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On the Hamiltonian structure of evolution equations

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Abstract. The theory of evolution equations in Hamiltonian form is developed by use of some differential complexes arising naturally in the formal theory of partial differential equations. The theory of integral invariants is extended to these infinite-dimensional systems, providing a natural generalization of the notion of a conservation law. A generalization of Noether's theorem is proved, giving a one-to-one correspondence between one-parameter (generalized) symmetries of a Hamiltonian system and absolute line integral invariants. Applications include a new solution to the inverse problem of the calculus of variations, an elementary proof and generalization of a theorem of Gel'fand and Dikiĭ on the equality of Lie and Poisson brackets for Hamiltonian systems, and a new hierarchy of conserved quantities for the Korteweg–de Vries equation.

INTRODUCTION

The many applications of variational methods to the study of non-linear partial differential equations has given new impetus to the study of equations in Hamiltonian form over infinite-dimensional spaces. This has been of special interest since the discovery (5) that certain physically interesting equations such as the Korteweg–de Vries equation can be interpreted as completely integrable Hamiltonian systems. In this paper it is shown how the classical Hamiltonian formalism of differential geometry can be generalized to the study of evolution equations. For simplicity we work in Euclidean space, although similar results for equations defined over smooth manifolds are immediate, and lead to interesting cohomology classes. These will not be touched on here; but see, for instance, (1), (25) and (29–31).

The motivation for this paper came from the observation (29), (17) that, whereas every conservation law of a system of p.d.e.'s having a variational principle can be constructed via Noether's theorem from a (generalized) symmetry group of the system, not every symmetry gives rise to a conserved quantity. The development of the notion of an integral invariant (3) in an infinite-dimensional context leads to an understanding of the conservational roles of these further symmetry groups. We show that every symmetry of an evolution equation in Hamiltonian form provides an invariant line integral of the equation, similar to the conservation of circulation in fluid dynamics. Those line integrals whose associated one-forms satisfy an additional assumption of closure with respect to a certain exterior derivative are then conservation laws of the usual sort. These ideas are applied to give a new hierarchy of invariant integrals of the KdV equation, and an additional integral invariant for the BBM equation (2). Further applications will be announced elsewhere.

The techniques to be used are in the spirit of the formal variational calculus of Gel'fand and Dikiĭ (6–9) and its subsequent developments by Manin (12), Kupershmidt (11), Sternberg (24) and the author (15–17). In the course of the development of the theory, other useful applications arise. Yet another solution of the inverse problem of the calculus of variations, i.e. finding necessary and sufficient conditions for a system of p.d.e.s to come from a variational principle, is found by use of a new resolution of the Euler operator. The Hamiltonian structure allows us to give an elementary proof and a generalization of a result of Gel'fand and Dikiĭ on the equality of certain Lie and Poisson brackets arising from equations having a Lax representation (8, 9). This reduces to a question concerning the closure of a certain two-form, and a method of proving this for general forms is discussed. The present generalization of Noether's theorem to include invariant integrals is an immediate consequence of this theory.

1. DIFFERENTIAL COMPLEXES

Consider the Euclidean spaces $X = \mathbf{R}^p$, with coordinates $x = (x^1, \ldots, x^p)^T$, and $U = \mathbf{R}^q$, with coordinates $u = (u^1, \ldots, u^q)^T$. (Here ^{T} denotes transpose, so x and u are column vectors). Viewing the u's as dependent and the x's as independent variables, let J_k denote the k-jet space, having coordinates $u^i_J = \partial^J u^i$. Here ∂^J is the partial derivative corresponding to the multi-index $J = (j_1, \ldots, j_p)$. There is a natural projection $\pi: J_k \to J_{k-1}$, and we let J_∞ be the inverse limit of the finite jet spaces J_k. The exterior powers of the cotangent space of J_∞ are similarly defined as inverse limits:

$$\wedge^k = \wedge^k T^* J_\infty = \operatorname*{inv\,lim}_j \wedge^k T^* J_j.$$

A k-differential form, meaning a smooth section of $\wedge^k$, thus consists of a finite sum of terms

$$P\, dx^I \wedge du^{\mathscr{J}} \equiv P(x, u^{(n)})\, dx^{i_1} \wedge \ldots \wedge dx^{i_l} \wedge du^j_J \wedge \ldots \wedge du^k_K. \tag{1·1}$$

Here $P \in \mathscr{A}$, the algebra of smooth differential functions, and depends only on finitely many derivatives of the u^j, with n indicating the degree of highest order derivative. Note that $\mathscr{A}$ is a partial differential algebra, the (total) derivative in the x^i coordinate being

$$D_i = \frac{d}{dx^i} = \frac{\partial}{\partial x^i} + \sum_j \sum_J u^j_{J,i} \frac{\partial}{\partial u^j_J},$$

where $J, i = (j_1, \ldots, j_{i-1}, j_i + 1, j_{i+1}, \ldots, j_p)$. In practice we often restrict our attention to the subalgebra $\mathscr{A}_p$ consisting of those P's which are polynomials in the u's and their derivatives, but this is not essential in what follows.

The exterior derivative $d: \underline{\wedge}^k \to \underline{\wedge}^{k+1}$, the underbar denoting spaces of sections, is defined as the inverse limit of the exterior derivatives on $\underline{\wedge^k T^* J_j}$. Therefore

$$d(P\, dx^I \wedge du^{\mathscr{J}}) = \left(\sum_i \frac{\partial P}{\partial x^i} dx^i + \sum_{j,J} \frac{\partial P}{\partial u^j_J} du^j_J \right) \wedge dx^I \wedge du^{\mathscr{J}}. \tag{1·2}$$

The finite-dimensional d-Poincaré lemma (cf. (23), theorem III.4.1) immediately implies the following:

THEOREM 1·1. *The complex*

$$0 \to \mathbf{R} \to \underline{\wedge}^0 \xrightarrow{d} \underline{\wedge}^1 \xrightarrow{d} \underline{\wedge}^2 \xrightarrow{d} \ldots,$$

is exact.

Next define the exterior powers of the tangent space to J_∞, denoted by $\wedge_k = \wedge_k T J_\infty$, to be the dual spaces to the cotangent spaces $\wedge^k$. Thus a *vector field* on J_∞ will be a (formal) infinite sum

$$\mathbf{v} = \sum_{i=1}^{p} P^i \frac{\partial}{\partial x^i} + \sum_J \sum_j Q^j_J \frac{\partial}{\partial u^j_J}, \tag{1·3}$$

where the P^i, Q^j_J are in the algebra $\mathscr{A}$. In particular, the total derivatives D_i can be viewed as vector fields. A vector field is *vertical* if all the x-components P^i vanish. A vector field is *special* if it is vertical and commutes with all the total derivatives. It is not hard to see (6) that every special vector field has the form

$$\mathbf{v}_K = \sum_{j,J} D^J K^j \frac{\partial}{\partial u^j_J}, \tag{1·4}$$

where $K = (K^1, \ldots, K^q)^{\mathrm{T}} \in \mathscr{A}^q$, and D^J is the total derivative corresponding to the multi-index J.

