On the classification of symmetrically-coupled integrable evolution equations

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1 Introduction.

This paper deals with the classification of a certain subclass of coupled integrable evolution equations. The equations we consider generalize several classical scalar integrable equations, including the Burgers, KdV, modified KdV (mKdV), potential KdV (pKdV), Kaup-Kupershmidt and Sawada-Kotera equations.

Our approach to the classification is based on the existence of higher generalized symmetries. This approach was successfully used by many authors and it led to the discovery of many new equations, both C-integrable and S-integrable in Calogero's terminology, [1]. Since the case of homogeneous, autonomous polynomial scalar evolution equations with linear leading terms was completely resolved by Sanders & Wang [10], we have investigated the next more general class: the two-component equations.

Integrability of coupled evolution equations was often dealt with in the literature. However, the only classes that were completely investigated from a symmetry point of view are the nonlinear Schrödinger-type and derivative Schrödinger-type equations [7, 8]. This is mostly due to the extremely large size of the systems of equations one has to solve to implement such a classification even in the simplest case. Yet it may be possible to do a complete classification for a certain subclass of coupled systems. We suggested considering the symmetrically-coupled systems of type

$$\begin{cases} u_t = F[u,v] \\ v_t = F[v,u] \end{cases} \tag{1.1}$$

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†Supported in part by NSF Grant DMS 98–03154

(where the square brackets denote the dependence on the x-derivatives) and we were able to implement the complete classification in several cases. In our work we use a MATHEMATICA package written by the second author. This package performs automatically a large part of the required computations.

2 Statement of the problem.

Let us consider an equation taking values in a commutative algebra $\mathcal{A}$ possessing an involution $u \to \bar{u}$. The general form of such an equation is $u_t = F[u, \bar{u}]$. If this equation has real coefficients, we obtain a symmetrically-coupled system of evolution equations of type (1.1) by taking $v = \bar{u}$ and appending the equation for v. Thus the classification of integrable equations of type (1.1) is equivalent to the classification of integrable equations involving an involution.

Definition 2.1 A second system of evolution equations

$$\begin{cases} u_t = Q_1[u, v] \\ v_t = Q_2[u, v] \end{cases} \tag{2.1}$$

is said to be a *generalized symmetry* of (1.1) if their flows formally commute

$$\frac{\partial \mathbf{Q}}{\partial t} + \mathbf{D}_K(\mathbf{Q}) - \mathbf{D}_Q(\mathbf{K}) = 0. \tag{2.2}$$

Here $\mathbf{D}_K$ denotes the Fréchet derivative of $\mathbf{K}[\mathbf{u}] = (F[u, v], F[v, u])$, where $\mathbf{u} = (u, v)$.

It was noticed that the right-hand sides of many important integrable equations, as well as of their symmetries, are homogeneous if we assign certain weights to both the dependent and the independent variables. Hereafter we restrict our attention to such homogeneous equations.

We introduce the following weighting scheme on the space of differential polynomials. We assign the weight $n = deg\, u = deg\, v$ to the dependent variables and $1 = deg\, x$ to the independent variable. Thus, the k-th derivative of u (or of v) with respect to x has weight $n + k$. The weight of a monomial is defined as the sum of weights of its factors, while the weight of a PDE $\mathbf{u}_t = \mathbf{K}$ is $weight\, \mathbf{K} - n$. A detailed description can be found in [3, 7].

The Burgers, mKdV and pKdV equations are homogeneous for $n = 1$ (we will call it the *Burgers weighting*), while the KdV, Kaup-Kupershmidt and Sawada-Kotera equations are homogeneous for $n = 2$ (we will call it the *KdV weighting*). The coupled equations considered in this article are homogeneous in either one of these two weightings. The work for the other types of integrable scalar evolution equations is still in progress.

Definition 2.2 A two-component system of evolution equations is called *decoupled* if either one of the two equations depends only on one dependent variable.

Definition 2.3 A decoupled two-component system of evolution equations of type (1.1) is called *integrable* if it possesses two generalized symmetries of higher order.

Remark: Fokas' conjecture [2] says that for a system in n dependent variables it is enough to produce exactly n higher symmetries to ensure the existence of an infinite number thereof. In a recent preprint, van der Kamp and Sanders [12] propose an example of a very complicated two-component system that possesses exactly two generalized symmetries. However, this equation is decoupled. It is thus not a "true" two-component system. Since all the known symmetry "pathologies" occur only for decoupled systems, it is reasonable to continue to rely on Fokas' conjecture for integrability of nondecoupled systems.

3 The classification.

In this section we present the classification of integrable nondecouplable equations of type (1.1) that are either of weight 2 or 3 in the Burgers weighting, or else of weight 3 or 5 in the KdV weighting. All the classes we obtain are one- or two-parameter families (up to a scaling). However, the number of parameters can be further reduced by 1 using a linear change of dependent variables that preserves the form (1.1) of the system. In this paper we will only show the reduced classes. The complete lists can be found in [3].

