# Continuous Calculus

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1. Introduction.

One evening not so long ago, for no apparent reason (or at least none I can now recall) I started thinking about the topological definition of continuity, [9]. In brief, a function is said to be continuous if and only if the inverse image of any open set is open. This sounds very simple — and certainly simpler than the limit-based definition used in calculus. I started wondering why not try to develop basic calculus using this as the starting point, and, possibly, eliminating all references to limits, epsilons, and deltas while still retaining rigor. And, after some thought, I realized it could be done. Continuity is basic, and limits, including limits of sequences, and derivatives follow from it in a reasonably straightforward manner, while bypassing epsilons and deltas entirely! You will see the results of this line of reasoning below.

Not only can the development be made completely rigorous, I believe it is more elementary and eminently more understandable by the beginning mathematics student, who will be better able to appreciate the rigor behind the calculational tools. Moreover, this approach introduces them to the basics of point set topology at an early stage in their mathematical career, rather than having to start from scratch in a later course in the subject or in preparation to study real analysis. The casual reader of these notes can, of course, skip over the proofs to assimilate the basic ideas and computational techniques, returning to them at a later stage as necessary. On the other hand, the reader already familiar with basic calculus may find new insight and appreciation for the subject using this alternative foundation of the subject.

In the second half of these notes, I have also adopted a somewhat non-traditional approach to integration — both single and double integrals — that is based entirely on a simplified definition of area, although it is not that far removed from the standard constructions of Riemann sums. The proof of the Fundamental Theorem of the Calculus, that the derivative of the integral of a continuous function is the function itself, is a straightforward consequence of the approach to differentiation. While there is much more in these notes than could be expected of a basic course, a suitably toned down subset could, I believe, form the basis for an introductory course in calculus, both more rigorous and more understandable to the student than our usual limit-based approach.

The reader is referred to the standard textbooks [1, 5, 6, 23] for traditional introductions to calculus as well the innovative text [24]. All rely on limits, and use ε’s and δ’s throughout, although in a typical introductory calculus course, we tend skip over the rigorous foundations entirely because they are simply too challenging for most students to come to grips with! The one precedent for this approach that I have been able to locate so far is the online booklet of Daniel J. Bernstein, [2], which I became aware of after realizing that the continuity-based definition of the derivative goes back to Carathéodory, [3, 10]; see Sections 7 and 11. For an alternative limit-free approach to the foundations of calculus, see [14].
**Prerequisites**: The reader should be familiar with elementary set theory, proof by induction, the real number line, and the concept of a function, including polynomials, rational, and algebraic functions, as well as the basics of exponential, logarithmic, and trigonometric functions. Otherwise these notes are self-contained, except for the proofs of a couple of the more difficult theorems that arise in multivariable calculus, where we refer to the literature. In particular, no familiarity with topology is assumed, and the concepts of open and closed sets, and continuity are developed from scratch. Perhaps a better description of these notes would be *Topological Calculus*. However, I refrained from using this more sophisticated title so as not to potentially scare off prospective readers by use of unfamiliar words at the outset.

I will employ standard notation for sets and functions. In particular, \( \{ F | C \} \) will denote a set, where \( F \) gives the formula for the members of the set and \( C \) is a list of conditions. For example, \( \{ x \in \mathbb{R} | 0 \leq x \leq 1 \} \) means the closed unit interval from 0 to 1, also written \([0,1]\) and also sometimes abbreviated as \( \{ 0 \leq x \leq 1 \} \). The empty set is denoted by the symbol \( \emptyset \). The notation \( f: X \rightarrow Y \) means that the function \( f \) maps the domain set \( X \) to the image set \( Y \), with formula \( y = f(x) \). The range of \( f \), meaning \( \{ y \in Y | y = f(x) \text{ for some } x \in X \} \) is, in general, a subset of \( Y \). Composition of functions is denoted by \( f \circ g \), so that \( f \circ g(x) = f[g(x)] \).

Finally *Q.E.D.* signifies the end of a proof, an acronym for “quod erat demonstrandum”, which is Latin for “which was to be demonstrated”.

### 2. Elementary Topology of the Real Line.

The setting of one variable calculus is the real line \( \mathbb{R} \), and we assume the reader is familiar with its basic properties. Multivariable calculus takes place in higher dimensional space and we will discuss how these constructions can be adapted there starting in Section 9. We will use the notation \( \mathbb{Z} \) for the integers, \( \mathbb{Q} \) for the rational numbers, and \( \mathbb{N} = \{ n \in \mathbb{Z} | n > 0 \} \) for the natural numbers, excluding 0. All are subsets of \( \mathbb{R} \).

The **basic open sets** of \( \mathbb{R} \) are the open intervals:

\[
(a, b) = \{ x \in \mathbb{R} | a < x < b \}.
\] (2.1)

Here \( a < b \) are any real numbers, the *endpoints* of the interval. We also allow \( a = -\infty \) or \( b = \infty \) or both, so that \( \mathbb{R} = (-\infty, \infty) \) is itself a basic open set. Two other important cases are \( \mathbb{R}^+ = (0, \infty) = \{ x > 0 \} \), the *positive real numbers*, and \( \mathbb{R}^- = (-\infty, 0) = \{ x < 0 \} \), the *negative real numbers*. The interval \((a, b)\) is *bounded* if both \(a, b \in \mathbb{R}\); otherwise it is *unbounded*.

**Definition 2.1.** An *open subset* is a union of basic open sets.

Thus, the open sets in \( \mathbb{R} \) are unions of open intervals. The notion of open set is said to define a **topology** on a space, in this case the real number line, [9]. The word “topology” was coined by mathematicians in the nineteenth century, and is based on the Greek work topos (\( \tau \omega \pi \omicron \sigma \)) that means “place”, hence topology is the “study of place”.

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Here are a few simple examples of open sets:

- Any open interval is of course open, being the union of a single basic open set, namely itself.
- The empty set is open, being the union of zero open intervals. Or if you don’t accept that, just take this fact as an addendum to the definition of open subset.
- The union of two interval is open, but not necessarily an interval. For example \((0, 1) \cup (2, 3)\) is the set of points \(x \in \mathbb{R}\) such that either \(0 < x < 1\) or \(2 < x < 3\). On the other hand, \((0, 2) \cup (1, 3) = (0, 3)\) is an open interval. Finally \((0, 1) \cup (1, 2)\) is the pair of intervals obtained by deleting the point 1 from the interval \((0, 2)\); in other words \((0, 1) \cup (1, 2) = (0, 2) \setminus \{1\}\). One can similarly take unions of any finite number of open intervals to obtain an open set that is, on occasion, an interval itself. The reader may enjoy writing down necessary and sufficient conditions for this to be the case.
- But we can also take unions of infinitely many intervals. For example, 
  \[ S = \bigcup_{n=-\infty}^{\infty} (n, n+1) \]
  is the open set obtained by deleting all the integers: \(S = \mathbb{R} \setminus \mathbb{Z}\).

A particularly interesting example of the last case is provided by the open set \(S\) obtained as the union of the intervals

\[
\left(\frac{3k-2}{3^n}, \frac{3k-1}{3^n}\right), \quad \text{where} \quad n \in \mathbb{N}, \quad \text{and} \quad k = 1, \ldots, 3^{n-1}.
\] (2.2)

Thus \(S \subset [0, 1]\) consists of an infinite collection of disjoint intervals of progressively smaller and smaller size. Its complement \(C = [0, 1] \setminus S\) is known as the *Cantor set*, named after the nineteenth century Russian-born, German-based mathematician Georg Cantor, the founder of modern set theory, although it was first discovered in 1874 by the British mathematician Henry John Stephen Smith. (Mathematics is full of objects discovered by X but named after Y.) The Cantor set plays a particularly important role in the study of fractals, [12], and in advanced real analysis, [21].

Let us state some basic facts about open sets.

**Proposition 2.2.** The union of any collection of open sets is open.

This is clear because each open set in the collection is itself a union of basic open sets, and hence the union of the collection is equal to the union of all the basic open sets associated with each set in the collection.

**Proposition 2.3.** The intersection of any finite collection of open intervals is either empty or an open interval.

*Proof:* Let us first prove the result for two open intervals: \(I_1 = (a, b)\), \(I_2 = (c, d)\), where \(a < b\) and \(c < d\). We also assume, without loss of generality, that \(a \leq c\) as otherwise
we interchange the two intervals: \( I_1 \leftrightarrow I_2 \). There are then three possible subcases:

(a) If \( a < b \leq c < d \) then \( I_1 \cap I_2 = \emptyset \); (b) if \( a \leq c < b \leq d \), then \( I_1 \cap I_2 = (c, b) \); (c) if \( a \leq c < d \leq b \), then \( I_1 \cap I_2 = (c, d) \). Thus in each case, the intersection is either empty or an open interval. By a straightforward induction, the same holds for any finite collection of open intervals. 

\[ \text{Q.E.D.} \]

**Proposition 2.4.** The intersection of any finite collection of open sets is open.

**Proof:** It suffices to prove this for two open sets \( S, T \) since then one can use an evident induction to prove it for any finite intersection. Let \( S = \bigcup I \) be a union of open intervals, and similarly for \( T = \bigcup J \). But then

\[
S \cap T = \bigcup I \cap \bigcup J = \bigcup (I \cap J),
\]

where the right hand side is the union of the intersections or all pairs of intervals forming \( S \) and \( T \) respectively. Proposition 2.3 implies that each term in the final union is either empty or an open interval, and hence, discarding all the empty sets, one expresses \( S \cap T \) as a union of open intervals. 

\[ \text{Q.E.D.} \]

The intersection of an infinite collection of open sets need not be open. For example,

\[
\bigcap_{n=1}^{\infty} \left( -\frac{1}{n}, \frac{1}{n} \right) = \{0\},
\]

and the latter set, being a single point, is not open. (Why?)

**Lemma 2.5.** A set \( S \subset \mathbb{R} \) is open if and only if for every \( x \in S \) there is a (bounded) open interval \( I \) such that \( x \in I \subset S \).

**Proof:** Since an open set \( S \) is, by definition, the union of open intervals, every \( x \in S \) belongs to one or more of them. If one requires boundedness, and the interval in question is unbounded, we can just choose any bounded interval that contains \( x \) and is contained within it. Conversely, if each \( x \in S \) belongs to a (bounded) open interval: \( x \in I_x \subset S \), then \( S = \bigcup_{x \in S} I_x \) can be expressed as their union. It may be an uncountable union, but that is still allowed in the original definition. 

\[ \text{Q.E.D.} \]

**Remark:** In Lemma 2.5, one could even impose a bound on the length of the intervals allowed, i.e., require \( b - a < r \) for some fixed \( r > 0 \).

Two sets \( A, B \) are said to be **disjoint** if their intersection is empty: \( A \cap B = \emptyset \). A collection of subsets is (pairwise) **disjoint** if any two members are disjoint. The proof of the following result will appear below.

**Proposition 2.6.** If \( S \subset \mathbb{R} \) is open, then it is the union of disjoint open intervals.

**Theorem 2.7.** The real line \( \mathbb{R} \) cannot be written as the disjoint union of two or more open sets.
Proof: We can use Proposition 2.6 to write each of the subsets as the disjoint union of open intervals, and hence it suffices to show that \( \mathbb{R} \) cannot be written as the disjoint union of two or more open intervals. Let \((a, b)\) be one of the intervals. Then, for any other interval \((c, d)\) in the collection, either \(d < a\) or \(b < c\) since the two intervals are assumed to be disjoint. But this implies neither \(a\) nor \(b\) lies in any of the intervals in the collection. At least one of \(a, b\) is finite, as otherwise \((a, b) = (-\infty, \infty) = \mathbb{R}\) and there would be no other intervals in the collection. Thus, a finite endpoint of any interval in the collection does not belong to any of the intervals, and hence does not lie in their union either. \( Q.E.D. \)

Remark: With a little more care, one can similarly prove that an open interval \((a, b)\) cannot be written as the disjoint union of two or more open subsets.

The set-theoretic complement of an open set is known as a closed set.

**Definition 2.8.** A set \( C \subset \mathbb{R} \) is closed if the set \( S = \mathbb{R} \setminus C \) is open.

For example, \( \mathbb{R} \) is closed because \( \mathbb{R} \setminus \mathbb{R} = \emptyset \) is open. The same holds for the empty set \( \emptyset \), both of which are also open. In fact, the only subsets of \( \mathbb{R} \) which are both open and closed are \( \mathbb{R} \) itself and the empty set \( \emptyset \).

**Proposition 2.9.** If \( \emptyset \neq S \subset \mathbb{R} \) and \( S \) is both open and closed, then \( S = \mathbb{R} \).

Proof: Suppose \( \emptyset \neq S \neq \mathbb{R} \). According to Proposition 2.6, \( S \) is the union of one or more disjoint open intervals. On the other hand, since \( S \) is also closed its complement \( C = \mathbb{R} \setminus S \) is open, not empty, and hence also the union of one or more disjoint open intervals. But this implies \( \mathbb{R} = S \cup C \) is the union of two or more disjoint open intervals, in contradiction to Theorem 2.7. \( Q.E.D. \)

A closed interval

\[
[a, b] = \{ x \in \mathbb{R} \mid a \leq x \leq b \}
\]  

where \( a \leq b \) are both finite, is, not surprisingly, closed. In particular, a single point \( \{a\} = [a, a] \) forms a closed set. Indeed, we can write its complement

\[
\mathbb{R} \setminus [a, b] = (-\infty, a) \cup (b, \infty)
\]

as the union of two unbounded open intervals.

Some sets are neither open nor closed, examples being the half-open intervals

\[
(a, b) = \{ a < x \leq b \}, \quad [a, b) = \{ a \leq x < b \},
\]  

when \( a < b \) are both finite. The definitions (2.4) make sense if \( a = -\infty \) in the first case, or \( b = \infty \) in the second, although the resulting unbounded intervals are closed sets. Indeed, the complement of \((-\infty, b)\) is the open unbounded interval \((b, \infty) = \mathbb{R} \setminus (-\infty, b)\), and similarly for \([a, \infty)\).

**Proposition 2.10.** The intersection of any collection of closed sets is closed. The union of any finite collection of closed sets is closed.
This result follows easily from Propositions 2.2 and 2.4. It allows us to construct further examples of closed sets, as follows. Since a single point \( \{ a \} \) forms a closed set, any finite set of points is closed. Discrete infinite sets of points need not be closed. For example \( \mathbb{Z} \) is closed since, as we noted above, its complement \( \mathbb{R} \setminus \mathbb{Z} \) is open. On the other hand, \( \tilde{A} = \{ 1/n | n \in \mathbb{N} \} \) is not closed since 0 is in its complement, but no interval containing 0 is because if \( a < 0 < b \), then \( \tilde{A} \cap (a, b) = \{ 1/n | 1/b < n \in \mathbb{N} \} \). On the other hand, \( A = \tilde{A} \cup \{ 0 \} \) is closed because its complement is the union of the open intervals \( (-\infty, 0) \cup (1, \infty) \cup (1/(n + 1), 1/n) \) for all \( n \in \mathbb{N} \). See Section 6 for further details on this important example.

**Lemma 2.11.** If \( S \) is open and \( C \) is closed then \( S \setminus C \) is open. Similar, if \( C \) is closed and \( S \) is open, then \( C \setminus S \) is closed.

Indeed, given that \( S \) is open, we can write \( S \setminus C = S \cap (\mathbb{R} \setminus C) \) as the intersection of two open sets, proving the first statement. The second statement is established by an analogous argument.