For each vector field $\mathbf{v}$, there is a corresponding *Lie derivative* acting on the space of differential forms, which we also denote by $\mathbf{v}$. For example if $\mathbf{v}$ is given by (1·3), and $\omega = P\, du^j_J \wedge du^k_K$, then

$$\mathbf{v}(\omega) = \mathbf{v}(P)\, du^j_J \wedge du^k_K + P\, dQ^j_J \wedge du^k_K + P\, du^j_J \wedge dQ^k_K.$$

The following formulae are easily proved:

$$\mathbf{v}(d\omega) = d[\mathbf{v}(\omega)], \tag{1·5}$$

$$\mathbf{v}(\omega \wedge \mu) = \mathbf{v}(\omega) \wedge \mu + \omega \wedge \mathbf{v}(\mu). \tag{1·6}$$

The space $\wedge^k$ can be decomposed as a direct sum

$$\wedge^k = \wedge^k_0 \oplus \wedge^{k-1}_1 \oplus \wedge^{k-2}_2 \oplus \cdots,$$

which terminates with either

$$\wedge^0_k \quad \text{if} \quad k \leqslant p, \quad \text{or} \quad \wedge^{k-p}_p \quad \text{if} \quad k > p.$$

Here $\wedge^{k-l}_l$ is the subspace spanned by the differential forms (1·1) with l, the number of dx's, fixed. A second exterior derivative $D: \underline{\wedge}^k_l \to \underline{\wedge}^k_{l+1}$ is defined by

$$D\omega = \sum_{i=1}^{p} dx^i \wedge D_i(\omega), \tag{1·7}$$

the D_i's acting as Lie derivatives.

THEOREM 1·2. *For each integer $k \geqslant 0$, the complex*

$$0 \to \delta^k_0 \mathbf{R} \to \underline{\wedge}^k_0 \xrightarrow{D} \underline{\wedge}^k_1 \xrightarrow{D} \cdots \xrightarrow{D} \underline{\wedge}^k_{p-1} \xrightarrow{D} \underline{\wedge}^k_p$$

is exact. (Here $\delta^k_0 \mathbf{R} = \mathbf{R}$ for $k = 0$ and 0 for $k > 0$.) Moreover, the derivatives d and D anticommute:

$$d.D + D.d = 0. \tag{1·8}$$

There now exist a number of different proofs of this important result. In the polynomial case, C. Shakiban's thesis (22) transforms the above complex to the standard

Hilbert syzygy complex and thereby proves exactness. Explicit, lengthy computations verifying exactness are to be found in Tulczyjew(27) (for $k > 0$), Takens (25), and Andersen and Duchamp(1). Proofs based on some deep results from algebraic topology are given by Vinogradov(29, 30) and Tsujishita(26).

The problem now is that d does not preserve the grading $\underline{\wedge}_j^k$. In fact $d = d_x + d_u$, where $d_x: \underline{\wedge}_j^k \to \underline{\wedge}_{j+1}^k$ gives the exterior derivative for just the x's, and $d_u: \underline{\wedge}_j^k \to \underline{\wedge}_j^{k+1}$ for just the u's. (In (1·2) the first summation is d_x and the second d_u.) Theorem 1·1 implies that the two subcomplexes

$$\underline{\wedge}_0^k \xrightarrow{d_x} \underline{\wedge}_1^k \xrightarrow{d_x} \underline{\wedge}_2^k \xrightarrow{d_x} \dots \xrightarrow{d_x} \underline{\wedge}_p^k,$$

and

$$\underline{\wedge}_j^0 \xrightarrow{d_u} \underline{\wedge}_j^1 \xrightarrow{d_u} \underline{\wedge}_j^2 \xrightarrow{d_u} \underline{\wedge}_j^3 \xrightarrow{d_u} \dots$$

are both exact. Furthermore, D, d_x and d_u all mutually anticommute. Define

$$\wedge_{j*}^k = \operatorname{Coker}(D: \underline{\wedge}_{j-1}^k \to \underline{\wedge}_j^k) = \underline{\wedge}_j^k / D\underline{\wedge}_{j-1}^k.$$

By anticommutativity, there is an induced derivative

$$d_*: \wedge_{j*}^k \to \wedge_{j*}^{k+1},$$

where, for $\pi_*: \underline{\wedge}_j^k \to \wedge_{j*}^k$ the natural projection,

$$d_* \pi_*(\omega) = \pi_*(d_u \omega).$$

THEOREM 1·3. *For each j the complex*

$$\wedge_{j*}^0 \xrightarrow{d_*} \wedge_{j*}^1 \xrightarrow{d_*} \wedge_{j*}^2 \xrightarrow{d_*} \dots$$

is exact.

Proof. This follows from standard spectral sequence arguments using the fact that D and d_u make $\underline{\wedge}_j^k$ into a 'double complex' [cf. (4)]. For completeness, we include a proof. Note that, for $j = 0$, the statement is trivial. Next, by induction, assume the complex for $j-1$ is exact. We must prove that if $\omega \in \underline{\wedge}_j^k$ and $d_u \omega = D\nu$ for some $\nu \in \underline{\wedge}_{j-1}^{k+1}$, then $\omega = d_u \mu + D\pi$ for some $\mu \in \underline{\wedge}_j^{k-1}$, $\pi \in \underline{\wedge}_{j-1}^k$. Now if $d_u \omega = D\nu$, then $0 = d_u D\nu = -Dd_u \nu$, hence $d_u \nu = D\lambda$ for some $\lambda \in \underline{\wedge}_{j-2}^{k+2}$ by Theorem 1·2. By induction, $\nu = -d_u \pi + D\rho$ for some $\pi \in \underline{\wedge}_{j-1}^k$ and $\rho \in \underline{\wedge}_{j-2}^{k+1}$. Finally

$$d_u(\omega - D\pi) = d_u \omega + Dd_u \pi = d_u \omega + D(D\rho - \nu) = 0.$$

Hence Theorem 1·1 implies that $\omega - D\pi = d_u \mu$ for some $\mu \in \underline{\wedge}_j^{k-1}$, which completes the proof.

Fixing the volume form $dx = dx^1 \wedge \dots \wedge dx^p$ on X allows us to identify $\wedge_p^k$ with $\wedge_0^k$, via $\omega \wedge dx \simeq \omega$. Note that, under this identification, the derivative $D: \underline{\wedge}_p^{k-1} \to \underline{\wedge}_p^k$ can be identified with the *total divergence operator*:

$$\operatorname{Div}(\omega_1, \dots, \omega_p) = D_1 \omega_1 + D_2 \omega_2 + \dots + D_p \omega_p.$$

Therefore, with

$$\wedge_*^k = \underline{\wedge}_0^k / \operatorname{im}(\operatorname{Div}) \simeq \underline{\wedge}_p^k / D(\underline{\wedge}_{p-1}^k),$$

Theorem 1·3 proves the exactness of

$$0 \to \wedge_*^0 \xrightarrow{d_*} \wedge_*^1 \xrightarrow{d_*} \wedge_*^2 \xrightarrow{d_*} \dots, \tag{1·9}$$

where it remains to show the exactness at the first stage. This will be done presently. Two forms ω, $\tilde{\omega} \in \underline{\Lambda}_0^k$ will be called *equivalent*, denoted by $\omega \sim \tilde{\omega}$, if they have the same image in Λ_*^k, i.e. $\omega = \tilde{\omega} + \mathrm{Div}\,(\mu)$ for some $\mu = (\mu_1, \ldots, \mu_p)$, with $\mu_j \in \underline{\Lambda}_0^k$.

A word of caution should be added regarding the spaces Λ_*^k. If $\omega_1 \sim \tilde{\omega}_1$ and $\omega_2 \sim \tilde{\omega}_2$ are equivalent forms, it does not follow that $\omega_1 \wedge \omega_2$ and $\tilde{\omega}_1 \wedge \tilde{\omega}_2$ are equivalent. For instance, if $p = q = 1$, then $du_x \sim 0$ in Λ_*^1, whereas $du \wedge du_x$ is not equivalent to 0 in Λ_*^2. In other words there is no well-defined exterior product on the spaces Λ_*^k.