Remark: We will write down here only the first components of the new integrable systems we obtained. The other components can be easily reconstructed using (1.1).

Theorem 3.1 *An integrable nondecouplable equation of type (1.1) and weight 3 in the KdV weighting is equivalent to one of the following up to a linear change of variables:*

$$u_t = \frac{1}{2}u_{xxx} + \frac{1}{2}v_{xxx} + 2uu_x + vv_x, \tag{3.1}$$

$$u_t = -\frac{1}{2}u_{xxx} + \frac{3}{2}v_{xxx} + uv_x + vv_x, \tag{3.2}$$

Remark: The system corresponding to (3.1) is the symmetrized version of the Ito equation, while (3.2) is the symmetrized version of the Hirota-Satsuma equation.

Theorem 3.2 *An interable nondecouplable equation of type (1.1) and weight 5 in the KdV weighting that is reduced by the substitution $v = u$ to either the Kaup-Kupershmidt or the Sawada-Kotera equation is equivalent to one of the following up to a linear change of variables:*

$$\begin{aligned} u_t = {} & -4u_{xxxxx} + 5v_{xxxxx} + 20vu_{xxx} - 40vv_{xxx} + 5u_xu_{xx} + 15v_xu_{xx} + 5u_xv_{xx} \\ & -75v_xv_{xx} + 50u^2u_x - 60uvu_x + 10v^2u_x - 20u^2v_x + 40uvv_x + 60v^2v_x, \end{aligned} \tag{3.3}$$

$$u_t = u_{xxxxx} + 5uu_{xxx} + 5vu_{xxx} + 15u_xu_{xx} + 10v_xv_{xx} + 5(u+v)^2u_x. \tag{3.4}$$

Remark: The equation (3.3) is the symmetrized version of the system considered by Zhou, Jiang & Jiang [13]. The other equation seems to be previously unknown. We remark that both systems reduce to the Kaup-Kupershmidt equation. Remarkably, there are no nondecouplable generalizations of the Sawada-Kotera equation.

Theorem 3.3 *An integrable nondecouplable equation of type (1.1) and weight 2 in the Burgers weighting is equivalent to one of the following up to a linear change of variables:*

$$u_t = \frac{1}{2}u_{xx} + \frac{1}{2}v_{xx} + 5uu_x + vu_x + 2vv_x, \tag{3.5}$$

$$u_t = \frac{1}{2}u_{xx} + \frac{1}{2}v_{xx} + 8uu_x - 2uv_x + 10vv_x - 9u^3 + 11u^2v - 11uv^2 + 9v^3, \tag{3.6}$$

$$\begin{aligned} u_t = {} & (1-\alpha)u_{xx} + \alpha v_{xx} + (7-8\alpha)uu_x + 5vu_x + (1+4\alpha)uv_x + (3+4\alpha)vv_x - \\ & -(3+2\alpha)u^3 + (1-10\alpha)u^2v + (-1+10\alpha)uv^2 + (3+2\alpha)v^3. \end{aligned} \tag{3.7}$$

Remark: The class (3.7) includes the symmetrized versions of the systems found by Svinolupov (equation (2.2.31) in [5]) and by Olver & Sokolov (equation (4.13) in [7]). The values of α are 0 and 1, correspondingly.

Remark: For the systems (3.5) and (3.6) we were able to find a recursion operator, thereby proving the exitence of an infinite number of generalized symmetries. The recursion operator for (3.6) depends on three arbitrary parameters.

Theorem 3.4 *An integrable nondecouplable equation of type (1.1) and weight 3 in the Burgers weighting is equivalent to one of the following up to a linear change of variables:*

$$u_t = u_{xxx} + 3uu_{xx} - 3vu_{xx} + 3u_x^2 + 3u^2u_x - 6uvu_x + 3v^2u_x \tag{3.8}$$

$$u_t = u_{xxx} + 3uu_{xx} - 3vu_{xx} + 3u_x^2 + 3u_xv_x + 3u^2u_x - 6uvu_x + 3v^2u_x \tag{3.9}$$

$$u_t = \frac{1}{2}u_{xxx} + \frac{1}{2}v_{xxx} + 2u_x^2 + v_x^2 \tag{3.10}$$

$$u_t = \frac{1}{2}u_{xxx} + \frac{1}{2}v_{xxx} + (u-v)u_{xx} + 3u_x^2 + 2u_xv_x + v_x^2 + 2(u-v)^2u_x + \beta(u-v)^4 \tag{3.11}$$

$$\begin{aligned} u_t = {} & -\frac{1}{2}u_{xxx} + \frac{3}{2}v_{xxx} + 6uu_{xx} - 6vu_{xx} + 3u_x^2 - 6u_xv_x - 3v_x^2 - \\ & -12(u-v)^2v_x - 3(u-v)^4 \end{aligned} \tag{3.12}$$