**Proposition 2.12.** If \( S, T \subseteq \mathbb{R} \) are disjoint, nonempty open sets, so \( S, T \neq \emptyset \) while \( S \cap T = \emptyset \), then \( S \cup T \neq \mathbb{R} \).

*Proof:* Suppose not, i.e., \( S \cup T = \mathbb{R} \). Since they are disjoint, \( S = \mathbb{R} \setminus T \), which implies \( S \) is also a closed subset. But then Proposition 2.9 implies either \( S = \mathbb{R} \) and hence \( T = \emptyset \), or \( S = \emptyset \), either of which contradicts our original assumption. \( \quad \text{Q.E.D.} \)

The *closure* of a set \( A \), written \( \overline{A} \), is defined as the intersection of all the closed sets containing it. Note that the closure \( \overline{A} \) is the smallest closed set containing \( A \); in other words, if \( C \supset A \) is closed, then \( \overline{A} \subseteq C \). For example, the closure of an open interval \( (a, b) \) is the closed interval \([a, b] \). The integers \( \mathbb{Z} \subseteq \mathbb{R} \) is already closed. Interestingly, the closure of the rational numbers \( \mathbb{Q} \) is the entire set of real numbers: \( \overline{\mathbb{Q}} = \mathbb{R} \). This is because every open interval \( I \subseteq \mathbb{R} \) contains infinitely many rational numbers, and hence \( \mathbb{R} \setminus \overline{\mathbb{Q}} = \emptyset \).

Finally, we mention the fundamental *completeness* property of the real line, that is a consequence of its axiomatic construction, [21]. A set \( A \subseteq \mathbb{R} \) is *bounded from above* if there exists \( a \in \mathbb{R} \) such that \( x \leq a \) for all \( x \in A \). Any number satisfying this is called an *upper bound* for \( A \). The *least upper bound* property states that, for any set bounded from above, there exists a minimal upper bound, meaning an upper bound \( b \in \mathbb{R} \) satisfying \( b \leq a \) for any upper bound \( a \). The least upper bound is written \( b = \sup A \), where sup is short for *supremum*, which is a fancier name for it. Moreover, if \( (x, b) \) is any open interval, where \( x < b \), then \( S \cap (x, b) \neq \emptyset \) since otherwise \( x < b \) would also be an upper bound of \( A \), and \( b \) would not be the least upper bound. The least upper bound may or may not be an element of \( A \). However, if \( A \) is open, then \( b = \sup A \notin A \) since otherwise an interval containing \( b \) would be contained in \( A \), and thus \( b \) would not be an upper bound. On the other hand, if \( A \) is closed then \( b = \sup A \in A \). Indeed, if \( b \in \mathbb{R} \setminus A \), because the latter set is open, there is an open interval \( b \in I \subseteq \mathbb{R} \setminus A \), but then \( A \cap I = \emptyset \) which would imply
that any point \( x \in I \) is an upper bound for \( A \) and hence \( b \) cannot be the the least upper bound.

Vice versa, a set \( A \subset \mathbb{R} \) is bounded from below if there exists a lower bound \( c \in \mathbb{R} \) such that \( c < x \) for all \( x \in A \). There also exists a greatest lower bound, written \( d = \inf A \), where \( \inf \) is short for \( \text{infimum} \), such that \( d \geq c \) for any lower bound \( c \). The above statements for the least upper bound also hold for the greatest lower bound.

By convention, if \( A \) is not bounded from above, we write \( \sup A = \infty \), while if it is not bounded from below, we write \( \inf A = -\infty \).

Remark: The least upper bound property is not true for the rational numbers \( \mathbb{Q} \). Indeed, the least upper bound of \( A = \{ r \in \mathbb{Q} \mid r^2 < 2 \} \) is \( \sup A = \sqrt{2} \) which is not rational. There is no smallest rational upper bound. Indeed, any rational number \( q \in \mathbb{Q} \) such that \( q > \sqrt{2} \) is an upper bound of \( A \), but there is no least such number. (Why?) This distinction between \( \mathbb{R} \) and \( \mathbb{Q} \) is one of the main reasons why one needs to extend the rational numbers to the real numbers in order to do calculus and mathematical analysis.

Lemma 2.13. Let \( A \subset \mathbb{R} \) be a subset such that whenever \( x < y < z \) and \( x, z \in A \), then \( y \in A \). Then \( A \) is an interval — open, closed, or half open.

Proof: Let \( m = \inf A \) and \( M = \sup A \). Thus, if both are finite, \( A \subset [m, M] \), with analogous statements if one or both are infinite. If \( m = M \), then \( A = \{ m \} = [m, M] \) is a singleton, which is a closed interval. Otherwise, suppose \( m < y < M \). Let \( x \in (m, y) \cap A \) and \( z \in (m, M) \cap A \), both of which are nonempty by the preceding remarks. Then \( x < y < z \), and hence \( y \in A \). Thus every point \( y \in (m, M) \) belongs to \( A \). If \( m, M \not\in A \), then \( A = (m, M) \) is an open interval. If \( m, M \in A \) then \( A = [m, M] \) is an closed interval. If one but not the other belong to \( A \), then \( A \) is the corresponding half open interval. Q.E.D.

Let us state a couple of elementary results about upper and lower bounds — suprema and infima — whose proofs are left as exercises for the reader.

Lemma 2.14. If \( A \subset B \subset \mathbb{R} \), then \( \sup A \leq \sup B \) and \( \inf A \geq \inf B \).

Lemma 2.15. Let \( A, B \subset \mathbb{R} \). Define their Minkowski sum to be the set

\[
A + B = \{ x + y \mid x \in A, \ y \in B \}.
\]

Then

\[
\sup(A + B) = \sup A + \sup B, \quad \inf(A + B) = \inf A + \inf B.
\]

The Minkowski sum is named after the late nineteenth century German/Lithuanian mathematician Hermann Minkowski, one of the founders of special relativity. In particular, if \( B = \{ b \} \) is a singleton, where \( b \in \mathbb{R} \), then we write \( A + b = A + \{ b \} = \{ x + b \mid x \in A \} \), which is the set obtained by translating the set \( A \) by a distance \( b \). Since \( \sup B = \inf B = b \), we deduce

\[
\sup(A + b) = \sup A + b, \quad \inf(A + b) = \inf A + b.
\]

Finally we supply an as yet unrealized proof.
Proof of Proposition 2.6: The case when $S = \emptyset$ is trivial, since it is the union of zero open intervals. Otherwise, given $x \in S$, let

$$a_x = \inf \{ a \mid x \in (a, b) \subset S \}, \quad b_x = \sup \{ b \mid x \in (a, b) \subset S \}.$$  

Then it is not hard to show that $(a_x, b_x) \subset S$ is the maximal interval belonging to $S$ that contains $x$. Furthermore $I_x = I_y$ if $y \in I_x$ or, equivalently, $x \in I_y$. On the other hand, $I_x \cap I_y = \emptyset$ if $x \not\in I_y$, or, equivalently, $y \not\in I_x$. Thus the collection of all such intervals (without repeats) is pairwise disjoint and their union is the entire set $S$. Q.E.D.

3. Functions.

We will now look at real-valued functions defined on $\mathbb{R}$ and subsets thereof. We assume that the reader is familiar with the elementary functions, by which we mean polynomials, rational functions, algebraic functions (involving roots), exponential functions, (natural) logarithms, and the trigonometric functions and their inverses. Elementary properties of these functions will be stated and used without proof.

Let $D \subset \mathbb{R}$ be the domain of a function $f: D \to \mathbb{R}$. Thus, to each point $x \in D$, the function $f$ assigns a value $y = f(x) \in \mathbb{R}$. If $A \subset D$ is any subset, we define its image under $f$ to be the subset

$$f(A) = \{ y = f(x) \mid x \in A \} \subset \mathbb{R}.$$  

In particular, the range of $f$ is the image of the entire domain set range $f = f(D)$. For example, the following functions all have domain $D = \mathbb{R}$. The range of $f(x) = x$ is $\mathbb{R}$; the range of $f(x) = x^2$ is $[0, \infty)$; the range of $f(x) = e^x$ is $(0, \infty)$; the range of $f(x) = \sin(x)$ is $[-1, 1]$. In general, if $A \subset D$, then one can restrict $f$ to $A$, defining $\tilde{f}: A \to \mathbb{R}$ by $\tilde{f}(x) = f(x)$ whenever $x \in A$, with range $\tilde{f}(A) = f(A)$. We will often drop the tilde and identify $f$ with its restriction to any subdomain, writing $f: A \to \mathbb{R}$ instead.

A function $f: D \to \mathbb{R}$ is said to be one-to-one if no two points have the same value; in other words, if $x \neq y$ then $f(x) \neq f(y)$. A one-to-one function has a well-defined inverse, namely $f^{-1}(y) = x$ whenever $y = f(x)$. The domain of $f^{-1}$ is the range of $f$ and vice versa, so $f(D) = R$ while $f^{-1}(R) = D$. Note that the inverse of the inverse function is the original function: $(f^{-1})^{-1} = f$.

Example 3.1.

- The constant function $f(x) \equiv 1$ is not one-to-one because every point has the same value.
- The function $f(x) = x$ is one-to-one and is its own inverse: $f^{-1}(y) = y$. More generally, any non-constant linear function† $f(x) = mx + l$ with $m \neq 0$ is one-to-one with inverse $f^{-1}(y) = y/m - l/m$.

† We will follow the calculus convention of calling such functions linear since their graphs are straight lines. This is at odds with the conventions of linear algebra, [19], where only those with $l = 0$ are linear. The general case, when $l \neq 0$, is called an affine function.
• The function \( f(x) = x^2 \) is not one-to-one as a function from \( \mathbb{R} \) to \([0, \infty)\) since \( f(-x) = f(x) \) for any \( x \). If we restrict to \( f \) to the domain \( D^+ = [0, \infty) \), then it is one-to-one, with range \( R^+ = f(D^+) = [0, \infty) \). Its inverse is given by the (positive) square root function: \( f^{-1}(y) = +\sqrt{y} \) for \( y \geq 0 \). On the other hand, \( f \) is also one-to-one when restricted to \( D^- = (-\infty, 0] \), with the same range: \( R^- = f(D^-) = [0, \infty) \); here its inverse is the negative square root function: \( f^{-1}(y) = -\sqrt{y} \) for \( y \geq 0 \).

• The function

\[
f(x) = \begin{cases} 
1/x, & x \neq 0, \\
0, & x = 0,
\end{cases}
\]

is one-to-one and is its own inverse:

\[
f^{-1}(y) = \begin{cases} 
1/y, & y \neq 0, \\
0, & y = 0.
\end{cases}
\]

• The exponential function \( f(x) = e^x \) is one-to one with range \( \mathbb{R}^+ = (0, \infty) \). The inverse function is the natural logarithm: \( f^{-1}(y) = \log y \) for \( y > 0 \).

Even though functions that are not one-to-one do not have (single-valued) inverses, we can nevertheless define the inverse image of a set. Namely, given \( f: D \to \mathbb{R} \) and a set \( S \subset \mathbb{R} \), we write

\[
f^{-1}(S) = \{ x \in \mathbb{R} \mid f(x) \in S \} \subset D.
\]

In particular, if \( R = \text{range } f \) and \( S \cap R = \emptyset \), then \( f^{-1}(S) = \emptyset \). Note that the set-valued inverse respects unions and intersections:

\[
f^{-1}(\bigcup S) = \bigcup f^{-1}(S), \quad f^{-1}(\bigcap S) = \bigcap f^{-1}(S).
\]

4. Continuity.

Now we reach the heart of our topological approach to calculus: the definition of continuity. To begin with, for simplicity, we assume the domain of our function \( f \) is all of \( \mathbb{R} \), although we do not require that its range be all of \( \mathbb{R} \).

**Definition 4.1.** A function \( f: \mathbb{R} \to \mathbb{R} \) is called **continuous** if, whenever \( S \) is open, then \( f^{-1}(S) \) is open.

Why this requirement is called “continuous” will gradually become evident as we explore its many consequences. Thus, in view of Lemma 2.5, to verify that \( f^{-1}(S) \) is open one needs to show that, every point \( x \in f^{-1}(S) \) is contained in an open interval \( x \in I \subset f^{-1}(S) \). Furthermore, since any open set is a union of (bounded) intervals, in view of property (3.2), to prove continuity one only needs to prove that if \( J \subset \mathbb{R} \) is an open (bounded) interval, then \( f^{-1}(J) \) is open, i.e., has the above property. Note that \( f^{-1}(J) \) need not itself be an open interval.

We also note that \( f \) is continuous if and only if whenever \( C \subset \mathbb{R} \) is closed, then \( f^{-1}(C) \) is closed. Indeed, setting \( S = \mathbb{R} \setminus C \), which is open, then \( f^{-1}(C) = \mathbb{R} \setminus f^{-1}(S) \) is closed, and vice versa.
Example 4.2.

- A constant function, e.g., \( f(x) \equiv 1 \) is continuous. Indeed, \( f^{-1}(S) = \mathbb{R} \) if \( 1 \in S \) while \( f^{-1}(S) = \varnothing \) if \( 1 \notin S \). In either case \( f^{-1}(S) \) is open (even if \( S \) itself is not open).

- The function \( f(x) = x \) is continuous. Indeed, if \( S \) is open so is \( f^{-1}(S) = S \).

- The function \( f(x) = x^2 \) is continuous. Note that the range of \( f \) is the closed half line \([0, \infty)\). To prove continuity, we use the preceding remark and thus only need to look at the effect of \( f^{-1} \) on open intervals \( J = (a, b) \) for \( a < b \). There are three cases: (a) If \( b \leq 0 \), then \( f^{-1}(J) = \varnothing \). (b) If \( a < 0 < b \), then \( f^{-1}(J) = (-\sqrt{b}, \sqrt{b}) \) is an open interval. (c) If \( 0 \leq a < b \), then \( f^{-1}(J) = (-\sqrt{b}, -\sqrt{a}) \cup (\sqrt{a}, \sqrt{b}) \) is the union of two open intervals. Since each one of these is open, the result follows. Later we will prove that all polynomial functions are continuous.

- The exponential function \( f(x) = e^x \) is continuous. Indeed, if \( J = (a, b) \) then \( f^{-1}(J) = (\log a, \log b) \) is an open interval. \textit{Remark:} Technically, this relies on the fact that, as \( y \) varies between \( a \) and \( b \), the corresponding logarithms \( x = \log y \) fill out the entire interval \( \log a < x < \log b \). While it is possible to prove this from basic principles, a slicker, less involved method is to first prove continuity of the natural logarithm via integration, and then deduce continuity of its inverse, the exponential function; see Example 13.17 for details.

- We remark, without proof, that the trigonometric functions \( \sin x \) and \( \cos x \) are continuous on all of \( \mathbb{R} \). The tangent function \( \tan x \) is continuous where defined, i.e. on \( D = \mathbb{R} \setminus \{ (n + \frac{1}{2}) \pi \mid n \in \mathbb{Z} \} \). (See below for continuity on subdomains.) As with the exponential and logarithm, the quickest proofs of these facts rely on integration.

- The \textit{sign function}

\[
\text{sign } x = \begin{cases} 
1, & x > 0, \\
0, & x = 0, \\
-1, & x < 0, 
\end{cases}
\]

is not continuous. Indeed, if \( S = (-\frac{1}{2}, \frac{1}{2}) \), then \( f^{-1}(S) = \{0\} \), which is not open. Thus, to prove that a function is not continuous, it suffices to find one open set \( S \subset \mathbb{R} \) such that \( f^{-1}(S) \) is not open. Moreover, it is not possible to redefine the sign function at \( x = 0 \) and make it continuous. For example, if we set

\[
f(x) = \begin{cases} 
1, & x \geq 0, \\
-1, & x < 0, 
\end{cases}
\]

then \( f^{-1}(0, 2) = [0, \infty) \) is closed, not open.