2. The inverse problem

Each sufficiently smooth extremal of the variational problem

$$I[u] = \int L(x, u^{(n)})\, dx,$$

with Lagrangian L satisfies the well-known Euler–Lagrange equations $E(L) = 0$, where $E = (E_1, \ldots, E_q)^{\mathrm{T}} : \mathscr{A} \to \mathscr{A}^q$ is the Euler operator, or variational derivative. Here

$$E_j = \sum_J (-D)^J \frac{\partial}{\partial u_J^j}. \tag{2·1}$$

The inverse problem of the calculus of variations is to characterize those systems of p.d.e.'s which arise as the Euler–Lagrange equations of some variational problem. Here we apply the differential complex (1·9) to give a new solution to the problem.

Lemma 2·1. *Every differential form in $\underline{\Lambda}_0^1$ is equivalent to a unique form*

$$P.du = \sum_{i=1}^{q} P_i\, du^i \tag{2·2}$$

where $P = (P_1, \ldots, P_q)^{\mathrm{T}} \in \mathscr{A}^q$, and $du = (du^1, \ldots, du^q)^{\mathrm{T}}$.

The proof is a straightforward application of integration by parts. The form $P.du$ will be called the *standard representative* of an element in Λ_*^1.

Lemma 2·2. *If $L \in \mathscr{A}$, then the standard representative of $d_* L$ is $E(L).du$.*

Proof. Integration by parts shows that

$$d_u L = \Sigma \frac{\partial L}{\partial u_J^j} du_J^j \sim \Sigma (-D)^J \frac{\partial L}{\partial u_J^j} du^j = E(L).du,$$

which proves the lemma.

Theorem 2·3. *A system of q p.d.e.'s in q dependent variables, $P = (P_1, \ldots, P_q)^{\mathrm{T}} = 0$ are the Euler–Lagrange equations for some variational problem if and only if $d_*\,(P.du) = 0$ in Λ_*^2.*

The proof is an immediate application of Lemma 2·1 to the exact complex (1·9). Seen in this light, (1·9) provides a new 'resolution' of the Euler operator, where $d_*: \Lambda_*^0 \to \Lambda_*^1$ is essentially the same as the Euler operator. This resolution is different from those appearing in Kuperschmidt (11), and Manin (12), where new independent variables are appended, Olver and Shakiban (18), (22) which is algebraic, and Vinogradov (29, 30), extending a theorem of Vainberg on potential operators to a full resolution.

As an example, consider the system

$$v_{xy} + uv_y = 0,$$
$$u_{xy} - uu_y = 0.$$

We must check the d_*-closure of the form

$$\omega = (v_{xy} + uv_y)\,du + (u_{xy} - uu_y)\,dv.$$

Now

$$\begin{aligned} d_*\omega &= dv_{xy} \wedge du + u\,dv_y \wedge du + du_{xy} \wedge dv - u\,du_y \wedge dv - u_y\,du \wedge dv \\ &\sim dv \wedge du_{xy} - u_y\,dv \wedge du - u\,dv \wedge du_y + du_{xy} \wedge dv - u\,du_y \wedge dv - u_y\,du \wedge dv \\ &= 0, \end{aligned}$$

so ω is d_*-exact. Indeed,

$$\omega \sim d_*(-u_x u_y + \tfrac{1}{2}u^2 v_y) = d_* L,$$

and L is the Lagrangian for the variational problem for which these are the Euler–Lagrange equations.

We proceed to analyse the operator $d_*: \bigwedge_*^1 \to \bigwedge_*^2$.

THEOREM 2·4. *Suppose $\omega = P.\,du$, as in* (2·2). *Then $d_*\omega \sim 0$ if and only if the $q \times q$ matrix differential operator $\mathscr{D}$ with entries*

$$\mathscr{D}_{ij} = \sum_J \frac{\partial P_i}{\partial u_J^j} D^J, \quad i,j = 1,\dots,q \tag{2·3}$$

is formally self-adjoint. In other words, $\mathscr{D}^* = \mathscr{D}$, where the (i,j)-th entry of $\mathscr{D}^*$ is the L^2-adjoint of the (j,i)-th entry of $\mathscr{D}$.

Proof. Note that

$$d_*\omega = -du^{\mathrm{T}} \wedge \mathscr{D}\,du.$$

Integration by parts shows that

$$du^{\mathrm{T}} \wedge \mathscr{D}\,du = -du^{\mathrm{T}} \wedge \mathscr{D}^*\,du,$$

hence, if $\mathscr{D}$ is self-adjoint, then $d_*\omega$ is zero. The proof of the converse is left until section 4.

This theorem, combined with Theorem 2·3, gives the following formal analog of Vainberg's theorem ((28), theorem 5·1) that an operator on a Banach space is the gradient of some potential operator if and only if its derivative is a symmetric operator.

COROLLARY 2·5. *The system $P = 0$ forms the Euler–Lagrange equations of some variational principle if and only if the operator $\mathscr{D}$ defined by* (2·3) *is self-adjoint.*

In light of the above considerations, it is of use to have an explicit, easily verifiable criterion for knowing when a given form ω is equivalent to 0 in $\bigwedge_*^k$. We conclude this section by describing one such criterion due to Gel'fand and Dikiĭ (6). The vector fields $\partial/\partial u_J^i$ and D_j all act on $\underline{\bigwedge}_0^k$ as Lie derivatives, so formula (2·1) for the Euler operator also defines an operator $E: \underline{\bigwedge}_0^k \to \underline{\bigwedge}_0^k$.

THEOREM 2·6. *Let $\omega \in \underline{\bigwedge}_0^k$ be a form. Then $\omega \sim 0$ in $\bigwedge_*^k$ if and only if $E(\omega) = 0$.*

3. INVARIANT INTEGRALS

Let $\mathscr{H}$ be the Hilbert space of L^2 functions $f: \mathbf{R}^p \to \mathbf{R}^q$ or the space of periodic L^2 functions f. Let $\mathscr{S} \subset \mathscr{H}$ be the Schwartz space of C^∞ functions rapidly decreasing for large $|x|$, or the space of periodic C^∞ functions. (Certain of the following results hold for more general $\mathscr{H}$ and $\mathscr{S}$.) Suppose $S \subset \mathscr{S}$ is a smoothly embedded k-dimensional manifold. A local coordinate patch S_λ on S is described by a smoothly parametrized family of functions $u(x, \lambda)$, where $x \in \mathbf{R}^p$, and $\lambda \in \mathbf{R}^k$. Given an n-form $\omega \in \wedge_0^n$, there is an induced n-form on S, denoted by $\omega|S \in \wedge^n T^*S$. In local coordinates, if

$$\omega = P\, du_J^j \wedge \ldots \wedge du_K^k,$$

then

$$(\omega|S)(\lambda) = \sum_I \left[\int P(u(x,\lambda)) \frac{\partial(u_J^j(x,\lambda), \ldots, u_K^k(x,\lambda))}{\partial(\lambda_{i_1}, \ldots, \lambda_{i_n})} dx\right] d\lambda_{i_1} \wedge \ldots \wedge d\lambda_{i_n}. \tag{3·1}$$

(Integration takes place over $\mathbf{R}^p$ or a fundamental period.) For instance, if $p = q = 1$ and $\omega = u_x du$, then

$$\omega|S = \sum_{i=1}^{k} \left[\int \frac{\partial u}{\partial x}\frac{\partial u}{\partial \lambda_i} dx\right] d\lambda_i.$$

It is always assumed that the integrals in (3·1) converge for each λ. Note that integration by parts shows that, if ω and $\tilde{\omega}$ are equivalent forms, then $\omega|S = \tilde{\omega}|S$, hence the restricted form $\omega|S$ only depends on the equivalence class of ω in $\wedge_*^n$.

