiltonian structures: Three are first order, given by the operators $\mathscr{D}_0$, $\mathscr{D}_1$, and $\mathscr{D}_2$ and three are third order, given by the operators $\mathscr{E}_0$, $\mathscr{E}_1$, and $\mathscr{E}_2$. Moreover $\mathscr{D}_i$ is compatible with $\mathscr{E}_j$ if and only if $i=j$. Whether there are yet more Hamiltonian structures, not trivially related to these, remains an open question!

VI. DIAGONALIZATION AND HIGHER ORDER CONSERVATION LAWS

As shown by Olver and Nutku[9], for a generalized gas dynamics Hamiltonian system there is an additional hierarchy of higher order conservation laws generalizing Verosky's rational first-order conserved density[17]:

$$\hat{H}_1[u,v] = v_x/(u_x^2 - v^{\gamma-3}v_x^2)\,. \tag{6.1}$$

The case of a Chaplygin gas, $\gamma = -1$, is distinguished in that it admits an infinite collection of distinct first-order conserved densities (i.e., they do not differ by a divergence): The easiest way to see this is to apply a diagonalization technique, described by Verosky[10] and Tsarev.[18]

Definition 7: A first-order quasilinear system is said to be in *diagonal form* if it has the form

$$p_t = A(p,q)p_x, \quad q_t = B(p,q)q_x. \tag{6.2}$$

We remark that the existence of a diagonal form for a quasilinear first-order system is related to the existence of Riemann invariants.[18]

Proposition 8: For the Chaplygin system (2.12), the transformation

$$p = u + 1/v, \quad q = u - 1/v \tag{6.3}$$

place it in the diagonal form[19]

$$p_t = -qp_x, \quad q_t = -pq_x. \tag{6.4}$$

Theorem 9[10]: A two-dimensional diagonal quasilinear system (6.2) has a first-order conservation law $D_tT + D_xX = 0$, with conserved density and flux of the form

$$T = \frac{F(p,q)}{p_x} + \frac{G(p,q)}{q_x}, \quad X = \frac{AF}{p_x} + \frac{BG}{q_x} \tag{6.5}$$

if and only if F and G satisfy the system of differential equations

$$\begin{aligned}&(A-B)G_p = 2GB_p, \quad (B-A)F_q = 2FA_q, \\ &FA_p + GB_q = 0.\end{aligned} \tag{6.6}$$

For the special case $A = -q$, $B = -p$, corresponding to the Born–Infeld equation, the third equation in (6.6) is vacuous; thus there are the solutions

$$F(p,q) = \alpha(p)/(p-q)^2, \quad G(p,q) = \beta(q)/(p-q)^2 \tag{6.7}$$

depending on the arbitrary functions $\alpha(p), \beta(q)$. There are similar expressions for other gas dynamics systems with $\gamma \neq -1$, but then the third equation in (6.6) is not vacuous; this restricts the corresponding functions to satisfying $\alpha = -\beta$ and thus both coefficients must be constant! Thus the Born–Infeld case is very special.

In terms of the gas dynamics variables, the conserved densities have the form

$$\begin{aligned}T[u,v] &= v^4\alpha(u+v^{-1})/(v^2u_x - v_x) \\ &\quad + v^4\beta(u-v^{-1})/(v^2u_x + v_x)\,.\end{aligned}$$

Note that the case $\alpha = \frac{1}{2}, \beta = -\frac{1}{2}$ reproduces the conserved density (6.1) when $\gamma = -1$. Under the transformation (2.11), these turn into the following conserved densities for the null version of the Born–Infeld equation:

$$T[z,w] = \frac{z^4w^4\tilde{\alpha}(z^{-1} - w^{-1})}{w^2z_x - z^2w_x} + \frac{z^4w^4\tilde{\beta}(z^{-1} + w^{-1})}{w^2z_x + z^2w_x},$$

where

$$\tilde{\alpha}(s) = \alpha(-s^2)/8s, \quad \tilde{\beta}(s) = \beta(-s^2)/8s.$$

For the particular choices $\tilde{\alpha}(s) = 1$, $\tilde{\beta}(s) = \pm 1$, i.e., $\alpha(s) = 8\sqrt{-s}, \beta(s) = 8\sqrt{-s}$, we obtain the conserved densities

$$z^6w^6w_x/(w^4z_x^2 - z^4w_x^2), \quad z^6w^6z_x/(w^4z_x^2 - z^4w_x^2),$$

which are more like the first-order densities discovered in Verosky[17]; see, also, Olver and Nutku.[9] It is interesting that the transformation (2.11) does *not* map the Verosky-type densities to each other.

It can be shown that the third-order evolution equations corresponding to the above two densities are each bi-Hamiltonian systems; hence the recursion operators lead to two further hierarchies of higher order conserved densities.

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