- Similarly, the function

\[
f(x) = \begin{cases} 
1/x, & x \neq 0, \\
0, & x = 0, 
\end{cases}
\]

is not continuous since, for example, \( f^{-1}(-1, 1) = \{0\} \cup (-\infty, -1) \cup (1, \infty) \). The second and third sets are open, but the first, being a single point, is not. Again, one cannot redefine \( f(x) \) at \( x = 0 \) in order to make a continuous function.
The next result states that basic algebraic operations on functions maintain continuity.

**Theorem 4.3.** If $f(x)$ and $g(x)$ are continuous, then

- Their sum $f(x) + g(x)$ and difference $f(x) - g(x)$ are both continuous.
- Their product $f(x)g(x)$ is continuous.
- Their quotient $f(x)/g(x)$ is continuous provided $g(x) \neq 0$.
- Their maximum $\max\{f(x), g(x)\}$ and minimum $\min\{f(x), g(x)\}$ are both continuous.
- Their composition $f \circ g(x) = f[g(x)]$ is continuous.

In particular, the function $-f(x)$, being the product of the constant function $-1$ and the continuous function $f(x)$ is continuous. We defer the proof of the first four facts until Section 9, when it will be easy. As for the composition, this is an immediate consequence of the fact that

$$(f \circ g)^{-1}(S) = g^{-1} \left[f^{-1}(S)\right] = g^{-1}(T), \quad \text{where } T = f^{-1}(S).$$

Thus, if $S$ is open, continuity of $f$ implies that $T = f^{-1}(S)$ is open. Furthermore, continuity of $g$ implies that $g^{-1}(T) = (f \circ g)^{-1}(S)$ is open, which proves continuity of $f \circ g$.

We can use sums, products, and quotients to build up complicated continuous functions. For instance, knowing that $x$ is continuous, and any constant function is continuous, we deduce $mx$ is continuous for any $m \in \mathbb{R}$, from which it follows $mx + l$ for $l \in \mathbb{R}$ is also continuous. Similarly, the product of $x$ with itself, $x^2 = xx$, is continuous, which implies $x^3 = xxx$ is also continuous, and, by induction, any power $x^n$ for $n \in \mathbb{N}$ is continuous. We can in turn multiply by any constant $c \in \mathbb{R}$, so $cx^n$ is also continuous. Then summing up such functions we arrive at the conclusion that any polynomial function is continuous. Moreover, if $p(x)$ and $q(x)$ are polynomials with $q(x) \neq 0$ — an example is $q(x) = x^2 + 1$ — then the rational function $r(x) = p(x)/q(x)$ is continuous.

The following result provides a useful test for continuity.

**Theorem 4.4.** Let $\lambda > 0$ be a positive constant. If $f: \mathbb{R} \to \mathbb{R}$ satisfies

$$|f(x) - f(y)| \leq \lambda |x - y| \quad (4.2)$$

for all $x, y \in \mathbb{R}$, then $f$ is continuous.

**Proof:** Let $(\alpha, \beta)$ be an open interval. Suppose $a \in f^{-1}(\alpha, \beta)$, so $\alpha < f(a) < \beta$. Choose $r > 0$ so that $\alpha < f(a) - r < f(a) + r < \beta$. Then if $|x - a| < r/\lambda$, condition (4.2) implies

$$|f(x) - f(a)| \leq \lambda |x - a| < r$$

which, by our choice of $r$, implies $f(x) \in (\alpha, \beta)$. Thus, the open interval $(a-r/\lambda, a+r/\lambda) \subset f^{-1}(\alpha, \beta)$. Since this applies to every $a \in f^{-1}(\alpha, \beta)$, we conclude that $f^{-1}(\alpha, \beta)$ is open, and hence that $f$ is continuous. 

Q.E.D.
A function that satisfies the condition (4.2) for some \( \lambda > 0 \) is called \textit{Lipschitz continuous}, after the nineteenth century German mathematician Rudolf Lipschitz. One can weaken the condition (4.2) by only requiring that it hold when \( |x - y| < s \) for some \( s > 0 \). Lipschitz continuity is stronger than mere continuity, although functions that are continuous but not Lipschitz continuous are badly behaved. See also Theorem 7.16 below.

\textit{Remark}: One can replace (4.2) by the condition

\[ |f(x) - f(y)| \leq \lambda |x - y|^{\alpha} \tag{4.3} \]

where \( \alpha > 0 \) is another constant; even more generally, one can replace \( |x - y|^{\alpha} \) by \( H(|x - y|) \) where \( H: [0, \infty) \to [0, \infty) \) is any continuous strictly increasing function with \( H(0) = 0 \). Again, one can weaken either condition by only requiring it when \( |x - y| < r \).

Functions that satisfy (4.3) are said to be \textit{Hölder continuous}, after Lipschitz’s younger German contemporary Otto Hölder. Since \( t^{\alpha} > t \) when \( 0 < t < 1 \), Hölder is weaker than Lipschitz, although not all continuous functions are Hölder continuous either.

\textbf{Lemma 4.5.} Let \( y \in \mathbb{R} \). Suppose \( f: \mathbb{R} \to \mathbb{R} \) is continuous, and \( f(x) \geq y \) for all \( x \neq a \). Then \( f(a) \geq y \). Similarly, if \( f(x) \leq y \) for \( x \neq a \), then \( f(a) \leq y \).

\textit{Proof}: We just prove the first statement, leaving the second as an exercise for the reader. Suppose \( f(a) < y \). Let \( b < f(a) < c < y \). Then \( f^{-1}(b, c) = \{ a \} \) which is not open, contradicting our assumption that \( f \) is continuous. \( Q.E.D. \)

\textit{Remark}: This result is not true for the strict inequality. Indeed, \( f(x) = x^2 \) satisfies \( f(x) > 0 \) for all \( x \neq 0 \) but \( f(0) = 0 \).

The next result is of fundamental importance, and is known as Intermediate Value Theorem. Roughly speaking, it says that if a continuous function takes on two values, then it necessarily takes on all intermediate values.

\textbf{Theorem 4.6.} Let \( f: \mathbb{R} \to \mathbb{R} \) be continuous. Suppose \( a < b \). If \( c \in \mathbb{R} \) satisfies either \( f(a) < c < f(b) \), or \( f(a) > c > f(b) \), then there exists \( x \in [a, b] \) such that \( f(x) = c \)

\textit{Proof}: Suppose not, i.e., \( c \) is not in the range of \( f \). Then \( S_- = f^{-1}(-\infty, c) \) and \( S_+ = f^{-1}(c, \infty) \) are nonempty open subsets since either \( a \in S_- \) and \( b \in S_+ \) or the reverse. Moreover, since for every \( x \in \mathbb{R} \), its image \( f(x) \) lies in either \( (-\infty, c) \) or \( (c, \infty) \), we have \( \mathbb{R} = S_- \cup S_+ \) with \( S_- \cap S_+ = \emptyset \). But this expresses \( \mathbb{R} \) as the disjoint union of two nonempty open sets, which, by Proposition 2.12, is not possible. \( Q.E.D. \)

The Intermediate Value Theorem 4.6 is the reason we informally say that a function is continuous if one can draw its graph without lifting the pencil from the paper. Indeed, the result says that to get from one point \( (a, f(a)) \) on the graph to another one \( (b, f(b)) \) one goes through all the intervening points \( (x, f(x)) \) in a continuous manner without missing any values.
In many situations, a function of interest is not defined on all of \( \mathbb{R} \), but we are nevertheless interested in its continuity on its domain of definition. We will assume at first that the domain of the function \( D \subset \mathbb{R} \) is an open subset. We then say \( f: D \rightarrow \mathbb{R} \) is **continuous on \( D \)** if \( f^{-1}(S) \subset D \subset \mathbb{R} \) is open whenever \( S \subset \mathbb{R} \) is open. Extensions to non-open domains are deferred until Section 5.

For example, the function \( f(x) = 1/x \) is continuous on its domain \( D = \mathbb{R} \setminus \{0\} \). Indeed, given a bounded open interval \((a, b)\), we have

\[
    f^{-1}(a, b) = \begin{cases} 
    (1/b, 1/a), & 0 < a < b, \\
    (1/b, \infty), & 0 = a < b, \\
    (-\infty, 1/a) \cup (1/b, \infty), & a < 0 < b, \\
    (-\infty, 1/a), & a < b = 0, \\
    (1/b, 1/a), & a < b < 0, 
    \end{cases}
\]

each of which is an open interval or the union of two open intervals, thus proving continuity.

Any rational function \( p(x)/q(x) \) is continuous on its domain \( D = \{ x \in \mathbb{R} | q(x) \neq 0 \} \), i.e., outside the zeros of \( q \).

**Remark:** The Intermediate Value Theorem 4.6 applies as stated when \( f \) is only defined on an open interval. It is not true on more general open sets; for example, the function \( f(x) = 1/x \) is continuous on \( \mathbb{R} \setminus \{0\} \), and \( f(-1) = -1 \), \( f(1) = 1 \), but there is no point \( x \in \mathbb{R} \setminus \{0\} \) such that \( f(x) = 0 \).

**Lemma 4.7.** Let \( I \subset \mathbb{R} \) be an open interval. A continuous function \( f:I \rightarrow \mathbb{R} \) is one-to-one if and only if it is strictly monotone, meaning either it is strictly increasing, so that \( f(x) < f(y) \) whenever \( x < y \), or strictly decreasing, so that \( f(x) > f(y) \) whenever \( x < y \).

**Proof:** Suppose \( f \) is not strictly monotone. If \( f(a) = f(b) \) for \( a < b \), then \( f \) is not one-to-one. So not being strictly monotone would imply that one can find \( a < b < c \) in \( I \) such that either \( f(a) < f(b) > f(c) \) or \( f(a) > f(b) < f(c) \). Let us deal with the first case, leaving the second to the reader. Let \( z \in \mathbb{R} \) satisfy both \( f(a) < z < f(b) \) and \( f(c) < z < f(b) \). By the Intermediate Value Theorem 4.6, we can find \( a < x < b \) such that \( f(x) = z \), and, again, \( b < y < c \) such that \( f(y) = z \). But this implies \( x \neq y \) whereas \( f(x) = f(y) \), in contradiction to our assumption that \( f \) is one-to-one. \( Q.E.D. \)

**Theorem 4.8.** Let \( I \subset \mathbb{R} \) be an open interval. If \( f:I \rightarrow \mathbb{R} \) is continuous and one-to-one, then its range \( J = f(I) \) is an open interval and its inverse \( f^{-1}:J \rightarrow I \) is continuous.

**Proof:** According to Lemma 4.7, \( f \) must be strictly monotone. Let us, for definiteness, assume that \( f \) is strictly increasing, so \( f(x) < f(y) \) whenever \( x < y \). Then, given an open interval \((a, b) \subset I\), if \( a < x < b \), then \( f(a) < f(x) < f(b) \). Moreover, by the Intermediate Value Theorem 4.6, given any \( c \) with \( f(a) < c < f(b) \) there exists \( x \in (a, b) \) such that \( f(x) = c \). Putting these together, we conclude that the image of the open interval \((a, b)\) under \( f \) is the open interval \( f(a, b) = (f(a), f(b)) \). This implies \( f \) maps open sets to open
sets. Since \( f = (f^{-1})^{-1} \) is the inverse of \( f^{-1} \), we conclude that \( f^{-1} \) satisfies the condition of Definition 4.1, and hence is continuous. \( Q.E.D. \)

**Example 4.9.** The function \( f_3(x) = x^3 \) is continuous and strictly monotone. Its inverse is the cube root function \( f_3^{-1}(y) = y^{1/3} \). Similar remarks apply to \( f_n(x) = x^n \) where \( n \) is any odd positive integer. On the other hand, the \( f_2(x) = x^2 \) is not monotone on \( \mathbb{R} \), and hence not one-to-one. Each \( y \) has two possible real square roots. If we restrict \( f_2 \) to the positive reals \( \mathbb{R}^+ = (0, \infty) \) then the resulting function \( f_2^+: \mathbb{R}^+ \to \mathbb{R}^+ \) is strictly monotone, and its inverse \( (f_2^+)^{-1}(y) = +\sqrt{y} \), the positive square root function, is continuous on \( \mathbb{R}^+ \). Similarly, the restriction of \( f_2 \) to the negative reals, denoted \( f_2^-: \mathbb{R}^- \to \mathbb{R}^+ \), where \( \mathbb{R}^- = \{ x < 0 \} \), is also strictly monotone and decreasing on its domain, and so its inverse \( (f_2^-)^{-1}(y) = -\sqrt{y} \) is a continuous function \( (f_2^-)^{-1}: \mathbb{R}^+ \to \mathbb{R}^- \).

5. **Topology and Continuity on Subsets of the Real Line.**

If \( A \subset \mathbb{R} \) is any subset, we can define a topology on \( A \) by declaring that any set of the form \( S \cap A \), where \( S \subset \mathbb{R} \) is open, is an open subset of \( A \). In particular, the basic open subsets of \( A \) are the intersections \( I \cap A \) of open intervals \( I = (a,b) \) with \( A \). As before, \( S \subset A \) is open if and only if every \( x \in S \) is contained in a basic open set: \( x \in I \cap A \subset S \).

A function \( f: A \to \mathbb{R} \) is continuous (when restricted to \( A \)) if, whenever \( S \subset \mathbb{R} \) is open, \( f^{-1}(S) \subset A \) is open in \( A \), i.e., a union of basic open subsets of \( A \). As before, one only needs to check this when \( S = J \subset \mathbb{R} \) is an open (bounded) interval. The following lemma states that the restriction of continuous functions to subsets remain continuous.

**Lemma 5.1.** Suppose \( B \subset A \subset \mathbb{R} \). Given \( f: A \to \mathbb{R} \), let \( \tilde{f}: B \to \mathbb{R} \) denote its restriction to \( B \), so that \( \tilde{f}(x) = f(x) \) whenever \( x \in B \subset A \). If \( f \) is continuous on \( A \), then its restriction \( \tilde{f} \) is continuous on \( B \).

**Proof:** Let \( S \subset \mathbb{R} \) be open. Continuity of \( f \) on \( A \) implies \( f^{-1}(S) \) is an open subset of \( A \), and hence of the form \( f^{-1}(S) = T \cap A \), where \( T \subset \mathbb{R} \) is open. But then \( \tilde{f}^{-1}(S) = f^{-1}(S) \cap B = T \cap A \cap B = T \cap B \) since \( B \subset A \). But the latter set is open in \( B \), and hence \( \tilde{f} \) is continuous. \( Q.E.D. \)

As above, when not potentially causing confusion, we will use the same letter \( f \) to denote a function and its restriction to a subset.

**Remark:** Conversely, given a continuous function on a subset, it is not always possible to extend it to a continuous function on a larger subset. As we already noted, the function \( f(x) = 1/x \) is continuous on \((0, \infty)\), but cannot be extended to a continuous function on any set containing the origin; this is because \( f(x) \) is not bounded, and a continuous extension would violate the Boundedness Theorem 5.13 below. Even bounded continuous functions may not have continuous extensions; an example is the sign function \( \text{sign} \, x \), as noted above. A more interesting function is \( \sin(1/x) \), which is continuous on \((0, \infty)\). It
also cannot be extended to any set containing the origin due to its increasingly rapid oscillations between $-1$ and $1$ as $x$ gets closer and closer to $0$. On the other hand, as we will see, a function that is continuous on a closed interval can easily be continuously extended to all of $\mathbb{R}$.