If S is oriented, n-dimensional, and $\omega \in \wedge_*^n$, then, assuming convergence, we can integrate ω over S:

$$\int_S \omega = \int_S \omega|S,$$

the latter integral being with respect to the volume form induced from $\mathscr{H}$. The generalization of Stokes' theorem is immediate:

THEOREM 3·1. *If S is a smooth n-dimensional manifold of functions with smooth boundary ∂S, and $\omega \in \wedge_*^{n-1}$, then*

$$\int_S d\omega = \int_{\partial S} \omega.$$

The proof is based on the elementary formula

$$(d\omega)\,|\,S = d(\omega|S). \tag{3·2}$$

We also note the following straightforward result.

LEMMA 3·2. *If $\int_S \omega = 0$ for all oriented manifolds S with boundary (of the appropriate dimension), then $\omega \sim 0$ in $\wedge_*^k$.*

Now consider a general evolution equation

$$u_t = K(x, u^{(n)}), \tag{3·3}$$

where $K = (K_1, \ldots, K_q)^{\mathrm{T}} \in \mathscr{A}^q$. It will be assumed that (3·3) is locally uniquely soluble in $\mathscr{S}$ for initial data $u(x, 0) = f(x) \in \mathscr{S}$. Thus (3·3) defines a *flow* $u(x,t) = \mathscr{K}_t[f(x)]$, $t > 0$, where $\mathscr{K}_t: \mathscr{S} \to \mathscr{S}$ forms a local one-parameter semi-group. The special vector field $\mathbf{v}_K$ is the 'infinitesimal generator' of this semi-group, so we may write $\mathscr{K}_t = \exp(t\mathbf{v}_K)$. Given a compact, oriented manifold $S_0 \subset \mathscr{S}$, let $S_t = \mathscr{K}_t(S_0)$, the image of S_0 under the flow. For t sufficiently small, S_t is again compact, oriented.

Suppose $\omega \in \wedge_*^n$ is an n-form. If

$$\int_{S_t} \omega = \int_{S_0} \omega \tag{3·4}$$

for all compact, oriented, n-dimensional submanifolds S_0 and all sufficiently small t, then ω is called an *absolute integral invariant* of the evolution equation (3·3). If (3·4) holds only for closed S_0 without boundary, then ω is called a *relative integral invariant*. In finite-dimensional systems, the notion of an integral invariant dates back to Poincaré(20). See also Cartan(3) for a comprehensive introduction. Integral invariants can be viewed as generalized circulations, although the integration takes place in function space. Stokes' theorem implies:

LEMMA 3·3. *A form ω is a relative integral invariant if and only if $d_* \omega$ is an absolute integral invariant.*

THEOREM 3·4. *A form $\omega \in \bigwedge_*^n$ is an absolute integral invariant of the evolution equation $u_t = K$ if and only if $\mathbf{v}_K(\omega) \sim 0$ in $\bigwedge_*^n$. Also, ω is a relative integral invariant if and only if $\mathbf{v}_K(\omega) = d_* \mu$ for some $\mu \in \bigwedge_*^{n-1}$.*

This follows from the characterization of $\mathbf{v}_K$ as the Lie derivative of the flow induced by the evolution equation. Note that the Lie derivative $\mathbf{v}_K$ is well defined in $\bigwedge_*^n$ because the special vector fields are precisely those which commute with the total derivatives D_i.

More generally, consider a one-parameter family of forms, $\omega(t)$ depending smoothly on time t. The analogue of (3·4) is

$$\int_{S_t} \omega(t) = \int_{S_0} \omega(0), \qquad (3\cdot 4_t)$$

which defines relative and absolute time-dependent integral invariants. The criterion of Theorem 3·4 now reads that $\omega(t)$ is an absolute invariant if and only if $\partial_t \omega - \mathbf{v}_K(\omega) \sim 0$, where $\partial_t = \partial/\partial t$.

Given a vector field $\mathbf{v}$ and a form $\omega \in \bigwedge_0^n$, there is an induced form $\mathbf{v} \lrcorner \omega \in \bigwedge_0^{n-1}$, the *interior product* of $\mathbf{v}$ and ω, satisfying $\langle \mathbf{V}, \mathbf{v} \lrcorner \omega \rangle = \langle \mathbf{V} \wedge \mathbf{v}, \omega \rangle$ for all $\mathbf{V} \in \bigwedge_{m-1}$. The following formulae are proven exactly as their differential-geometric counterparts ((23); pp. 102–3).

LEMMA 3·5. *Let $\mathbf{v}$, $\mathbf{w}$ be vector fields, $\omega \in \bigwedge_0^n$, $\omega' \in \bigwedge_0^m$. Then*

(i) *The interior product $\mathbf{v} \lrcorner$ is an antiderivation:*

$$\mathbf{v} \lrcorner (\omega \wedge \omega') = (\mathbf{v} \lrcorner \omega) \wedge \omega + (-1)^n \omega \wedge (\mathbf{v} \lrcorner \omega'), \qquad (3\cdot 5)$$

(ii) $\mathbf{v}(\omega) = \mathbf{v} \lrcorner d\omega + d(\mathbf{v} \lrcorner \omega)$, (3·6)

(iii) $\mathbf{v}(\mathbf{w} \lrcorner \omega) = \mathbf{w} \lrcorner \mathbf{v}(\omega) + [\mathbf{v}, \mathbf{w}] \lrcorner \omega$. (3·7)

COROLLARY 3·6. *The interior product of special vector fields is well defined on $\bigwedge_*^n$. Moreover the analogues of* (3·6, 7) *hold:*

$$\mathbf{v}(\omega) = \mathbf{v} \lrcorner d_* \omega + d_*(\mathbf{v} \lrcorner \omega), \qquad (3\cdot 6_*)$$

$$\mathbf{v}(\mathbf{w} \lrcorner \omega) = \mathbf{w} \lrcorner \mathbf{v}(\omega) + [\mathbf{v}, \mathbf{w}] \lrcorner \omega, \qquad (3\cdot 7_*)$$

for $\mathbf{v}$, $\mathbf{w}$ special vector fields and $\omega \in \bigwedge_*^n$.

4. Hamiltonian structure

Let $\Omega \in \wedge^2_*$ be nondegenerate, d_*-closed. Integration by parts shows that Ω can be put into the standard form

$$\Omega = -\tfrac{1}{2} du^{\mathrm{T}} \wedge \mathscr{D}\, du, \tag{4·1}$$

where $\mathscr{D}$ is a nondegenerate skew-adjoint matrix of differential operators, uniquely determined by Ω. To see that $\mathscr{D}$ is unique, if $\Omega = du^{\mathrm{T}} \wedge \mathscr{D}\, du \sim 0$, then corollary 3·6 implies that, for any $K \in \mathscr{A}^q$, $0 \sim \mathbf{v}_K \lrcorner\, \Omega = (\mathscr{D} - \mathscr{D}^*) K . du$, hence $\mathscr{D}K = \mathscr{D}^*K$ for all K, so $\mathscr{D}$ must be self-adjoint. This, incidentally, completes the proof of Theorem 2·4. If $\mathbf{v}_K$ is a special vector field, then define the one-form

$$\omega_K = \mathbf{v}_K \lrcorner\, \Omega = \mathscr{D}(K) . du \in \wedge^1_*.$$

Conversely, if $\omega \sim P . du$ is a one-form, then since Ω is non-degenerate, for $P \in \operatorname{im} \mathscr{D}$, there is a uniquely defined special vector field $\mathbf{v}_\omega = \mathbf{v}_K$ such that $\omega = \mathbf{v}_\omega \lrcorner\, \Omega$; in fact $\mathscr{D}K = P$. We will often enlarge our class of such forms Ω to include cases when $\mathscr{D}$ is a skew-adjoint formal pseudo-differential operator in the sense of Gel'fand and Dikiĭ (8). Usually, either $\mathscr{D}$ or $\mathscr{D}^{-1}$ will be a genuine matrix differential operator.