$$\begin{aligned} u_t = {} & \frac{5}{2}u_{xxx} - \frac{3}{2}v_{xxx} + 6uu_{xx} - 6vu_{xx} + 6u_x^2 + 12u_xv_x - 6v_x^2 - \\ & -12(u-v)^2(u_x - v_x) - 3(u-v)^4 \end{aligned} \tag{3.13}$$

$$u_t = u_{xx} + \frac{1}{3}u^2v \tag{3.14}$$

$$u_t = u_{xxx} + 3uvu_x + u^2v_x \tag{3.15}$$

$$u_t = u_{xxx} + 3((u-v)u_{xx} + u_x^2 - u_xv_x + (u^2 - 4uv + v^2)u_x - 2u^2v_x + 2uvv_x) \tag{3.16}$$

$$u_t = u_{xxx} + 3((u-v)u_{xx} + u_x^2 - u_xv_x + (u^2 - 8uv + v^2)u_x - 4u^2v_x + 2uvv_x) \tag{3.17}$$

$$\begin{aligned} u_t = {} & \frac{1}{2}u_{xxx} + \frac{1}{2}v_{xxx} + uu_{xx} - vu_{xx} - uv_{xx} + vv_{xx} - 2u_xv_x + 2v_x^2 + \\ & +4u^2u_x - 20uvu_x - 4u^2v_x - 4uvv_x \end{aligned} \tag{3.18}$$

$$u_t = \frac{1}{2}u_{xxx} + \frac{1}{2}v_{xxx} + uu_{xx} - vu_{xx} + u_x^2 - u_xv_x - 4uvu_x - 2u^2v_x \tag{3.19}$$

$$u_t = \frac{1}{2}u_{xxx} + \frac{1}{2}v_{xxx} + uu_{xx} - vu_{xx} - uv_{xx} + vv_{xx} - 2u_xv_x + 2v_x^2 - 12uvu_x -$$

$$-4v^2u_x - 12uvv_x + 4v^2v_x + 4u^4 - 12u^3v + 12u^2v^2 - 4uv^3 \quad (3.20)$$

$$u_t = \frac{1}{2}u_{xxx} + \frac{1}{2}v_{xxx} + 2uu_{xx} - 2vu_{xx} + (u_x - v_x)^2 - 2u^2u_x - 36uvu_x - 10v^2u_x - \\ -14u^2v_x - 28uvv_x - 6v^2v_x - 6u^4 - 32u^3v + 20u^2v^2 + 16uv^3 + 2v^4 \quad (3.21)$$

$$u_t = -\frac{1}{2}u_{xxx} + \frac{3}{2}v_{xxx} + 3uu_{xx} - 3vu_{xx} + 3u_x^2 - 3u_xv_x - 6v^2v_x \quad (3.22)$$

Remark: The systems (3.8) - (3.13) are the generalizations of the pKdV equation, while the systems (3.14) - (3.22) generalize the mKdV equation. (3.8), (3.14), (3.16) are symmetries of integrable Schrödinger-type equations from [7]. (3.10) is the potential form of the symmetrized Ito equation (3.1). The other 11 equations seem to be new.

Remark: We were able to find several Hamiltonian formulations for the newly-found equations, as well as a biHamiltonian formulation for the system (3.19). For details see [3].

4 Coupled equations in noncommutative variables.

The general theory of integrable noncommutative equations goes back to [4, 11] and was further developed in [7, 3, 9]. We use this theory to consider the symmetrically-coupled equations in an associative noncommutative algebra $\mathcal{A}$.

The main difference from the commutative case lies in the fact that there are two kinds of involutions in a noncommutative algebra: order-preserving (say $\bar{u}$) and order-reversing (say u^t). Thus we can consider two classes of symmetrically-coupled systems with real coefficients:

$$\begin{cases} u_t = F[u, v] \\ v_t = F[v, u] \end{cases} \quad (4.1)$$

and

$$\begin{cases} u_t = F[u, v] \\ v_t = F^t[v, u] \end{cases} \quad (4.2)$$

The former class (4.1) is equivalent to a complex equation involving the complex conjugate $u_t = F[u, \bar{u}]$, while the latter class (4.2) is equivalent to a matrix equation involving the transpose $u_t = F[u, u^t]$.

Due to a huge amount of computations, we were only able so far to complete the classification of integrable KdV- and Burgers-type equations. The equations we obtain are generalizations of integrable commutative equations of type (1.1). They are rather lengthy and we refer the reader to [3] for the complete list of them.

As to the case of commutative symmetrically-coupled equation of type (1.1) that generalize Kaup-Kupershmidt, Sawada-Kotera, pKdV and mKdV equations, we showed that neither of these equations has a noncommutative analog of type (4.1) or (4.2). These results suggest that the Sokolov's conjecture [7] about the noncommutative generalizations of the Kaup-Kupershmidt and the Sawada-Kotera equation may be false, at least for equations with real-valued coefficients.

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