If $A$ is itself an open interval then the basic open sets are just the open subintervals $(a, b) \subset A$. Indeed, Proposition 2.3 states that the intersection of any two open intervals is either empty or an open interval. In this case, the continuity conditions are the same as the local version discussed above, when the domain of the function is restricted to the open interval.

Let us next discuss the case of a half-open interval $A = (a, b]$. The basic open subsets are the open subintervals $(c, d)$ with $a \leq c < d \leq b$, and the half-open subintervals $(c, b]$, for $a \leq c < b$, including $(a, b]$ itself. If $f: (a, b] \to \mathbb{R}$ is continuous when restricted to $(a, b]$, then we say that $f$ is continuous from the right at the endpoint $b$. On the other hand, according to Lemma 5.1 and the preceding paragraph, the restriction of $f$ to the open subinterval $(a, b) \subset (a, b]$ coincides with ordinary continuity there. The analogous “mirror image” results hold for a half-open interval $B = [a, b)$, and one speaks of continuity from the left at the endpoint $a$ of a continuous function $f: [a, b) \to \mathbb{R}$. Finally, if $C = [a, b]$ is a closed interval, then its basic open sets are the open subintervals $(c, d)$ where $a \leq c < d \leq b$, the half-open subintervals $[a, c)$ and $(c, b]$ for $a \leq c \leq b$, and the entire closed interval $[a, b]$. If $f: [a, b] \to \mathbb{R}$ is continuous, we say it is continuous from the left at the endpoint $a$ and continuous from the right at the endpoint $b$. Its restriction $f: (a, b) \to \mathbb{R}$ to the open subinterval is continuous in the usual sense.

The following result allows one to stitch together continuous functions on subdomains to obtain a continuous function on a larger domain. First we state a basic lemma.

**Lemma 5.2.** Let $a < b < c$. Suppose $U$ is open in $(a, b]$ and $V$ is open in $[b, c)$. Then $U \cup V \subset (a, c)$ is open (also as a subset of $\mathbb{R}$) if and only if either $b \notin U$ and $b \notin V$, or $b \in U$ and $b \in V$.

**Proof:** The first case implies $U \subset (a, b)$ and $V \subset (b, c)$, which implies that both $U$ and $V$ are open subsets of $\mathbb{R}$ and hence their union is open. In the second case, since $b \in U \subset (a, b]$, there is a half open interval $(x, b] \subset U$ for some $a < x < b$. Similarly, there is a half open interval $[b, y) \subset V$ for some $b < y < c$. But then the open interval $(x, y) \subset U \cup V$ contains $b$. Thus, every point in $U \cup V$, including $b$, belongs to an open interval that is contained therein, which, by Lemma 2.5, proves $U \cup V$ is open. **Q.E.D.**

**Theorem 5.3.** Suppose $f: (a, b] \to \mathbb{R}$ and $g: [b, c) \to \mathbb{R}$ are continuous and $f(b) = g(b) = z$. Then the combined function $h: (a, c) \to \mathbb{R}$ given by

\[
h(x) = \begin{cases} 
  f(x), & a < x < b, \\
  z, & x = b, \\
  g(x), & b < x < c,
\end{cases}
\]
is continuous. The same holds when one or both domains are closed intervals.

Proof: Let $S \subset \mathbb{R}$ be open, and let $U = f^{-1}(S)$, which is open in $(a, b]$, and $V = g^{-1}(S)$, which is open in $[b, c)$. If $z \notin S$, then $b \notin U$ and $b \notin V$; on the other hand, if $z \in S$, then $b \in U$ and $b \in V$. In either case, Lemma 5.2 implies $U \cup V = h^{-1}(S)$ is open in $(a, c)$, and hence $h$ is continuous. Q.E.D.

Remark: Under the set-up of the Stitch Theorem 5.3, if $f(b) \neq g(b)$, no matter how one defines $h(b)$, one cannot stitch the two functions together to form a continuous function on the combined interval $(a, c)$.

Example 5.4. The function $g(x) = x$ is continuous on the positive half line $(0, \infty]$. The function $f(x) = -x$ is continuous on the negative half line $[-\infty, 0)$. Moreover $f(0) = g(0) = 0$. Thus, Theorem 5.3 implies that the combined absolute value function $h(x) = \lfloor x \rceil$ is continuous on all of $\mathbb{R}$. Alternatively, one can write $\lfloor x \rceil = \max \{x, -x\}$ and use Theorem 4.3 to prove continuity.

There are, of course, many other ways to extend the function $g(x)$ to the negative half line while maintaining continuity, besides setting $h(x) = x$ or $\lfloor x \rceil$. For example, setting $f(x) \equiv 0$ for all $x \leq 0$ leads to the continuous extension

$$h(x) = \max \{0, x\} = \begin{cases} x, & x \geq 0, \\ 0, & x < 0, \end{cases} \quad (5.1)$$

known as the ramp function, or in modern data analysis of neural networks, the \textit{rectified linear unit} or \textit{ReLU function}, [11].

The last example indicates one how one can easily extend a continuous function defined on a closed interval to the entire real line. Its proof follows by applying Theorem 5.3 to both endpoints.

Proposition 5.5. Suppose $f: [a, b] \to \mathbb{R}$ is continuous. Then

$$h(x) = \begin{cases} f(a), & x < a, \\ f(x), & a \leq x \leq b, \\ f(b), & x > b, \end{cases} \quad (5.2)$$

defines a continuous function $h: \mathbb{R} \to \mathbb{R}$.

Of course, as before, (5.2) is but one of many possible extensions. More generally, suppose $A = [a, b] \cup [c, d]$ is the disjoint union of two closed intervals, with $a < b < c < d$. Then

$$h(x) = \begin{cases} f(a), & x < a, \\ f(x), & a \leq x \leq b, \\ \frac{f(c)(x-b) - f(b)(x-c)}{c-b}, & b < x < c, \\ f(x), & c \leq x \leq d, \\ f(d), & x > d, \end{cases} \quad (5.3)$$
defines a continuous function \( h: \mathbb{R} \to \mathbb{R} \). Here one has extended \( f \) by constant values on the two complementary unbounded intervals \((-\infty, a]\) and \([d, \infty)\), but on the intermediate interval \([b, c]\) one uses the linear interpolating function whose graph is the straight line passing through the points \((b, f(b))\) and \((c, f(c))\); see Figure 1. One can clearly adapt this construction to a finite collection of disjoint closed intervals, but it does not necessarily work on an infinite collection.

All results, both above and below, that were stated when the domain of the function is all of \( \mathbb{R} \) also hold, suitably rephrased, when the domain of \( f \) is an open interval. They also hold, provided care is taken at the endpoints, on half-open and closed intervals. However, they do not necessarily apply to more general subsets, which must be handled on their own merits, or lack thereof.

There are, of course, many other subsets \( A \subset \mathbb{R} \) besides intervals, and the continuity of functions \( f: A \to \mathbb{R} \) thereon is also of interest. A particularly important case is when the subset \( A \subset \mathbb{R} \) is a countably discrete set of points possessing a “limit point”.

**Example 5.6.** Consider the countably infinite closed (as per the remarks after Proposition 2.10) subset

\[
A = \{0\} \cup \tilde{A}, \quad \text{where} \quad \tilde{A} = \{1/n \mid n \in \mathbb{N}\}.
\]  

Let us determine the basic open subsets of \( A \). Let \( a < b \) be real numbers. If \( 0 < a < 1 \), then \((a, b) \cap A \) consists of a finite number of adjacent points \(1/n \in \tilde{A}\), namely those for which \(1/b < n < 1/a\). In particular, given \( n \in \mathbb{N} \), if we choose any \( a, b \) such that \( n - 1 < a < n < b < n + 1 \), then \((a, b) \cap A \) is the single point \( \{1/n\} \in A \). Thus, in this case, unlike \( \mathbb{R} \) or subintervals of \( \mathbb{R} \), singleton sets can be open! If \( a = 0 < b \) then \((a, b) \cap A \) consists of an infinite collection of adjacent points, namely those for which \( n > 1/b \). On the other hand, if \( a < 0 < b \) then \((a, b) \cap A \) consists of an infinite collection of adjacent points, namely those for which \( n > 1/b \), combined with the point 0. Finally if \( a \geq 1 \) or \( b \leq 0 \), then \((a, b) \cap A = \emptyset \). The general open subsets are unions of these basic ones.
In particular, any subset of \( \tilde{A} = \{ 1/n \mid n \in \mathbb{N} \} \), either finite or infinite, is open. On the other hand, the singleton set \{0\} is not open. Indeed, if the open set \( S \subset A \) contains 0, then it contains a basic open set \((a, b) \cap A \subset S\) for some \( a < 0 < b \), which means that it contains the infinite subset \( \tilde{A}_b = \{ 1/n \mid 1/b \leq n \in \mathbb{N} \} \subset S \). Note that \( S \) can also contain any collection of points in \( \tilde{A} \setminus \tilde{A}_b = \{ 1/n \mid 1/b > n \in \mathbb{N} \} \); since all subsets of \( \tilde{A} \) are open, this does not affect the openness of \( S \).

Consider a function \( f: A \to \mathbb{R} \), where \( A \) is as in (5.4). Continuity of \( f \) requires that \( f^{-1}(c, d) \subset A \) be open for any (bounded) open interval \((c, d)\). Let \( z = f(0) \). If \( z \not\in (c, d) \), then \( 0 \not\in f^{-1}(c, d) \) and, since any subset of \( \tilde{A} \) is open, there are no conditions imposed for the continuity of \( f \). On the other hand, if \( z \in (c, d) \), then continuity requires that \( f^{-1}(c, d) \) is an open subset of \( A \) containing 0, and hence, by the preceding paragraph, must contain an infinite subset of the form \( \tilde{A}_b = \{ 1/n \mid 1/b \leq n \in \mathbb{N} \} \) for some \( b > 0 \). In other words, \( f: A \to \mathbb{R} \) is continuous if, for every \( c < f(0) < d \), there exists a natural number \( m = m_{c,d} \in \mathbb{N} \) such that \( c < f(1/n) < d \) for all \( n \geq m \). Note that the number \( m \) depends on \( c, d \), and we expect that the smaller \( d - c \) is, the larger \( m \) needs to be.

Keep in mind that if \( f: \mathbb{R} \to \mathbb{R} \) is continuous, then its restriction to \( A \) is automatically continuous as a function \( f: A \to \mathbb{R} \). Conversely, given a continuous function \( f: A \to \mathbb{R} \), one can construct a continuous extension \( h: \mathbb{R} \to \mathbb{R} \) by adapting the construction in (5.3). Namely, on the interval \((1/(n + 1), 1/n)\) one uses the linear interpolating function between the values of \( f \) at the endpoints, and then extends \( h \) to be constant for \( x \geq 1 \) and for \( x < 0 \). In other words,

\[
h(x) = \begin{cases} 
  f(0), & x \leq 0, \\
  \frac{f(1/n)(x - 1/(n + 1)) - f(1/(n + 1))(x - 1/n)}{1/n - 1/(n + 1)} & \frac{1}{n + 1} \leq x < \frac{1}{n}, \\
  n f \left( \frac{1}{n} \right) \left( (n + 1)x - 1 \right) - (n + 1) f \left( \frac{1}{n + 1} \right) (nx - 1), & x \geq 1,
\end{cases}
\]

for all \( n \in \mathbb{N} \). The proof that this function is continuous on the entire real line is left as an exercise for the reader. We will return to this interesting and important example when we discuss limits in Section 6.

Let us use this example to motivate the definition of a limit point (also known as an accumulation point) of a set.

**Definition 5.7.** Let \( A \subset \mathbb{R} \). A point \( \ell \in \mathbb{R} \) is called a **limit point** of \( A \) if, whenever \( S \) is an open subset containing \( \ell \), we have \( A \cap (S \setminus \{ \ell \}) \neq \emptyset \).

We delete \( \ell \) from \( S \) in the limit point condition since otherwise every point in \( A \) would be a limit point, which is not what we want. It is, however, possible that some points in \( A \) are limit points. For example, if \( A = (a, b) \) is an open interval with \( a < b \) finite, then every
point in the closed interval \([a, b]\) is a limit point of \(A\). On the other hand, the singleton set \(\{a\}\) has no limit points.

One need only check the limit point condition when \(S = I\) is a (sufficiently small) interval containing \(l\). Thus, in the case of Example 5.6, \(l = 0\) is a limit point for the sets \(A\) and \(\tilde{A}\). Indeed, given any interval \(0 \in (a, b)\), so \(a < 0 < b\), then

\[
A \cap [(a, b) \setminus \{0\}] = \{1/n \mid 1/b < n \in \mathbb{N}\} \neq \emptyset.
\]

On the other hand, there are no other limit points, because if \(l \neq 0\), then any sufficiently small interval containing \(l\) either contains no points in \(A\), or, when \(l = 1/n \in A\), contains just one point in \(A\), namely \(l = 1/n\) itself.

**Lemma 5.8.** If \(A \subset B\) then every limit point of \(A\) is also a limit point of \(B\).

Of course, \(B\) may have more limit points than \(A\). One situation where this is not the case is the following:

**Lemma 5.9.** Let \(F \subset A\) be a finite subset. Then every limit point of \(A\) is a limit point of \(A \setminus F\) and vice versa.

Another fundamental property of the real line is the Limit Point Theorem, also known as the Bolzano–Weierstrass Theorem after two of the nineteenth century developers of modern rigorous mathematical analysis: the Czech precursor Bernard Bolzano and the influential German founder Karl Weierstrass.

**Theorem 5.10.** Let \(A \subset \mathbb{R}\) be a bounded infinite subset. Then the set of limit points of \(A\) is not empty.

**Proof:** Suppose first that \(A \subset [0, 1]\). We will construct the binary expansion of a limit point \(l = .b_1b_2b_3\ldots\), so that each binary digit \(b_i\) after the decimal point is either 0 or 1. First, we divide the interval into two half sized intervals, \([0, \frac{1}{2}]\) and \([\frac{1}{2}, 1]\). At least one of the two contains infinitely many points in \(A\). If it is the left hand one \([0, \frac{1}{2}]\), we choose it and record \(b_1 = 0\); if it is the right hand one \([\frac{1}{2}, 1]\), we choose it and record \(b_1 = 1\); if both do, choose either one and record the corresponding value for the first binary digit \(b_1\).

Note that any number \(x \in [0, 1]\) whose first binary digit is the recorded one must lie in the chosen interval. Next divide the chosen interval interval in half, so that each half has length 1/4. For example, if we choose \([\frac{1}{2}, 1]\), then the next two intervals are \([\frac{1}{4}, \frac{3}{4}]\) and \([\frac{3}{4}, 1]\). Again, choose one of the two that contains infinitely many points in \(A\). If it is the left half, record \(b_2 = 0\); if it is the right half record \(b_2 = 1\). Again, if the first two binary digits of \(0 \leq x \leq 1\) are as recorded, then \(x\) belongs to the chosen interval.