Define the Ω-*Poisson bracket* of forms ω, $\omega' = \wedge^1_*$ as

$$\{\omega', \omega\} = [\mathbf{v}_{\omega'}, \mathbf{v}_\omega] \lrcorner\, \Omega. \tag{4·2}$$

If $\omega \sim \mathscr{D}P . du$, $\omega' \sim \mathscr{D}P' . du$, then $\{\omega', \omega\} \sim \mathscr{D}Q . du$, where $Q = \mathbf{v}_{P'}(P) - \mathbf{v}_P(P')$, the vector fields acting component-wise.

Lemma 4·1. *If ω and ω' are d_*-closed, then $\{\omega', \omega\}$ is d_*-exact.*

Proof. By ($3·6_*$), if ω is d_*-closed, then

$$\mathbf{v}_\omega(\Omega) = \mathbf{v}_\omega \lrcorner\, d_* \Omega + d_*(\mathbf{v}_\omega \lrcorner\, \Omega) = d_* \omega = 0$$

Therefore, by ($3·7_*$)

$$\begin{aligned} \mathbf{v}_\omega(\omega') &= \mathbf{v}_\omega(\mathbf{v}_{\omega'} \lrcorner\, \Omega) \\ &= [\mathbf{v}_\omega, \mathbf{v}_{\omega'}] \lrcorner\, \Omega + \mathbf{v}_{\omega'} \lrcorner\, \mathbf{v}_\omega(\Omega) \\ &= -\{\omega', \omega\}. \end{aligned}$$

On the other hand, since ω' is closed, $\mathbf{v}_\omega(\omega') = d_*(\mathbf{v}_\omega \lrcorner\, \omega')$, completing the proof.

In the special case $\omega = d_* P$, $\omega' = d_* P'$ for $P, P' \in \mathscr{A}$,

$$\{\omega', \omega\} = \{d_* P', d_* P\} = \mathbf{v}_\omega(d_* P') = d_*[\mathbf{v}_\omega(P')].$$

Therefore we can define the Ω-Poisson bracket of P and P' by

$$\{P', P\} = \mathbf{v}_\omega(P') = -\mathbf{v}_{\omega'}(P), \quad \omega = d_* P, \quad \omega' = d_* P'. \tag{4·3}$$

Thus

$$\{d_* P', d_* P\} = d_*\{P', P\}, \tag{4·4}$$

where we have used the exactness of (1·9) in (4·3). Furthermore, since $d_* P \sim E(P) . du$, $\mathbf{v}_\omega = \mathbf{v}_K$ where $K = \mathscr{D}^{-1}E(P)$. Thus

$$\{P', P\} = E(P)^{\mathrm{T}} \mathscr{D}^{-1} E(P'), \tag{4·5}$$

since, for any $K \in \mathscr{A}^q, P \in \mathscr{A}$,

$$\mathbf{v}_K(P) - K.E(P) \in \operatorname{im}\operatorname{Div}.$$

Equation (4·4), when written out in full detail, yields the following generalization of a result of Gardner (5) and Gel'fand–Dikiĭ (8,9) on the relationship between the formal Poisson and Lie brackets:

THEOREM 4·2. *Suppose $\mathscr{D}$ is a non-degenerate skew-adjoint matrix of (pseudo)-differential operators such that the two form $\Omega = -\frac{1}{2}du^{\mathrm{T}} \wedge \mathscr{D}^{-1}du$ is d_*-closed. For any q-tuples of functions Q, $Q' \in \operatorname{im} E$,*

$$\mathbf{v}_{P'}(P) - \mathbf{v}_P(P') = \mathscr{D}E(Q^{\mathrm{T}}\mathscr{D}Q'), \quad P = \mathscr{D}Q, \quad P' = \mathscr{D}Q'. \tag{4·6}$$

For a counter-example to (4·6) when Ω does not happen to be closed, let $p = q = 1$ and consider the skew-adjoint operator $\mathscr{D} = 2u_{xx}D_x + u_{xxx}$. Let $Q = u$, $Q' = u^2$, which are certainly in the image of E. Then

$$E(Q\mathscr{D}Q') = 6uu_x u_{xx} + 2u_x^3.$$

It is easy to see that the right-hand side of (4·6) contains the monomial $2u_x^3 u_{xxx}$; however, inspection of the left-hand side shows that this monomial occurs with the coefficient 20, so (4·6) cannot hold.

DEFINITION 4·3. *A quasi-Hamiltonian system* is an evolution equation of the form

$$u_t = \mathscr{D}E(H), \tag{4·7}$$

where $\mathscr{D}$ is a skew-adjoint matrix (pseudo-)differential operator and $H \in \mathscr{A}$ is the Hamiltonian. If the associated two-form $\Omega = -\frac{1}{2}du^{\mathrm{T}} \wedge \mathscr{D}^{-1}du$ is d_*-closed, then (4·7) will be called a *Hamiltonian system* and Ω the *fundamental two-form*.

THEOREM 4·4. *For a Hamiltonian system, the fundamental two-form is an absolute integral invariant. Conversely, given an evolution equation with a non-degenerate closed two-form for an absolute integral invariant, then the equation is a Hamiltonian system.*

Proof. Since $d_* \Omega = 0$, by ($3·6_*$),

$$\mathbf{v}_K(\Omega) = d_*(\mathbf{v}_K \lrcorner \Omega) = d_*(E(H).du) = 0,$$

where $K = \mathscr{D}E(H)$. This proves the first statement. Conversely, if the invariant of $u_t = K$ is in the standard form $\Omega = -\frac{1}{2}du^{\mathrm{T}} \wedge \mathscr{D}^{-1}du$, with $\mathscr{D}$ skew adjoint, then $\mathbf{v}_K \lrcorner \Omega = \mathscr{D}^{-1}K.du$ must be d_*-closed. By the exactness of (1·9), $\mathscr{D}^{-1}K = E(H)$ for some $H \in \mathscr{A}$.

In Manin's treatise (12) equation (4·6) is taken as the *definition* of a Hamiltonian operator $\mathscr{D}$. See also Vinogradov (31) for a coordinate-free version. Manin also gives a rather cumbersome criterion – his theorem I. 7. 13 – for checking whether a given operator is Hamiltonian. The present definition depending only on the d_*-closure of the fundamental two-form is a much more natural generalization of differential-geometric theory of finite-dimensional Hamiltonian mechanics and the fundamental forms to the evolution equations under consideration; see Sternberg (23) for the finite-dimensional situation. Also, we have an immediate proof of the important result that any skew-adjoint matrix of linear, constant coefficient differential operators is Hamiltonian; no further calculations are necessary.

The objection could be raised that since, in practice, $\mathscr{D}$ is usually a skew-adjoint matrix of genuine differential operators, checking the d_* closure of the form

$$du^{\mathrm{T}} \wedge \mathscr{D}^{-1} du \quad .$$

is not easy, and might in fact be just as cumbersome as Manin's criterion. We therefore provide an easily verifiable criterion for the operator $\mathscr{D}$ to be Hamiltonian, based on the closure of the *associated two-form* $\tilde{\Omega} = \frac{1}{2} du^{\mathrm{T}} \wedge \mathscr{D} du$ with respect to a suitably modified exterior derivative.