The process proceeds in the evident manner, and thus produces the binary expansion of \(0 \leq l \leq 1\). Note that if \(l = \frac{1}{2}\), the method could produce either of its binary expansions .10000 \ldots\) or .01111 \ldots depending on how the intervals are chosen. Similar remarks apply to any rational \(l\) of the form \(k2^{-n}\) for \(n, k \in \mathbb{N}\) with \(n \leq 2^k\). By the preceding
remarks, \( \ell \) belongs to all of the chosen intervals, of successive lengths \( 2^{-n} \) for \( n \in \mathbb{N} \). To prove that \( \ell \) is a limit point, let \( r > 0 \). All of the chosen intervals of lengths \( 2^{-n} < r \) for \( n \in \mathbb{N} \) contain \( \ell \), and hence are all contained in \((\ell - r, \ell + r)\), which has length \( 2r \). Since they all contain infinitely many points in \( A \), the same is true of \((\ell - r, \ell + r)\). But this holds for all \( r > 0 \), which proves that \( \ell \) is indeed a limit point of \( A \subset [0,1] \).

Now, in general, if \( A \subset [a, b] \), then there is at least one unit interval \([n, n+1]\), for some \( n \in \mathbb{N} \) satisfying \( a - 1 \leq n \leq b + 1 \), that contains infinitely many points in \( A \). Let \( B = \{a - n | n \leq a \leq n+1\} \subset [0,1] \). Since \( B \) is infinite, it contains a limit point \( \ell \in [0,1] \).

Then it is easily verified that \( l = n + \ell \) is a limit point for \( A \). \[ Q.E.D. \]

**Remark:** The result in Theorem 5.10 is false if \( A \) is unbounded. For example \( \mathbb{N} \) and \( \mathbb{Z} \) have no limit points.

**Proposition 5.11.** A subset \( C \subset \mathbb{R} \) is closed if and only if all its limit points \( l \) belong to it: \( l \in C \).

**Proof:** If \( C \) is closed, then \( S = \mathbb{R} \setminus C \) is open. If \( x \in S \), then, since \( S \cap C = \varnothing \), the point \( x \) does not satisfy the limit point criterion using \( S \) itself as the open set. Conversely, if every limit point of \( C \) belongs to \( C \), then given \( x \in S = \mathbb{R} \setminus C \), there must exist an open interval \( I_x \) containing \( x \) with \( I_x \cap C = \varnothing \), as otherwise \( x \) would be satisfy the limit point criterion. But this means \( I_x \subset S \), and hence Lemma 2.5 implies \( S \) is open. \[ Q.E.D. \]

**Warning:** While every limit point of a closed subset \( C \) is contained in it, not every point of \( C \) is a limit point. For example, the singleton set \( C = \{a\} \) has no limit points. Similarly, the only limit point of the closed set \( A \) in Example 5.6 is 0.

**Proposition 5.12.** Let \( A \subset \mathbb{R} \) and let \( L \) be the set of all its limit points. Then \( C = A \cup L \) is closed and in fact \( C = \overline{A} \) is the closure of \( A \).

**Proof:** If \( x \notin C \), then \( x \notin A \) and \( x \notin L \) is not a limit point of \( A \). This implies that there is an open interval \( I \) containing \( x \) such that \( I \cap C = \varnothing \) as otherwise \( x \) would either be in \( A \) or would be a limit point. But this implies \( \mathbb{R} \setminus C \) is open, and hence \( C \) is closed. Thus \( \overline{A} \subset C \) because the closure is the smallest closed subset containing \( A \). On the other hand, if \( \tilde{C} \supset A \) is any other closed subset, then Lemma 5.8 implies \( L \subset \tilde{C} \), and hence \( C \subset \tilde{C} \). Thus, \( C = \overline{A} \). \[ Q.E.D. \]

In particular, if \( C \) is a closed subset, Proposition 5.11 implies that it is its own closure \( C = \overline{C} \). Combining Propositions 5.11 and 5.12, we deduce that a limit point of limit points is itself a limit point. In other words, the set of limit points of \( C = A \cup L \) also equals \( L \), the set of limit points of \( A \).

The closure of an open interval \((a, b)\) as well as the corresponding half open intervals \([a, b)\) or \((a, b]\) is the closed interval \([a, b]\), which, by the above remarks, is its own closure. The only limit point of the set \( \tilde{A} \) in Example 5.6 is 0, and hence its closure is \( A = \tilde{A} \cup \{0\} \).
A particularly important case are the continuous functions on bounded closed intervals, also known as compact intervals. The next result, known as the Boundedness Theorem, says that, on such a domain, a continuous function cannot go off to infinity.

**Theorem 5.13.** Let $a < b$ be finite, and suppose that $f: [a, b] \to \mathbb{R}$ is continuous. Then $f$ is bounded, meaning that its range $f([a, b])$ is a bounded subset of $\mathbb{R}$, and so there exist $\alpha \leq \beta$ such that $\alpha \leq f(x) \leq \beta$ for all $x \in [a, b]$.

**Remark:** It is essential that the interval be closed and hence bounded for this result to be valid. For example, the continuous function $f(x) = x$ is not bounded on any unbounded interval. Similarly, the function $f(x) = 1/x$ is continuous but not bounded on the open interval $(0, 1)$ or on the half open interval $(0, 1]$.

**Proof:** Suppose that $f(x)$ is not bounded from above for $x \in [a, b]$. In this case, for every $n \in \mathbb{N}$ there is a point $x_n \in [a, b]$ such that $f(x_n) \geq n$. According to the Limit Point Theorem 5.10, the infinite set $\{x_n\}$ has a limit point $\ell \in [a, b]$. Let $c < f(\ell) < m \in \mathbb{N}$. Then, by continuity of $f$, the open subset $f^{-1}(c, m)$ contains $\ell$ and hence contains infinitely many points $x_n$. But this implies $f(x_n) < m$ for all $x_n \in f^{-1}(c, m)$, which contradicts our specification of the points $x_n$. The proof of the existence of a lower bound is similar. Q.E.D.

As a consequence of the preceding developments, we arrive at the Extreme Value Theorem, which states that a continuous function on a bounded closed interval is not only bounded from above and below, but actually achieves its maximum and minimum values, as well as every value in between. Again, the fact that the interval is both closed and bounded is essential; on open or unbounded intervals there are many counterexamples, as noted above.

**Theorem 5.14.** Let $a < b$ be real numbers. If $f: [a, b] \to \mathbb{R}$ is continuous, then its range is a closed interval: $f([a, b]) = [m, M]$, where $M = \sup f[a, b]$ and $m = \inf f[a, b]$. This means that, given any $m \leq y \leq M$, there exists at least one point $x \in [a, b]$ such that $f(x) = y$.

**Proof:** By definition of supremum, $f(x) \leq M$ for all $x \in [a, b]$. Suppose that $f(x) < M$ for all $x \in [a, b]$. Then the function $g(x) = M - f(x) > 0$ for all $x \in [a, b]$. Thus, by Theorem 4.3, the function $h(x) = 1/g(x)$ is continuous. The Boundedness Theorem 5.13 implies $h(x) \leq \beta$ for all $x \in [a, b]$ for some $0 < \beta \in \mathbb{R}$. But this implies $g(x) \geq 1/\beta$, and hence $f(x) = M - g(x) \leq M - 1/\beta$ for all $x \in [a, b]$. Thus, $M - 1/\beta < M$ is an upper bound for $f[a, b]$, which contradicts the assumption that $M$ is the least upper bounds. A similar proof works for the greatest lower bound $m$; alternatively one can replace $f(x)$ by $\tilde{f}(x) = -f(x)$ whose least upper bound is $-m$. In this manner, we have proved the existence of at least one point $x_M \in [a, b]$ with $f(x_M) = M$ and at least one point $x_m \in [a, b]$ with $f(x_m) = m$. The final statement of the theorem then follows from the Intermediate Value Theorem 4.6. Q.E.D.
On the other hand, suppose \( f: (a, b) \to \mathbb{R} \) is continuous on an open interval. The Intermediate Value Theorem 4.6 combined with Lemma 2.13 implies that its range \( f(a, b) \) is an interval, but, unless \( f \) is one-to-one in which case Theorem 4.8 applies, then it is not necessarily true that the image \( f(a, b) \) is an open interval. For example, if \( f(x) = x^2 \) and \( a < 0 < b \) then the image of the open interval \((a, b)\) is the half open interval \([0, c) = f(a, b)\) where \( c = \max \{a^2, b^2\} \).


Interestingly, we will reverse the traditional approach to the subject, and define limits using continuity rather than the other way around. While it would thus be possible to avoid limits entirely in the presentation, the language and intuition remains useful for assimilating the continuity-based constructions.

**Definition 6.1.** Let \( f: D_a \to \mathbb{R} \) be continuous on the punctured domain \( D_a = \mathbb{R} \setminus \{a\} \) for some \( a \in \mathbb{R} \). We say that \( f(x) \) has limiting value \( z \in \mathbb{R} \) at \( x = a \) if the extended function

\[
\hat{f}: \mathbb{R} \to \mathbb{R} \quad \text{given by} \quad \hat{f}(x) = \begin{cases} f(x), & x \neq a, \\ z, & x = a, \end{cases}
\]

is continuous, in which case we write

\[
\lim_{x \to a} f(x) = z.
\]

The Stitch Theorem 5.3 and the remark following it imply that the limiting value of the function \( f \), if it exists, is unique. In other words, there is at most one possible value \( z \) that makes the the extended function (6.1) continuous.

In this case, one says that such a function \( f(x) \) has a *removable discontinuity* at \( x = a \) because one can remove the apparent discontinuity by appropriately defining its value at \( a \), namely by setting \( f(a) = z \). In such cases, we will often, to avoid notational clutter, use the same symbol \( f \) for the function and its extension. In particular, if \( f: \mathbb{R} \to \mathbb{R} \) is continuous, then

\[
\lim_{x \to a} f(x) = f(a)
\]

at all points \( a \in \mathbb{R} \). Many discontinuities are not removable, examples being the discontinuity at \( x = 0 \) of the functions \( \text{sign} x, 1/x, \) and \( \sin(1/x) \), for example.

**Example 6.2.** Consider the rational function

\[
f(x) = \frac{x^2 - 1}{x - 1},
\]

which, ostensibly, has a discontinuity at \( x = 1 \) because its denominator vanishes there. However, the discontinuity is removable, with

\[
\lim_{x \to 0} f(x) = 1.
\]
Indeed, factorizing the numerator, one immediately finds that, when \( x \neq 0 \),

\[
f(x) = \frac{(x - 1)(x + 1)}{x - 1} = x + 1,
\]

and hence the extended continuous function is merely \( \hat{f}(x) = x + 1 \) with \( \hat{f}(0) = 1 \).

On the other hand, the singularity of the rational function

\[
f(x) = \frac{x^2 + 1}{x - 1}
\]

at \( x = 1 \) is not removable. Indeed, we can write

\[
f(x) = x + 1 + \frac{2}{x - 1}
\]

and the last term is unbounded as \( x \) tends to 1.

One can also consider one-sided limits at a point. Let \( a < b \). If \( f: (a, b) \to \mathbb{R} \) is continuous, and \( a \) is finite, the extended function

\[
\hat{f}: [a, b) \to \mathbb{R} \quad \text{given by} \quad \hat{f}(x) = \begin{cases} f(x), & a < x < b, \\ z, & x = a, \end{cases}
\]

is continuous on the indicated half-open interval, then we say that \( z \) is the left-handed limit of \( f \) at the point \( a \), and write

\[
\lim_{x \to a^+} f(x) = z. \tag{6.4}
\]

Similarly, if

\[
\tilde{f}: (a, b] \to \mathbb{R} \quad \text{given by} \quad \tilde{f}(x) = \begin{cases} f(x), & a < x < b, \\ z, & x = b, \end{cases}
\]

is continuous, then we say that \( z \) is the right-handed limit of \( f \) at the point \( b \), and write

\[
\lim_{x \to b^-} f(x) = z. \tag{6.5}
\]

As a consequence of Lemma 5.2, we deduce that a function maintains continuity at a point if and only if its left- and right-handed limits are the same there:

**Lemma 6.3.** Let \( a < b < c \). If \( f: (a, b] \to \mathbb{R} \) and \( g: [b, c) \to \mathbb{R} \) are continuous on their respective domains, then the combined function

\[
h(x) = \begin{cases} f(x), & a < x < b, \\ z, & x = b, \\ g(x), & b < x < c, \end{cases}
\]

is continuous if and only if \( z = f(a) = g(b) \). In other words,

\[
\lim_{x \to b^-} h(x) = z \quad \text{if and only if} \quad z = \lim_{x \to b^-} h(x) = \lim_{x \to b^+} h(x).
\]

The next result is known as the Squeeze Theorem, since it says that a function squeezed between two continuous functions at a point must itself be continuous there. It is very useful for analyzing removable discontinuities and limits.
**Theorem 6.4.** Suppose $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$ are continuous, while $h: D_a \to \mathbb{R}$ is continuous on the punctured domain $D_a = \mathbb{R} \setminus \{a\}$ for some $a \in \mathbb{R}$. Suppose in addition that $f(x) \leq h(x) \leq g(x)$ for all $x \neq a$, and, further, $f(a) = g(a)$. Then $a$ is a removable discontinuity of $h$, with

$$\lim_{x \to a} h(x) = f(a) = g(a).$$

**Proof:** We set

$$\tilde{h}(x) = \begin{cases} h(x), & x \neq a, \\ z, & x = a, \end{cases}$$

to be the extension of $h(x)$ with $\tilde{h}(x) = z = f(a) = g(a)$. We need to prove that $\tilde{h}$ is continuous. Given an open subset $S \subset \mathbb{R}$, if $z \not\in S$, then $\tilde{h}^{-1}(S) = h^{-1}(S)$ is open by continuity of $h$. On the other hand, suppose $z \in S$. Let $(\alpha, \beta) \subset S$ be an open interval containing $z$, so that $\alpha < z < \beta$. Consider the open set $U = f^{-1}(\alpha, \beta) \cap g^{-1}(\alpha, \beta)$. Note that $a \in U$ since $f(a) = g(a) = z \in (\alpha, \beta)$. If $x \in U$, then $\alpha < f(x) < \beta$ and $\alpha < g(x) < \beta$, and therefore,

$$\alpha < f(x) \leq \tilde{h}(x) \leq g(x) < \beta$$

for all $x \in U$, including $x = a$. Thus $U \subset \tilde{h}^{-1}(\alpha, \beta)$, and hence $\tilde{h}^{-1}(S) = h^{-1}(S) \cup U$ is open, thereby proving continuity of $\tilde{h}$.

**Q.E.D.**

**Remark:** The result remains valid if the inequality only holds on an open interval $I$ that contains the point $a \in I$. The inequalities can, of course, be strict when $x \neq a$ without affecting the result.

Another option is to do a one-sided squeeze, in which one imposes the conditions of Theorem 6.4 on only one side of the point $a$, leading to the right or left hand limit of the squeezed function $h$. As usual, if they agree, then $h$ has their common value as its limit.

**Example 6.5.** Consider the following trigonometric ratio:

$$f(x) = \frac{\sin x}{x},$$

which is defined for $x \neq 0$. We claim that $x = 0$ is a removable discontinuity with limiting value

$$\lim_{x \to 0} f(x) = 1. \quad (6.6)$$

In Figure 2, we plot the unit circle, centered at the origin $O = (0,0)$ and a line at angle $0 < x < \frac{1}{2}\pi$ with respect to the horizontal, which intersects the unit circle at the point $B = (\cos x, \sin x)$. The point $C = (1, \tan x)$ also lies on this line, directly above $A = (1, 0)$. The area of the triangle $OAB$, which has base 1 and height $\sin x$ is $\frac{1}{2}\sin x$. This triangle is contained in the circular sector (pizza slice) bounded by the two line segments $OA$ and

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† See Section 12 for a full discussion of the concept of area.
Figure 2. Sine and Tangent Inequalities.