LEMMA 4·5. *Let* $\mathscr{D} = (\mathscr{D}_{ij})$ *be a skew-adjoint matrix of differential operators. Define the modified exterior derivative* $d_{\mathscr{D}}: \underline{\wedge}_0^k \to \underline{\wedge}_0^{k+1}$ *by the following properties:*

$$d_{\mathscr{D}}.d_u = 0, \quad d_{\mathscr{D}}.D_i = D_i.d_{\mathscr{D}},$$
$$d_{\mathscr{D}}(\omega \wedge \tilde{\omega}) = d_{\mathscr{D}}(\omega) \wedge \tilde{\omega} + (-1)^n \omega \wedge d_{\mathscr{D}}(\tilde{\omega}), \quad n = \deg \omega,$$
$$d_{\mathscr{D}}(u^i) = \sum_j \mathscr{D}_{ij}(du^j).$$

Then $\mathscr{D}$ *is Hamiltonian, meaning that the two-form* $\Omega = -\frac{1}{2} du^{\mathrm{T}} \wedge \mathscr{D}^{-1} du$ *is* d_**-closed, if and only if* $d_{\mathscr{D}}(\tilde{\Omega}) \sim 0$ *in* $\wedge_*^3$, *where* $\tilde{\Omega} = \frac{1}{2} du^{\mathrm{T}} \wedge \mathscr{D} du$.

Proof. Define a map $F: \underline{\wedge}_0^k \to \underline{\wedge}_0^k$,

$$F(P) = P, \quad P \in \mathscr{A}, \quad F(du^i) = d_{\mathscr{D}}(u^i),$$
$$F(\omega \wedge \tilde{\omega}) = F(\omega) \wedge F(\tilde{\omega}), \quad F(D_i \omega) = D_i F(\omega).$$

F is then well-defined, and, moreover, $F(\Omega) = \tilde{\Omega}$. From the last formula, $\omega \sim 0$ if and only if $F(\omega) \sim 0$, so F also defines a map on $\wedge_*^k$. Furthermore, using the definitions of $d_{\mathscr{D}}$ and F, for any $P \in \mathscr{A}$, $F(d_u P) = d_{\mathscr{D}} P$. It is *not* true that $F(d_u \omega) = d_{\mathscr{D}} F(\omega)$ for arbitrary forms ω, since in particular $d_{\mathscr{D}}.d_{\mathscr{D}} \neq 0$. However, we claim that

$$F(d_* \Omega) \sim d_{\mathscr{D}} \tilde{\Omega}$$

in $\wedge_*^3$, from which the lemma follows. To verify the claim, we use the usual formula for the differential of an inverse matrix function to compute

$$d_* \Omega = \tfrac{1}{2}(\mathscr{D}^{-1} du)^{\mathrm{T}} \wedge d\mathscr{D} \wedge \mathscr{D}^{-1} du.$$

Here, the notation $d\mathscr{D}$ stands for the matrix of one-form operators induced by the differential d_u acting on the coefficients of $\mathscr{D}$. For instance, if $\mathscr{D}$ is the scalar operator $D_x^3 + 2u_x D_x + u_{xx}$, then $d\mathscr{D}$ is an operator whose action on any form ω is given by $d\mathscr{D} \wedge \omega = 2du_x \wedge D_x(\omega) + du_{xx} \wedge \omega$. Applying F to the above expression yields

$$F(d_* \Omega) = \tfrac{1}{2} du^{\mathrm{T}} \wedge d_{\mathscr{D}} \mathscr{D} \wedge du = d_{\mathscr{D}} \tilde{\Omega},$$

with obvious notation. This completes the proof.

An elementary illustration of this result is provided by the non-Hamiltonian operator $\mathscr{D} = 2u_{xx} D_x + u_{xxx}$ considered previously. The associated two-form is

$$\tilde{\Omega} = \tfrac{1}{2} du \wedge (2u_{xx} du_x + u_{xxx} du) \sim u_{xx} du \wedge du_x.$$

Then
$$d_{\mathscr{D}}\tilde{\Omega} = D_x^2(2u_{xx}du_x + u_{xxx}du) \wedge du \wedge du_x$$
$$= (5u_{xxx}du_{xx} + 2u_{xx}du_{xxx}) \wedge du \wedge du_x$$
$$\sim 3u_{xxx}du \wedge du_x \wedge du_{xx}.$$

Theorem 2·6 then shows that this form is not equivalent to 0 in $\wedge^3_*$, hence $\mathscr{D}$ is not a Hamiltonian operator, as we had previously established.

For a less trivial application, we look at some operators arising in the work of Gel'fand and Dikiĭ (8). Consider the $(n-1)\times(n-1)$ matrix of differential operators with entries
$$\mathscr{D}_{ij} = \sum_{k=0}^{n-i-j-1} \binom{k+i}{i} u^{i+j+k+1}D^k - \binom{k+j}{j}(-D)^k u^{i+j+k+1},$$
with $D = D_x$, so $p = 1$. (Here i, j run from 0 to $n-2$.)

THEOREM 4·6. *If $\mathscr{D}$ is the operator with entries $\mathscr{D}_{ij}$ as above, then $\Omega = -\frac{1}{2}du^{\mathrm{T}} \wedge \mathscr{D}^{-1}du$ is d_*-closed.*

Proof. Here
$$\tilde{\Omega} = -\sum_{i,j=0}^{n-2}\sum_{k=0}^{n-i-j-1}\binom{k+i}{i}u^{i+j+k+1}du^i \wedge du^j_k.$$
Then
$$d_{\mathscr{D}}(\tilde{\Omega}) = \sum_{i,j,k}\binom{k+i}{i}\sum_{l=0}^{n-2}\mathscr{D}_{i+j+k+1,l}\,du^l \wedge du^i \wedge du^j_k$$
$$= \sum_{i,j,k,l,m}\binom{k+i}{i}u^{i+j+k+l+m+1}\left\{\binom{i+j+k+m+1}{m}du^l_m \wedge du^i \wedge du^j_k - \sum_{p=0}^{m}\binom{l+m}{m}\binom{m}{p}du^l \wedge du^i_p \wedge du^j_{k+m-p}\right\}.$$
The coefficient of $u^{i+j+k+l+m+1}du^l_m \wedge du^i \wedge du^j_k$ is
$$\binom{k+i}{i}\binom{i+j+k+m+1}{m} - \binom{m+i}{i}\binom{i+l+m+k+1}{k} +$$
$$+\sum_{q=0}^{l}\left\{\binom{q+l}{l}\binom{m+k+i-q}{i}\binom{n+k-q}{m} - \binom{q+j}{j}\binom{i+k+m-q}{i}\binom{k+m-q}{k}\right\},$$
which is zero. This is true because
$$\binom{m+k+i-q}{i}\binom{m+k-q}{m} = \binom{m+k+i-q}{m+i}\binom{m+i}{i},$$
so the first sum is
$$\binom{m+i}{i}\sum_{q=0}^{l}\binom{m+k+i-q}{m+i}\binom{q+l}{l} = \binom{m+i}{i}\binom{i+l+m+k+1}{k},$$
by a standard binomial identity. Therefore Lemma 4·5 implies Theorem 4·6.

As a direct consequence, formula (4·6) holds for the operator $\mathscr{D}$, which is Gel'fand and Dikiĭ's main Lemma. An intriguing question is why these particular operators should arise from inverse scattering. Are there corresponding completely integrable

Hamiltonian systems corresponding to other skew-adjoint operators $\mathscr{D}$ such that the corresponding two-form Ω is closed, and, if so, what do they look like?

One final remark on this subject is that, by virtue of Lemma 4·5, we can completely dispense with the use of the inverse differential operator $\mathscr{D}^{-1}$ in our original definition of Hamiltonian operators and systems, and work exclusively with the associated two-form rather than the fundamental two-form. This is vital in a fully rigorous treatment of the case of more than one independent variable, since there is as yet no commonly accepted rigorous definition of the inverse of a matrix of partial differential operators. We leave it to the interested reader to fill in any missing details in this alternative approach which has the advantage of full rigour, but suffers from a corresponding lack of intuition.

5. Noether's theorem

A *conservation law* of an evolution equation is a p.d.e. of the special form

$$D_t T + \operatorname{Div} X = 0, \tag{5·1}$$

satisfied for all solutions of the evolution equation. Here $T \in \mathscr{A}$ is the *conserved density* and $X \in \mathscr{A}^q$ the associated *fluxes.* Equivalently, the quantity $\int T\,dx$ is independent of time t for solutions such that the integral converges. This in turn means that the 0-form $T \in \wedge^0_*$ is a relative integral invariant of the equation since

$$\int T(u(t))\,dx - \int T(\tilde{u}(t))\,dx = \int T(u(0))\,dx - \int T(\tilde{u}(0))\,dx$$

for any pair of solutions $u, \tilde{u}$. Conversely, any relative integral invariant $T \in \wedge^0_*$ such that $\int T(f)\,dx = 0$ for at least one function f in $\mathscr{S}$ gives rise to a conservation law of the usual type.

Noether's theorem (14) relates the symmetries of a Hamiltonian system to its conservation laws. Recall, (15), (17), that a (generalized) one-parameter symmetry group of $u_t = K$ is given by the flow of a second evolution equation $u_t = P$ which commutes with the flow of the first equation. The corresponding infinitesimal criterion is that the two associated vector fields commute:

$$[\mathbf{v}_K, \mathbf{v}_P] = 0. \tag{5·2}$$

Lemma 5·1. *Suppose Ω is an absolute integral invariant of the evolution equation $u_t = K$. If $u_t = P$ is a symmetry, then the form $\mathbf{v}_P \lrcorner\, \Omega$ is also an absolute integral invariant.*

Proof. By (3·7$_*$), (5·2),

$$\mathbf{v}_K(\mathbf{v}_P \lrcorner\, \Omega) = \mathbf{v}_P \lrcorner\, \mathbf{v}_K(\Omega) + [\mathbf{v}_K, \mathbf{v}_P] \lrcorner\, \Omega = 0,$$

proving the lemma.

We relate the above result to the more usual point transformational symmetry groups of the Lie–Ovsjannikov theory (19). If G is a one-parameter local group of transformations acting on $X \times U$ with infinitesimal generator

$$\mathbf{v} = \sum_{i=1}^{p} \xi^i(x,u)\frac{\partial}{\partial x^i} + \sum_{j=1}^{q} \phi^j(x,u)\frac{\partial}{\partial u^j}, \tag{5·3}$$

then the corresponding standard vector field is $\mathbf{v}_P$, with

$$P_j = \phi^j - \sum_{i=1}^{p} \xi^i u_i^j, \tag{5·4}$$

where $u_i^j = \partial u^j / \partial x^i$. It is easily checked that if G is a symmetry group, then so is the group generated by $\mathbf{v}_P$.

Given a Hamiltonian system, its fundamental two-form is an absolute integral invariant, so every one-parameter symmetry group gives rise to an absolutely invariant one-form. If, moreover, this one-form is d_*-closed, then (1·9) and Lemma 3·3 show that there is a corresponding relative integral invariant 0-form, which, by the previous remarks, is just a classical conservation law. This is the essence of Noether's theorem. Note that whereas not every symmetry group gives rise to a conserved quantity, since the d_*-closure condition must be satisfied, there is always a corresponding conserved one-form. This resolves the observation on the lack of one-to-one correspondence between symmetries and conservation laws.

THEOREM 5·2. (*Generalized Noether's Theorem*). *Suppose* $u_t = \mathscr{D}E(H)$ *is a Hamiltonian system with fundamental two-form* $\Omega = -\frac{1}{2} du^{\mathrm{T}} \wedge \mathscr{D}^{-1} du$. *If* $u_t = P$ *is a one-parameter symmetry group, then the one-form* $\omega_P = \mathbf{v}_P \lrcorner \Omega = \mathscr{D}^{-1} P . du$ *is an absolute integral invariant. Conversely, given an absolute invariant one-form* $\omega = Q.du$, *the evolution equation* $u_t = \mathscr{D}Q$ *is a symmetry. Moreover,* $\omega_p = d_* T$ *for some conserved density* T *if and only if* $P = \mathscr{D}E(T)$, i.e. Ω *must also be an invariant of* $u_t = P$.

More generally, we can allow a time-dependent flow, $u_t = P(t)$, which is a symmetry of $u_t = K$ if it preserves the solution set. The analogous infinitesimal criterion is

$$[\mathbf{v}_K, \mathbf{v}_P] = \mathbf{v}_{P_t}. \tag{5·2$_t$}$$

The analogues of Lemma 5·1 and Theorem 5·2 now hold with no change in the statements, although the proofs must be slightly modified. This we leave to the reader.

6. SOME APPLICATIONS

(A) *The Korteweg–de Vries equation.* Consider the Korteweg–de Vries equation

$$u_t = u_{xxx} + uu_x.$$

This can be written in Hamiltonian form $u_t = DE(H)$, with $H = \int (\frac{1}{6}u^3 - \frac{1}{2}u_x^2)\, dx$. (Here $D = D_x$.) Thus the fundamental form associated with the KdV equation is $\Omega = -\frac{1}{2} du \wedge D^{-1} du$, which is of course d_*-closed. Then theorem 5·2 implies that if $\mathbf{v}_P$ is the infinitesimal generator of a one-parameter symmetry group of the KdV equation, then $\mathbf{v}_P \lrcorner \Omega = D^{-1}P.du$ is an absolute integral invariant. Moreover, if $P = E(T)$ for some T, then $D^{-1}P.du = d_* T$, so T is a conserved density for the KdV equation.

The connection between the higher-order analogues of the KdV equation and its infinite family of conservation laws is well known (17). Here we instead consider other symmetries of the KdV equation and derive a new family of absolute integral invariants which are not conservation laws of the usual kind. The point-transformational symmetry group of the KdV equation is four-parameter, generated by the vector fields

$$\tilde{\mathbf{v}}_1 = \partial_x,$$
$$\tilde{\mathbf{v}}_2 = \partial_t,$$
$$\tilde{\mathbf{v}}_3 = t\,\partial_x - \partial_u,$$
$$\tilde{\mathbf{v}}_4 = x\,\partial_x + 3t\,\partial_t - 2u\,\partial_u.$$

These represent invariance under space translations, time translations, Galilean boosts and scale transformations respectively. In order to apply our Hamiltonian formalism, we must first put these vector fields into standard form, namely $\mathbf{v}_{P_j}$, where

$$P_1 = u_x,$$
$$P_2 = u_t = u_{xxx} + uu_x$$
$$P_3 = 1 - tu_x,$$
$$P_4 = 2u + xu_x + 3tu_t = 2u + xu_x + 3t(u_{xxx} + uu_x).$$

The corresponding invariants are

$$\omega_1 = u\,du = d_*(\tfrac{1}{2}u^2),$$
$$\omega_2 = (u_{xx} + \tfrac{1}{2}u^2)\,du = d_*(-\tfrac{1}{2}u_x^2 + \tfrac{1}{6}u^3),$$
$$\omega_3 = (x + tu)\,du = d_*(xu + \tfrac{1}{2}tu^2),$$
$$\omega_4 = (xu + D^{-1}u + 3t(u_{xx} + \tfrac{1}{2}u^2))\,du$$
$$= d_*(\tfrac{1}{2}xu^2 + 3t(-\tfrac{1}{2}u_x^2 + \tfrac{1}{6}u^3)) + D^{-1}u\,du.$$