$OB$ and the circular arc from $A$ to $B$. Thus, the area of the triangle $OAB$ is strictly less than the area of the circular sector, which is $\frac{1}{2}x$, since it forms a fraction of $x/(2\pi)$ of the area of the unit disk, which is $\pi$. Finally, the circular sector is contained in the triangle $OAC$ and hence its area is strictly less than the area of this right triangle, which is $\frac{1}{2}\tan x$. Combining the preceding deductions, and cancelling the common factor of $\frac{1}{2}$, we have established the following inequalities:

$$\sin x < x < \tan x = \frac{\sin x}{\cos x} \quad \text{for} \quad 0 < x < \frac{1}{2}\pi. \quad (6.7)$$

Rearranging these two inequalities produces

$$\cos x < \frac{\sin x}{x} < 1. \quad (6.8)$$

 Initially, owing to (6.7), the inequalities (6.8) are only valid for $0 < x < \frac{1}{2}\pi$, but the first two functions are even and so replacing $x$ by $-x$ does not change either inequality. Thus, (6.8) is valid for $-\frac{1}{2}\pi < x < \frac{1}{2}\pi$. Moreover, both $\cos x$ and the constant function 1 are continuous and have value 1 at $x = 0$. Thus, the Squeeze Theorem 6.4 implies (6.6).

**Example 6.6.** Consider the trigonometric function

$$f(x) = x \sin \frac{1}{x},$$

which is defined and continuous for $x \neq 0$. Since $|\sin(1/x)| \leq 1$ for $x \neq 0$, we have

$$-|x| \leq f(x) \leq |x| \quad \text{for all} \quad x \neq 0.$$

The functions $\pm |x|$ are both continuous, as noted above, and have limiting value 0 at $x = 0$. Thus, applying the Squeeze Theorem 6.4, we conclude that

$$\lim_{x \to 0} f(x) = 0. \quad (6.9)$$
The above definition of limit adapts immediately to the case when the function is defined on a general subset \( A \subset \mathbb{R} \); just replace \( \mathbb{R} \) by \( A \) in Definition 6.1. If \( A \) is an open interval, the limiting behavior is identical to the real line situation. If \( A \) is a half open or closed interval, the limits at the endpoint(s) reduce to the appropriate left- and right-handed limits.

**Example 6.7.** Finally, let us look at the discrete setting of Example 5.6:

\[
A = \{ 0 \} \cup \tilde{A}, \quad \text{where} \quad \tilde{A} = \{ 1/n \mid n \in \mathbb{N} \}.
\]

We can identify a function \( f: \tilde{A} \to \mathbb{R} \) with a sequence, setting \( f_n = f(1/n) \) for \( n \in \mathbb{N} \). First of all, since any subset of \( \tilde{A} \) is open, any function \( f: \tilde{A} \to \mathbb{R} \) is continuous, and so, unlike the case of functions on \( \mathbb{R} \) or subintervals thereof, the limit at any point \( 1/n \in \tilde{A} \) is *not* uniquely defined. In other words, given \( n \in \mathbb{N} \) and a function \( \tilde{f}: \tilde{D}_{1/n} \to \mathbb{R} \) on the punctured domain \( \tilde{D}_{1/n} = \tilde{A} \setminus \{ 1/n \} \), *any* value \( z = f(1/n) \in \mathbb{R} \) will make the extended function \( \tilde{f}: \tilde{A} \to \mathbb{R} \) continuous, and hence satisfy the condition \( \lim_{x \to 1/n} f(x) = z \) on \( \tilde{A} \). This is strange, and will not be employed in the sequel.

More interesting is what happens at the limit point \( \ell = 0 \). Here we say the limit of the sequence \( \{ f_n \mid n \in \mathbb{N} \} \) defined by \( f: A \setminus \{ 0 \} = \tilde{A} \to \mathbb{R} \) is

\[
\lim_{n \to \infty} f_n = \lim_{1/n \to 0} f(1/n) = z
\]

(6.10) if the extended function

\[
\tilde{f}: A \to \mathbb{R} \quad \text{given by} \quad \tilde{f}(x) = \begin{cases} f(x), & x \in \tilde{A}, \\ z, & x = 0, \end{cases}
\]

(6.11) is continuous on \( A \). In this case, the limiting value, if it exists, is unique. Indeed, suppose, to the contrary, that \( z \neq w \) and both

\[
\tilde{f}(x) = \begin{cases} f(x), & x \in \tilde{A}, \\ z, & x = 0, \end{cases} \quad \tilde{f}(x) = \begin{cases} f(x), & x \in \tilde{A}, \\ w, & x = 0, \end{cases}
\]

are continuous. Choose disjoint open intervals, \( I \cap J = \emptyset \), with \( z \in I \) and \( w \in J \). Then, by continuity, both \( \tilde{f}^{-1}(I) \) and \( \tilde{f}^{-1}(J) \) are open and contain 0. This means that there exist \( m_1, m_2 \in \mathbb{N} \), such that \( M_1 = \{ 1/n \mid m_1 \leq n \in \mathbb{N} \} \subset \tilde{f}^{-1}(I) \) and \( M_2 = \{ 1/n \mid m_2 \leq n \in \mathbb{N} \} \subset \tilde{f}^{-1}(J) \). But if \( n \geq \max\{m_1, m_2\} \) then \( 1/n \in M_1 \cap M_2 \) and hence \( \tilde{f}(1/n) \in I \), and \( \tilde{f}(1/n) \in J \). However, \( \tilde{f}(1/n) = \tilde{f}(1/n) = f(1/n) \), which contradicts the assumption that \( I \cap J = \emptyset \).

Thus, the continuity characterization of limits includes the limiting behavior of a sequence of real numbers \( f_1, f_2, f_3 \ldots \in \mathbb{R} \), for which we identify \( f_n = f(1/n) \) with the values of a function on the points \( 1/n \in \tilde{A} \). Alternatively, we can identify the sequence with a suitably continuous extension of the function \( f: \tilde{A} \to \mathbb{R} \), defining a continuous function \( h: \mathbb{R}^+ \to \mathbb{R} \) such that \( h(1/n) = f(1/n) \) for \( n \in \mathbb{N} \). The sequence has a limit \( z \in \mathbb{R} \).
as in (6.10), if and only if one can extend \( h \) to a continuous function \( \tilde{h}: \mathbb{R} \to \mathbb{R} \) such that \( \tilde{h}(x) = h(x) \) for all \( x > 0 \) while \( \tilde{h}(x) = z \) for all \( x \leq 0 \), cf. (5.5).

It is also worth noting that this construction extends to other countably infinite subsets with one or more limit points. If a continuous function defined on such a subset can be extended to a continuous function on the closure — the union of the set and all its limit points — then the extension serves to define the limiting values of the original function at all the limit points. Details are left to the reader. We also leave it for the reader to speculate and/or investigate what the continuity and limiting behavior of functions \( f: \mathbb{Q} \to \mathbb{R} \) on the rational numbers might look like.

**Example 6.8.** Consider the function

\[
f(x) = \sin \frac{\pi}{x}, \quad x \neq 0,
\]

which is continuous on the punctured line \( \mathbb{R} \setminus \{0\} \). The corresponding sequence is \( f_n = f(1/n) = \sin \pi n = 0 \) for all \( n \in \mathbb{N} \), which is identically zero and hence has limit 0 at \( x = 0 \). However, there is no continuous extension of \( f(x) \) to \( x = 0 \), which is not a removable discontinuity. Indeed, suppose \( f(0) = z > -1 \). Let \( c > \max \{1, z\} \). Then \( A = f^{-1}(-1, c) = \mathbb{R} \setminus \{1/(2n + \frac{3}{2}) \mid n \in \mathbb{Z}\} \), which is not open because \( 0 \in A \) but there is no open interval \( I \subset A \) that contains 0. A similar construction applies when \( z < 1 \).

7. Derivatives.

We are now in a position to define the derivative directly using continuity instead of via the usual limiting process — although of course according to the previous section they are equivalent concepts. We begin with the simplest case of the derivative of a function at the origin.

**Definition 7.1.** Let \( f: \mathbb{R} \to \mathbb{R} \) be continuous and satisfy \( f(0) = 0 \). We say that \( f(x) \) has a derivative at \( x = 0 \) if there exists \( z \in \mathbb{R} \) such that the function

\[
q(x) = \begin{cases} 
  f(x)/x, & x \neq 0, \\
  z, & x = 0,
\end{cases}
\]

is continuous. In this case we write \( z = f'(0) \).

According to Theorem 4.3, the quotient \( q(x) = f(x)/x \) is continuous except at \( x = 0 \). The existence of a derivative implies that it has a removable discontinuity at \( x = 0 \), and hence by extending it by setting \( q(0) = f'(0) \) produces a continuous function on all of \( \mathbb{R} \).

**Example 7.2.** The function \( f(x) = 3x \) satisfies \( f(0) = 0 \). To find its derivative, we note that, for \( x \neq 0 \),

\[
q(x) = \frac{f(x)}{x} = \frac{3x}{x} = 3.
\]

Thus, \( x = 0 \) is a removable discontinuity, and we immediately conclude that \( f'(0) = 3 \).
Next consider the function \( f(x) = x^2 \). In this case
\[
q(x) = \frac{f(x)}{x} = \frac{x^2}{x} = x,
\]
again initially for \( x \neq 0 \), but the limiting value that makes \( q(x) \) continuous at \( x = 0 \) is \( q(0) = 0 = f'(0) \). The same clearly holds for any power function \( f(x) = x^n \) for \( 2 \leq n \in \mathbb{N} \).

We can similarly define derivatives at other points for functions with nonzero values. Suppose \( f(a) = c \). In Definition 7.1, we replace \( f(x) \) by \( f(x) - f(a) = f(x) - c \), which vanishes when \( x = a \); similarly we replace \( x \) by \( x - a \) which has the same property. In this manner, we arrive at the definition of the derivative.

**Definition 7.3.** Let \( f : \mathbb{R} \to \mathbb{R} \) be continuous. We say that \( f(x) \) has a derivative at \( a \in \mathbb{R} \) if there exists \( z \in \mathbb{R} \) such that the difference quotient function
\[
q(x) = \begin{cases} 
\frac{f(x) - f(a)}{x - a}, & x \neq a, \\
z, & x = a.
\end{cases}
\]
is continuous. In this case, we write \( f'(a) = z = q(a) \), or, equivalently
\[
f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.
\]

**Remark:** The preceding definition of derivative via continuity of the difference quotient function \( q(x) \) dates back to the early twentieth century Greek mathematician Constantin Carathéodory, who states an equivalent version in his textbook on complex analysis, [3], namely, that \( f : \mathbb{R} \to \mathbb{R} \) has a derivative at \( a \in \mathbb{R} \) if there exists a continuous function \( q : \mathbb{R} \to \mathbb{R} \) such that
\[
f(x) = f(a) + q(x) (x - a).
\]
Again, \( f'(a) = q(a) \). Clearly, if (7.4) holds, \( f(x) \), being a combination of continuous functions and constants, if itself necessarily continuous. Interestingly, Carathéodory does not use this definition in his real analysis text, [4]; see also [10]. Also, he initially defines continuity through limits, and not in the topological terms we use here.

If \( f(x) \) has a derivative at all points in its domain, we say \( f \) is *differentiable* and write \( f'(x) \) for the value of its derivative at the point \( x \).

**Example 7.4.** The preceding function \( f(x) = 3x \) has constant derivative: \( f'(x) = 3 \) at all \( x \in \mathbb{R} \). Indeed, the difference quotient is
\[
\frac{f(x) - f(a)}{x - a} = \frac{3x - 3a}{x - a} = 3 \quad \text{when} \quad x \neq a,
\]
and hence has limiting value \( f'(a) = 3 \).

**Example 7.5.** The derivative of any constant function \( f(x) = c \) is zero everywhere. Indeed,
\[
\frac{f(x) - f(a)}{x - a} = \frac{c - c}{x - a} = 0 \quad \text{when} \quad x \neq a.
\]
Later we will see that these are essentially the only functions with identically zero derivative.

**Example 7.6.** Let \( f(x) = \sin x \). The difference quotient at \( x = 0 \) is \( q_0(x) = \sin x/x \), whose limit \( q(0) = 1 = f'(0) \) we computed in Example 6.5. Thus the derivative of \( \sin x \) at \( x = 0 \) is 1. At a general point \( x = a \), the difference quotient can be rewritten using a standard trigonometric identity:

\[
q_a(x) = \frac{\sin x - \sin a}{x-a} = \frac{2 \cos \frac{1}{2}(x + a) \sin \frac{1}{2}(x - a)}{x-a}
= \frac{\sin \frac{1}{2}(x-a)}{\frac{1}{2}(x-a)} \cos \frac{1}{2}(x + a) = q_0 \left[ \frac{1}{2}(x-a) \right] \cos \frac{1}{2}(x + a).
\]

Thus, at the point \( x = a \),

\[
f'(a) = q_a(a) = q_0(0) \cos a = \cos a.
\]

Replacing \( a \) by \( x \) we conclude that the derivative of \( \sin x \) is \( \cos x \). Similarly, to compute the derivative of \( \cos x \), we write the corresponding difference quotient

\[
\frac{\cos x - \cos a}{x-a} = \frac{-2 \sin \frac{1}{2}(x + a) \sin \frac{1}{2}(x - a)}{x-a} = -q_0 \left[ \frac{1}{2}(x-a) \right] \sin \frac{1}{2}(x + a),
\]

which has the limiting value \( -\sin a \) at \( x = a \), and hence the derivative of \( \cos x \) is \( -\sin x \).

**Example 7.7.** As note above, the absolute value function \( f(x) = |x| \) is continuous when restricted to \( \mathbb{R}^+ \) it has the derivative \( f'(a) = +1 \), for all \( a > 0 \), because the quotient

\[
\frac{f(x) - f(a)}{x-a} = \frac{x-a}{x-a} = 1, \quad \text{provided} \quad a > 0, \quad x > 0, \quad x \neq a,
\]

On the other hand,

\[
\frac{f(x) - f(a)}{x-a} = \frac{-x+a}{x-a} = -1, \quad a \neq x < 0,
\]

and hence \( f'(a) = -1 \) when \( a < 0 \). Thus, on \( \mathbb{R} \setminus \{0\} \), the derivative of the absolute value function is the sign function, \( f'(x) = \text{sign} x \) when \( x \neq 0 \). However, the absolute value function does not have a derivative at \( a = 0 \) because the difference quotient

\[
q(x) = \frac{|x|}{x} = \text{sign} x
\]

does not have a removable discontinuity there; see the discussion following (4.1). Thus, the derivative of a continuous function, even when it exists, need not be continuous.