Also $\omega_0 = du$ is conserved. (This reflects the arbitrary constant in D^{-1}.) Now $\omega_0, \omega_1, \omega_2$ give rise to the first three conserved densities of the traditional hierarchy: u, $\tfrac{1}{2}u^2$, $-\tfrac{1}{2}u_x^2 + \tfrac{1}{6}u^3$. The conserved density $xu - \tfrac{1}{2}tu^2$ corresponding to ω_3 yields the anomalous conservation law found by Miura, Gardner and Kruskal (13). Note that, since $\tfrac{1}{2}u^2$ is a conserved density, this shows that $\int xu\,dx = \alpha t + \beta$ for constants α and β. Finally the form ω_4 is not closed. The fact that ω_4 is an absolute invariant amounts to the following property: if $u(x, t, \lambda)$, $0 \leqslant \lambda \leqslant 1$, is any one-parameter family of solutions of the KdV equation, then

$$\int_0^1 \int_{-\infty}^{\infty} \int_{-\infty}^{x} u\,d\tilde{x}.\frac{\partial u}{\partial \lambda}\,dx\,d\lambda + \int_{-\infty}^{\infty} \tfrac{1}{2}x[u^2(x,t,\lambda)]\Big|_{\lambda=0}^{\lambda=1} dx = \alpha' t + \beta'$$

for constants α', β', where $\alpha' = \int -\tfrac{1}{2}u_x^2 + \tfrac{1}{6}u^3\,dx$. Note that ω_4 contains the term $\omega = D^{-1}u\,du$, and $d_*\,\omega = -2\Omega$. By lemma 3·3 ω is already a relative integral invariant, so ω_4 can be considered as a 'completion' of ω to an absolute invariant.

Further, more complicated integral invariants can be obtained by use of the recursion operator for the KdV equation. Recall, (15), that, if $\mathbf{v}_P$ is a symmetry of the KdV equation, so is $\mathbf{v}_{P'}$ where $P' = \mathscr{D}P$, and

$$\mathscr{D} = D^2 + \tfrac{2}{3}u + \tfrac{1}{3}u_x D^{-1}.$$

Successive applications of the recursion operator $\mathscr{D}$ to the symmetries $\mathbf{v}_1$, $\mathbf{v}_2$ just gives the usual hierarchy of KdV-type equations. Here we apply $\mathscr{D}$ to the other point-transformational symmetries. Now

$$\mathscr{D}(1+tu_x) = tu_{xxx}+\tfrac{2}{3}u+\tfrac{2}{3}tuu_x+\tfrac{1}{3}xu_x+\tfrac{1}{3}tuu_x = \tfrac{1}{3}P_4,$$

so we just recover the symmetry $\mathbf{v}_4$. Applying $\mathscr{D}$ to P_4 gives

$$\mathscr{D}P_4 = 5u_{xx}+xu_{xxx}+\tfrac{4}{3}u^2+xuu_x+\tfrac{1}{3}u_x D_x^{-1}u+3tK_2,$$

where $K_2 = u_{xxxxx}+\tfrac{5}{3}uu_{xxx}+\tfrac{10}{3}u_x u_{xx}+\tfrac{5}{6}u^2u_x$, which is the next polynomial in the usual KdV hierarchy. The corresponding absolute integral invariant is

$$\begin{aligned}\omega_5 &= (4u_x+xu_{xx}+\tfrac{1}{2}xu^2+\tfrac{1}{3}uD^{-1}u+\tfrac{1}{4}D^{-1}(u^2)+3tD^{-1}K_2)\,du\\ &= d_*[x(\tfrac{1}{2}u_x^2+\tfrac{1}{6}u^3)+\tfrac{1}{6}u^2D^{-1}u+3tT_2]+(3u_x+\tfrac{5}{12}D^{-1}(u^2))\,du;\end{aligned}$$

where $T_2 = \tfrac{1}{2}u_{xx}^2-\tfrac{5}{6}uu_x^2+\tfrac{5}{72}u^4$ is the next conserved density in the usual hierarchy. Therefore, if $u(x,t,\lambda)$, $0 \leqslant \lambda \leqslant 1$, is a one-parameter family of solutions of the KdV equation,

$$\int_0^1\int_{-\infty}^{\infty}\left(3u_x+\frac{5}{12}\int_{-\infty}^{x}u^2\,d\tilde{x}\right)\frac{\partial u}{\partial\lambda}\,dx\,d\lambda+\int_{-\infty}^{\infty}\left\{(-\tfrac{1}{2}u_x^2+\tfrac{1}{6}u^3+\tfrac{1}{6}u^2)\int_{-\infty}^{x}u\,d\tilde{x}\right\}\Bigg|_{\lambda=0}^{1}dx = \alpha''t+\beta'',$$

for constants α'', β''. Further applications of $\mathscr{D}$ to the polynomial P_3 gives a new hierarchy of generalized symmetries $\mathbf{v}_{P_k}$, with $P_{k+3} = \mathscr{D}^k(1+tu_x)$, and thus a hierarchy of absolute integral invariants. I conjecture that none of these new invariants are d_*-exact, and hence do not give rise to conservation laws of the classical type. Applications of these invariants will be considered elsewhere.

(B) *The BBM equation.* The equation

$$u_t-u_{xxt} = uu_x+u_x$$

was proposed by Benjamin, Bona and Mahoney (2) as an alternative equation for describing long waves in shallow water. This can be put into Hamiltonian form $u_t = \mathscr{D}E(H)$, with $\mathscr{D} = (1-D^2)^{-1}D$ and

$$H = \int_{-\infty}^{\infty}(\tfrac{1}{6}u^3+\tfrac{1}{2}u^2)\,dx.$$

The fundamental two-form is then

$$\begin{aligned}\Omega &= -\tfrac{1}{2}du\wedge(1-D^2)D^{-1}du\\ &= \tfrac{1}{2}(du\wedge du_x-du\wedge D^{-1}du).\end{aligned}$$

Now if $\mathbf{v}_R$ is the standard form of a symmetry, then the corresponding invariant one-form is $\omega_P = \mathbf{v}_P \lrcorner\, \Omega = (D^{-1}P-DP)du$. It is easily shown that the symmetry group of the BBM equation has the three generators

$$\partial_x,\quad \partial_t,\quad t\partial_t-(u+1)\partial_u,$$

which have standard representatives (up to sign)

$$u_x \partial_u,$$
$$u_t \partial_u = (1-D^2)^{-1}(uu_x+u_x)\partial_u,$$
$$(u+1+tu_t)\partial_u = [u+1+t(1-D^2)^{-1}(uu_x+u_x)]\partial_u.$$

The corresponding absolute integral invariants are

$$\omega_1 = (u-u_{xx})\,du = d(\tfrac{1}{2}u^2+\tfrac{1}{2}u_x^2),$$
$$\omega_2 = (\tfrac{1}{2}u^2+u)\,du = d(\tfrac{1}{6}u^3+\tfrac{1}{2}u^2),$$
$$\omega_3 = [D^{-1}u+x-u_x+t(\tfrac{1}{2}u^2+u)]\,du = (D^{-1}u-u_x)\,du+d(xu)+t\omega_2.$$

Also $\omega_0 = du$ is an invariant. Now ω_0, ω_1, ω_2 are just the usual three conserved quantities of the BBM equation, and it can be shown that there are only these three quantities (16). Invariance of ω_3 implies that if $u(x,t,\lambda)$ is a one-parameter family of solutions of the BBM equation, then

$$\int_0^1\int_{-\infty}^{\infty}\left(\int_{-\infty}^{x} u\,d\tilde{x}+u_x\right)\frac{\partial u}{\partial\lambda}\,dx\,d\lambda+\int_{-\infty}^{\infty} xu\Big|_{\lambda=0}^{1}\,dx = \alpha t+\beta$$

for constants α, β. It can be shown by a fairly tedious calculation that there are no further symmetries of the BBM equation, and thus no further invariant one-forms.

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* *Note added in proof.* A recent paper of Gel'fand and Dorfman (32) also analyzes the Hamiltonian formalism and obtains results similar to those in section 4.

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