After one learns the basic rules for differentiation, the derivative of any combination of elementary functions can be calculated directly in terms of the derivatives of their constituents without the need to revert to the difference quotient. Included in each of the following Rules is the statement that if the function \( f \) and \( g \) are differentiable at a point \( a \) then the prescribed combination is also differentiable at \( a \) and has the indicated formula.
Continuity of the required combinations follows from Theorem 4.3 (which remains to be proved). Under these assumptions, as in (7.2), we set

\[ F(x) = \begin{cases} \frac{f(x) - f(a)}{x - a}, & x \neq a, \\ f'(a), & x = a, \end{cases} \quad G(x) = \begin{cases} \frac{g(x) - g(a)}{x - a}, & x \neq a, \\ g'(a), & x = a. \end{cases} \tag{7.5} \]

**Sum Rule:** The first rule is that the derivative of the sum of two functions equals the sum of their derivatives. Given

\[ s(x) = f(x) + g(x), \quad \text{then} \quad s'(a) = f'(a) + g'(a). \tag{7.6} \]

Indeed, since

\[ \frac{s(x) - s(a)}{x - a} = \frac{f(x) + g(x) - f(a) - g(a)}{x - a} = \frac{f(x) - f(a)}{x - a} + \frac{g(x) - g(a)}{x - a} = F(x) + G(x). \]

The right hand side is continuous, being the sum of two continuous functions, with limiting value \( F(a) + G(a) = f'(a) + g'(a) \) at \( x = a \). Thus, the left hand side is also continuous with limiting value \( s'(a) \), which implies (7.6) holds. The same holds for their difference, with an identical proof. Setting

\[ d(x) = f(x) - g(x), \quad \text{implies} \quad d'(a) = f'(a) - g'(a). \tag{7.7} \]

**Product Rule:** The derivative of the product of two functions is *not* the product of their derivatives! The correct formula says that the derivative of

\[ p(x) = f(x)g(x), \quad \text{is} \quad p'(a) = f'(a)g(a) + f(a)g'(a). \tag{7.8} \]

The proof follows from the algebraic formula

\[ \frac{p(x) - p(a)}{x - a} = \frac{f(x)g(x) - f(a)g(a)}{x - a} = \frac{f(x) - f(a)}{x - a} g(x) + f(a) \frac{g(x) - g(a)}{x - a} = F(x)g(x) + f(a)G(x). \]

The right hand side is continuous with limiting value

\[ F(a)g(a) + f(a)G(a) = f'(a)g(a) + f(a)g'(a) \]

at \( x = a \), thereby establishing (7.8).

**Reciprocal Rule:** The next formula is for the reciprocal of a nonzero continuous function. Let

\[ r(x) = \frac{1}{g(x)} \quad \text{with} \quad g(x) \neq 0. \quad \text{Then} \quad r'(a) = -\frac{g'(a)}{g(a)^2}. \tag{7.9} \]

To prove this, use the identity

\[ \frac{r(x) - r(a)}{x - a} = \frac{1}{g(x) - g(a)} = -\frac{1}{g(x)g(a)} \frac{g(x) - g(a)}{x - a} = -\frac{G(x)}{g(x)g(a)}, \]

which is valid provided \( x \neq a \) and \( g(x) \neq 0, \ g(a) \neq 0. \) The limiting value of the final expression at \( x = a \) is \(-g'(a)/g(a)^2\), thus producing the final formula (7.9).
Thus, the formula for \( g \)

\[ q(x) = \frac{f(x)}{g(x)} \quad \text{with} \quad g(x) \neq 0. \]

Then

\[ q'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{g(a)^2}. \]

This is proved by combining the product rule (7.8) and the reciprocal rule (7.9). Indeed, \( q(x) = f(x) r(x) \), and hence

\[ q'(a) = f'(a) r(a) + f(a) r'(a) = \frac{f'(a)}{g(a)} - f(a) \frac{g'(a)}{g(a)^2} = \frac{f'(a)g(a) - f(a)g'(a)}{g(a)^2}. \]

**Composition or Chain Rule:** Finally consider the composition of continuous functions:

\[ c(x) = f \circ g(x) = f(g(x)) \quad \text{Then} \quad c'(a) = f'(g(a)) g'(a). \quad (7.10) \]

To prove this we use the formula

\[ \frac{c(x) - c(a)}{x - a} = \frac{f(g(x)) - f(g(a))}{x - a} = \frac{f(g(x)) - f(g(a))}{g(x) - g(a)} \frac{g(x) - g(a)}{x - a}, \]

which is valid provided \( g(x) \neq g(a) \). Since the function \( F(x) \) given in (7.5) is continuous, its composition with \( g(x) \), namely

\[ H(x) = F \circ g(x) = \begin{cases} \frac{f(g(x)) - f(g(a))}{g(x) - g(a)}, & g(x) \neq g(a), \\ f'(g(a)), & g(x) = g(a), \end{cases} \]

is also continuous. But this implies continuity of

\[ \frac{c(x) - c(a)}{x - a} = H(x) G(x), \quad \text{with limit} \quad c'(a) = H(a) G(a) = f'(g(a)) g'(a), \]

as claimed.

As a simple application, we find that, for any \( a \in \mathbb{R} \),

\[ g(x) = f(x + a) \quad \text{has derivative} \quad g'(x) = f'(x + a), \\ h(x) = f(a - x) \quad \text{has derivative} \quad h'(x) = -f'(a - x). \quad (7.11) \]

To see this, note that \( g = f \circ l \) where \( l(x) = x + a \) has derivative \( l'(x) = 1 \) everywhere. Thus, the formula for \( g'(x) = f'(l(x)) l'(x) = f'(x + a) \) follows immediately from the Chain Rule. A similar calculation establishes the second formula in (7.11).

**Example 7.8.** Let us prove that

the derivative of \( f(x) = x^n \) is \( f'(x) = n x^{n-1} \) for \( n \in \mathbb{N} \). \( (7.12) \)

We will prove this by induction, with the cases \( n = 0, 1 \) already known. Given that (7.12) holds for a given \( n \), consider the function

\[ g(x) = x^{n+1} = x x^n = x f(x). \]

Applying the Product Rule (7.8) and the fact that the derivative of \( x \) is 1,

\[ g'(x) = f(x) + x f'(x) = x^n + x (n x^{n-1}) = (n + 1) x^n, \]
thus completing the induction step. More generally, let
\[ p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0 \]
where \( a_0, \ldots, a_n \) are constants, be a polynomial of degree \( n \). Then we can repeatedly employ the Sum and Product Rules to compute its derivative:
\[ p'(x) = n a_n x^{n-1} + (n-1) a_{n-1} x^{n-1} + \cdots + 2 a_2 x + a_1, \]
which is a polynomial of degree \( n - 1 \).

**Theorem 7.9.** Suppose \( f: \mathbb{R} \to \mathbb{R} \) is a continuous strictly monotone function, with continuous inverse \( g = f^{-1}: \mathbb{R} \to \mathbb{R} \). If \( f'(x) \neq 0 \), then \( g \) is differentiable at \( y = f(x) \) and
\[ g'(y) = \frac{1}{f'(g(y))}. \tag{7.13} \]

*Proof:* By definition of inverse,
\[ g \circ f(x) = x. \]
Taking the derivative of both sides, and using the Chain Rule (7.10), yields
\[ g'[ f(x) ] f'(x) = 1. \]
Under the assumption \( f'(x) \neq 0 \) we can divide both sides of the latter equation by it. Replacing \( x = g(y) \), so \( y = f(x) \), produces (7.13).

Q.E.D.

For example, consider the function \( f(x) = x^3 \). Its inverse is the function \( g(y) = y^{1/3} \). Since \( f'(x) = 3x^2 \), we find
\[ g'(y) = \frac{1}{3x^2} = \frac{1}{3} y^{-2/3}, \]
provided \( y \neq 0 \). In fact, \( g(y) = y^{1/3} \) is not differentiable at \( y = 0 \) since the quotient function \( q(y) = g(y)/y = y^{-2/3} \) is unbounded there. Note that this formula is in accordance with our earlier rule (7.12), where in this case we take \( n = \frac{1}{3} \). Indeed, it can be shown that the rule holds for all \( n \in \mathbb{R} \), where we exclude \( x = 0 \) when \( n < 1 \). Of course, when \( n = 0 \), we have \( x^0 = 1 \), with derivative 0 for all \( x \). More generally, by applying the Chain Rule (7.10), we deduce that if \( f(x) \) is differentiable, then its \( n \)th power
\[ F(x) = [ f(x) ]^n \]
is differentiable provided, when \( 0 \neq n < 1 \), that \( f(x) \neq 0 \), with
\[ F'(x) = n [ f(x) ]^{n-1} f'(x). \tag{7.14} \]

Given \( b \neq a \), the value of the difference quotient \( q(b) = \frac{f(b) - f(a)}{b - a} \) equals the slope of the secant line passing through the two points \((a, f(a))\) and \((b, f(b))\) on the graph of \( f \), which can be identified as the graph of the linear function
\[ s_b(x) = \frac{f(b) - f(a)}{b - a} (x - a) + f(a). \tag{7.15} \]
As $b \to a$, the difference quotient has the limiting value $q(a) = f'(a)$, and so the secant line functions (7.15) converge, in a continuous manner, to the linear function

$$t(x) = f'(a)(x - a) + f(a),$$

whose graph can be identified with the tangent line to the graph of $f$ at the point $(a, f(a))$. This is illustrated in Figure 3. In other words, as $b$ gets closer and closer to $a$, the secant lines approach the tangent line, whose slope equals the derivative $f'(a)$.

If $f: [a, b] \to \mathbb{R}$ is a function on a closed interval, then one can introduce the notions of left- and right-handed derivatives at the endpoints. Specifically, we define the left-handed derivative $f'(a^+)$ at the right hand endpoint $a$ to be the value (if such exists) that makes the extended function

$$F_a(x) = \begin{cases} 
\frac{f(x) - f(a)}{x - a}, & a < x \leq b, \\
 f'(a^+), & x = a,
\end{cases}$$

continuous on $[a, b]$. Similarly, we define the right-handed derivative $f'(b^-)$ at the left hand endpoint $b$ as the value that makes

$$F_b(x) = \begin{cases} 
\frac{f(x) - f(b)}{x - b}, & a \leq x < b, \\
 f'(b^-), & x = b,
\end{cases}$$

continuous on $[a, b]$. This extends in the obvious manner to half open intervals. Applying Lemma 6.3, if $f$ is defined on an open interval containing $a$, then $f'(a)$ exists if and only if $f'(a^+) = f'(a^-)$, in which case all three are equal.

The next two results state that the derivative of a function at a (local) minimum or maximum must be zero, provided it exists. We start with a simple version.
Figure 4. Tangent Lines at Minimum and Maximum.

Lemma 7.10. Suppose $f: \mathbb{R} \to \mathbb{R}$ is continuous, $f(0) = 0$, and $f(x) \geq 0$ for all $x \neq 0$. If $f$ is differentiable at 0 then $f'(0) = 0$.

Proof: As in the definition, set $q(x) = f(x)/x$. If $f(x) > 0$ for all $x \neq 0$, then $q(x) > 0$ for $x > 0$ while $q(x) < 0$ for $x < 0$. By the Intermediate Value Theorem 4.6, there is a point $c$ where $q(c) = 0$, but the only possibility is $c = 0$ since $q(x) \neq 0$ when $x \neq 0$.

If we only know $f(x) \geq 0$, then we must work a little harder to establish the result, because $q(x)$ will vanish whenever $f(x) = 0$ and so we cannot immediately conclude that $c = 0$. However, we do know $q(x) \geq 0$ for all $x > 0$. According to Lemma 4.5, adapted to one-sided limits, this implies $\lim_{x \to 0^+} q(x) = f'(0^+) \geq 0$. On the other hand, $q(x) \leq 0$ for all $x < 0$, which implies $\lim_{x \to 0^-} q(x) = f'(0^-) \leq 0$. Since we are assuming $q(x)$ is continuous, its right and left limits must coincide, and this implies

$$\lim_{x \to 0} q(x) = f'(0) = f'(0^-) = f'(0^+) = 0.$$  \hspace{1cm} Q.E.D.

Warning: This argument does not apply to right- and left-handed derivatives of a function at a minimum or maximum when they do not agree. For example, $f(x) = |x|$ has a minimum at $x = 0$, but $f'(0^-) = -1$, $f'(0^+) = +1$.

Note that the assumption of Lemma 7.10 implies that $f(0) = 0$ is a (global) minimum of the function $f(x)$ since everywhere else it is non-negative. Adapting the preceding proof to general local maxima and minima, we are able to establish the following general result.

Theorem 7.11. Suppose that $m = f(a)$ is a local minimum for $f(x)$, meaning that $f(x) \geq m$ for all $x \in I$ where $I$ is an open interval containing the point $a$. Then $f'(a) = 0$. Similarly, if $M = f(b)$ is a local maximum, so $f(x) \leq M$ for all $x \in J$, where $b \in J$ is an open interval, then $f'(b) = 0$.

One can interpret this result as saying that the tangent line at a local maximum or a local minimum must be horizontal; that is, it is the graph of a constant function.
Warning: If \( f: [a, b] \rightarrow \mathbb{R} \) is defined on a closed interval, and its maximum and/or minimum occurs at one of the endpoints, say \( a \), then it is not true that \( f'(a) = 0 \). For example, the function \( f(x) = 3x \) on the closed interval \([1, 2]\) has minimum \( 3 = f(1) \) and maximum \( 6 = f(2) \), but \( f'(x) \equiv 3 \) everywhere, including the endpoints.

The next result is sometimes known as Rolle’s Theorem, in honor of the seventeenth century French mathematician Michel Rolle.

**Theorem 7.12.** Let \( a < b \). Suppose that \( f: [a, b] \rightarrow \mathbb{R} \) is differentiable and \( f(a) = f(b) = 0 \). Then there exists \( a < c < b \) such that \( f'(c) = 0 \).

**Proof:** Let \( M = \sup f[a, b] \geq 0 \) and \( m = \inf f[a, b] \leq 0 \), where the inequalities follow from the assumption \( f(a) = f(b) = 0 \). According to the Extreme Value Theorem 5.14, there exists at least one point \( x_M \in [a, b] \) such that \( f(x_M) = M \) and at least one point \( x_m \in [a, b] \) such that \( f(x_m) = m \). If \( M > 0 \), then \( a < x_M < b \). Theorem 7.11 implies \( f'(x_M) = 0 \) and we can set \( c = x_M \). Similarly, if \( m < 0 \), then \( a < x_m < b \) and \( f'(x_m) = 0 \), so that \( c = x_m \) works. Finally, the only way we can have both \( M = m = 0 \) is if \( f(x) \equiv 0 \), but then \( f'(x) = 0 \) for all \( a < x < b \).

Q.E.D.

Rolle’s Theorem is a special case of a very useful technical result known as the Mean Value Theorem. It states that if \( f \) is differentiable on \([a, b]\), then there is at least one point on its graph such that the tangent line has the same slope as the secant line passing through the points \((a, f(a))\) and \((b, f(b))\); see Figure 5.

**Theorem 7.13.** Let \( a < b \) and suppose \( f: [a, b] \rightarrow \mathbb{R} \) is differentiable. Then there exists \( a < c < b \) such that

\[
    f'(c) = \frac{f(b) - f(a)}{b - a}.
\]

**Proof:** Define

\[
    g(x) = f(x) - s(x), \quad \text{where} \quad s(x) = \frac{f(b) - f(a)}{b - a} (x - a) + f(a)
\]
is, according to (7.15), the linear function whose graph is the secant line passing through the points \((a, f(a))\) and \((b, f(b))\). Thus, \(g(x)\) is continuous on \([a, b]\), and satisfies \(g(a) = 0\) and \(g(b) = 0\), which are the conditions of Rolle’s Theorem. This implies there is a point \(a < c < b\) such that

\[
0 = g'(c) = f'(c) - s'(c) = f'(c) - \frac{f(b) - f(a)}{b - a},
\]

which establishes (7.17).

**Proposition 7.14.** Let \(f: \mathbb{R} \to \mathbb{R}\) be differentiable. Then \(f'(x) \equiv 0\) for all \(x\) if and only if \(f(x) \equiv c\) is a constant function.

**Proof:** We already know that the derivative of a constant function is everywhere 0. To prove the converse, let \(a < b\). Then according to (7.17)

\[
\frac{f(b) - f(a)}{b - a} = f'(c) = 0
\]

for some point \(a < c < b\). This implies \(f(b) = f(a)\). Since this holds for all \(a, b\), it implies that \(f\) has the same constant value everywhere. \(Q.E.D.\)

**Remark:** The result holds when \(f\) is defined on an interval — open or closed. It does not hold on more general subsets. For example, the function \(f(x) = \text{sign } x\) has \(f'(x) = 0\) for all \(x \neq 0\), but is not constant. It is, however, constant on each subinterval \((-\infty, 0)\) and \((0, \infty)\) of its domain \(\mathbb{R} \setminus \{0\}\).

**Corollary 7.15.** The differentiable functions \(f(x)\) and \(g(x)\) have the same derivative, \(f'(x) = g'(x)\) if and only if \(f(x) = g(x) + c\) for some constant \(c \in \mathbb{R}\).

To prove this, merely apply Proposition 7.14 to the difference \(f(x) - g(x)\), using (7.7).

Let us next show that continuously differentiable functions are Lipschitz continuous, as characterized in (4.2).

**Theorem 7.16.** If \(f: [a, b] \to \mathbb{R}\) and its derivative \(f' : [a, b] \to \mathbb{R}\) are continuous, then \(f\) satisfies the Lipschitz continuity condition (4.2) for \(x, y \in [a, b]\).

**Proof:** Continuity of \(f'(x)\) and Theorem 5.13 imply that \(|f'(x)| \leq \lambda\) has an upper bound \(\lambda \geq 0\) for all \(x \in [a, b]\). By the Mean Value Theorem 7.13, if \(a \leq x < y \leq b\), then

\[
f(y) - f(x) = f'(z)(y - x)
\]

for some \(x < z < y\). Thus,

\[
|f(y) - f(x)| = |f'(z)||y - x| \leq \lambda|y - x|
\]

which is exactly (4.2). \(Q.E.D.\)

The following result, named after Guillaume de l’Hôpital, another seventeenth century French mathematician, is useful for computing limits of ratios when both numerator and denominator vanish.
Proposition 7.17. Suppose $f(x)$ and $g(x)$ have derivatives at $x = 0$. Suppose $f(0) = g(0) = 0$, but $g'(0) \neq 0$. Then

$$
\lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} \frac{f'(x)}{g'(x)} = \frac{f'(0)}{g'(0)}.
$$

(7.18)

Proof: Let

$$
F(x) = \begin{cases} 
\frac{f(x)}{x}, & x \neq 0, \\
 f'(0), & x = 0,
\end{cases} \quad G(x) = \begin{cases} 
\frac{g(x)}{x}, & x \neq 0, \\
 g'(0), & x = 0.
\end{cases}
$$

Both of these are continuous owing to the existence of $f'(0)$ and $g'(0)$. Moreover, since $G(0) = g'(0) \neq 0$, continuity implies $G(x) \neq 0$ for $x \in (-a, a)$ for some $a > 0$. Thus, the quotient function

$$
H(x) = \frac{F(x)}{G(x)}
$$

is continuous on $(-a, a)$, and $H(0) = \frac{F(0)}{G(0)} = \frac{f'(0)}{g'(0)}$.

On the other hand, given $0 \neq x \in (-a, a)$,

$$
H(x) = \frac{f(x)/x}{g(x)/x} = \frac{f(x)}{g(x)}.
$$

l'Hôpital’s Rule (7.18) is thus an immediate consequence of the continuity of $H(x)$ and the definition of limit. Q.E.D.

Example 7.18. The ratio

$$
\frac{e^x - 1}{x}
$$

satisfies the conditions of l'Hôpital’s Rule, with $f(x) = e^x - 1$ and $g(x) = x$. Since $f'(x) = e^x$, $g'(x) = 1$, the limiting value at $x = 0$ is

$$
\lim_{x \to 0} \frac{e^x - 1}{x} = \frac{f'(0)}{g'(0)} = 1.
$$

Next, consider the ratio

$$
\frac{e^x - 1 - x}{x^2}
$$

where now $f(x) = e^x - 1 - x$ and $g(x) = x^2$. In this case, $f'(x) = e^x - 1$, $g'(x) = 2x$, and so we cannot apply (7.18) as stated because $g'(0) = 0$. However, the first equation continues to hold, and so

$$
\lim_{x \to 0} \frac{e^x - 1 - x}{x^2} = \lim_{x \to 0} \frac{e^x - 1}{2x} = \frac{1}{2} \lim_{x \to 0} \frac{e^x - 1}{x} = \frac{1}{2},
$$

using the previous limiting formula.

† See (13.51) below for a justification that the derivative of the exponential function $e(x) = e^x$ is itself: $e'(x) = e^x$. 
8. Higher Order Derivatives and Taylor’s Theorem.

Suppose \( f(x) \) is a continuous function and possesses a derivative \( g(x) = f'(x) \) at each point. If \( g(x) \) is also continuous and has a derivative, then we call its derivative the second derivative of \( f \) and write \( f''(x) = g'(x) \). And so on for third and higher order derivatives. The \( n^{th} \) derivative is often denoted by \( f^{(n)}(x) \). Just as not every continuous function has a derivative everywhere, not every differentiable function has a second derivative, and so on.

As we saw, if \( f(0) = 0 \), the derivative \( f'(0) \) exists if the function \( q(x) = f(x)/x \) has a removable discontinuity. Similarly, if \( f'(0) = 0 \), the second derivative \( f''(0) \) exists if the function \( r(x) = f'(x)/x \) has a removable discontinuity. Somewhat surprisingly the second derivative \( f''(0) \) is not obtained from the value of \( f(x)/x^2 \) at its removable discontinuity.

To see what is going on, let us first write
\[
f(x) = x \, q(x) \quad \text{where} \quad q(0) = f'(0) = 0. \tag{8.1}
\]
Similarly, to find the derivative \( q'(0) \), we set
\[
q(x) = x \, r(x) \quad \text{where} \quad r(0) = q'(0). \tag{8.2}
\]
On the other hand, if we differentiate (8.1) using the product rule followed by (8.2), we find
\[
f'(x) = x \, q'(x) + q(x) = x \left[ q'(x) + r(x) \right] = x \, s(x), \quad \text{where} \quad s(x) = q'(x) + r(x).
\]
Since we are assuming \( f'(x) \) has a derivative at \( x = 0 \), the function \( s(x) \) has a removable discontinuity at \( x = 0 \) with value
\[
f''(0) = s(0) = q'(0) + r(0) = 2 \, r(0).
\]
Recapitulating, if \( f(0) = f'(0) = 0 \), then the removable discontinuity of the function
\[
r(x) = \frac{f(x)}{x^2} \quad \text{is given by} \quad r(0) = \frac{1}{2} \, f''(0).
\]
An alternative means of verifying this result is to apply l’Hôpital’s Rule (7.18) twice to the ratio \( r(x) \).

More generally, suppose \( f \) is differentiable, and \( f'(x) \) has derivative \( f''(a) \) at \( x = a \). Recall the Carathéodory formulation (7.4) of the derivative:
\[
f(x) = f(a) + (x-a) \, q(x), \quad \text{where} \quad q(a) = f'(a).
\]
Similarly, we can write
\[
q(x) = q(a) + (x-a) \, r(x), \quad \text{where} \quad r(a) = q'(a).
\]
On the other hand, differentiating the first expression and substituting the second yields
\[
f'(x) = q(x) + (x-a) \, q'(x) = g(a) + (x-a) \left[ q'(x) + r(x) \right] = f'(a) + (x-a) \, s(x),
\]
where
\[
s(x) = q'(x) + r(x).
\]
Thus,

$$f''(a) = s(a) = q'(a) + r(a) = 2r(a).$$

Putting the pieces together, we deduce that if \( f(x) \) and \( f'(x) \) are continuous, then \( f''(a) \) exists if and only if the function \( r(x) \) in

$$f(x) = f(a) + f'(a)(x-a) + (x-a)^2 r(x)$$

is continuous, in which case \( r(a) = \frac{1}{2} f''(a) \).

The general version of this result is known as Taylor’s Theorem, named after the early eighteenth century British mathematician Brook Taylor. Suppose \( f(x) \) is a function whose first \( n \) derivatives \( f(a), f'(a), f''(a), \ldots, f^{(n)}(a) \) exist. Consider the polynomial function

$$P_n(x) = f(a) + f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2} + f'''(a)\frac{(x-a)^3}{6} + \cdots + f^{(n)}(a)\frac{(x-a)^n}{n!}$$

(8.3)

known as the Taylor polynomial of degree \( n \) for the function \( f(x) \) at the point \( x = a \). Here

$$n! = n(n-1)(n-2) \cdots 3 \cdot 2 \cdot 1$$

(8.4)

denotes the factorial of \( n \in \mathbb{N} \). Note that, using (7.11, 12), the basic monomial function

$$p_n(x) = \frac{(x-a)^n}{n!}$$

has derivative

$$p_n'(x) = \frac{n(x-a)^{n-1}}{n!} = \frac{(x-a)^{n-1}}{(n-1)!} = p_{n-1}(x).$$

(8.5)

This implies that the derivatives of the Taylor polynomial (8.3), up to order \( n \), coincide with the derivatives of \( f \) at the point \( x = a \):

$$P_n(a) = f(a), \quad P_n'(a) = f'(a), \quad P_n''(a) = f''(a), \quad \ldots \quad P_n^{(n)}(a) = f^{(n)}(a).$$

On the other hand, since it has degree \( n \), its \( (n+1)^{\text{st}} \) derivative \( P_{n+1}^{(n+1)}(x) \equiv 0 \) for all \( x \).

**Theorem 8.1.** Suppose the first \( n \) derivatives \( f(x), f'(x), f''(x), \ldots, f^{(n)}(x) \) are continuous. Let \( P_n(x) \) be its Taylor polynomial of degree \( n \) at the point \( x = a \), (8.3). Then the \( (n+1)^{\text{st}} \) derivative \( f^{(n+1)}(a) \) exists at \( a \in \mathbb{R} \) if and only if the remainder function \( R_{n+1}(x) \) defined by the Taylor formula

$$f(x) = P_n(x) + R_{n+1}(x) \frac{(x-a)^{n+1}}{(n+1)!}$$

(8.6)

is continuous, in which case its value is \( R_{n+1}(a) = f^{(n+1)}(a) \).

To prove the Theorem, we use (8.5) to write

$$R_{n+1}(x) = \frac{f(x) - P_n(x)}{p_{n+1}(x)},$$

and then repeatedly apply l’Hôpital’s Rule (7.18) \( n+1 \) times to arrive at its limiting value \( R_{n+1}(a) = f^{(n+1)}(a) \).

Furthermore, there is a formula for the Taylor remainder function that generalizes the Mean Value Theorem 7.13.
Theorem 8.2. Given the Taylor expansion (8.6), with continuous \( R_{n+1}(x) \), then for \( b > a \) there exists \( a < y < b \), while for \( b < a \), there exists \( b < y < a \), such that
\[
R_{n+1}(b) = f^{(n+1)}(y). \tag{8.7}
\]

The case \( n = 0 \) of Theorem 8.2 coincides with the Mean Value Theorem 7.13, because \( P_0(x) = f(a) \) and so the Taylor expansion (8.6) reduces to
\[
f(x) = f(a) + (x-a) R_1(x), \quad \text{whereby} \quad R_1(x) = \frac{f(x) - f(a)}{x-a}
\]
is the difference quotient.

Proof: The following proof is modeled on that in [6]. Let us assume \( a < b \); the proof for \( b < a \) is essentially identical. Consider the function
\[
Q_n(x) = f(x) + f'(x)(b-x) + f''(x) \frac{(b-x)^2}{2} + f'''(x) \frac{(b-x)^3}{6} + \cdots + f^{(n)}(x) \frac{(b-x)^n}{n!}.
\tag{8.8}
\]
Comparing with (8.3) we see that \( Q_n(x) \) is the Taylor polynomial based at \( x \) evaluated at the point \( b \). It is a polynomial in \( b \), but a complicated function of the initial point \( x \). In particular, at \( x = a \) it coincides with the Taylor polynomial (8.3) based at \( a \) evaluated at the point \( x = b \):
\[
Q_n(a) = P_n(b), \quad \text{while} \quad Q_n(b) = f(b). \tag{8.9}
\]
Let us compute its derivative, using the product and chain rules. By (7.11, 12), the derivative of
\[
q_n(x) = \frac{(b-x)^n}{n!} \quad \text{is} \quad q'_n(x) = - \frac{(b-x)^{n-1}}{(n-1)!} = - q_{n-1}(x).
\]
Thus,
\[
Q'_n(x) = f'(x) + \left[ - f'(x) + f''(x)(b-x) \right] + \left[ - f''(x)(b-x) + f'''(x) \frac{(b-x)^2}{2} \right] + \cdots
\]
\[
\cdots + \left[ - f^{(n)}(x) \frac{(b-x)^{n-1}}{(n-1)!} + f^{(n+1)}(x) \frac{(b-x)^n}{n!} \right].
\]
Observe that the first and second terms cancel; the third and fourth terms also cancel; and so on; indeed, all the terms cancel out except the very last one, and so
\[
Q'_n(x) = f^{(n+1)}(x) \frac{(b-x)^n}{n!}. \tag{8.10}
\]
Now consider the function
\[
h(x) = f(b) - Q_n(x) + C \frac{(b-x)^{n+1}}{(n+1)!}, \quad \text{where} \quad C = \frac{(n+1)! \left[ f(b) - Q_n(a) \right]}{(b-a)^{n+1}} \tag{8.11}
\]

is a constant, chosen so that
\[
0 = h(a) = f(b) - Q_n(a) - C \frac{(b-a)^{n+1}}{(n+1)!} = f(b) - P_n(b) - C \frac{(b-a)^{n+1}}{(n+1)!}, \tag{8.12}
\]
where the second formula follows from the first equation in (8.9). In addition, in view of the second equation in (8.9),
\[ h(b) = f(b) - Q_n(b) = 0. \]

Thus, on the interval \([a, b]\), the function \(h(x)\) satisfies the conditions \(h(a) = h(b) = 0\) of Rolle’s Theorem 7.12, and we conclude that there is a point \(a < y < b\) such that
\[ 0 = h'(y) = -Q'_n(y) + C \frac{(b - y)^n}{n!} = -f^{(n+1)}(y) \frac{(b - y)^n}{n!} + C \frac{(b - y)^n}{n!}, \]
using (8.10). Thus, \(C = f^{(n+1)}(y)\). Substituting back into (8.12) yields
\[ f(b) = P_n(b) + f^{(n+1)}(y) \frac{(b - a)^{n+1}}{(n+1)!}, \]
proving (8.7).

Q.E.D.


Let us now generalize the preceding constructions and results to functions of two variables. Some aspects are more challenging than the one variable case considered so far, but once this case is properly understood, the further extension to functions of more than two variables is reasonably straightforward. Some of the applications will include new results for functions of a single variable, as well as supplying the missing proofs of results in the preceding sections.

Recall that the two-dimensional plane \(\mathbb{R}^2\) is the set of all ordered pairs of real numbers \(v = (x, y)\) where \(x, y \in \mathbb{R}\), also known as vectors. The basic open sets in \(\mathbb{R}^2\) are the open rectangles
\[ R_{a,b,c,d}^h = \{ (x, y) \mid a < x < b, \ c < y < d \} \]  
(9.1)
where \(a < b\) and \(c < d\) are real numbers, in which case the rectangle is bounded, or \(\pm \infty\) when unbounded. When the side lengths are equal and finite, so \(d - c = b - a\), the rectangle is a square. We note that the intersection of any finite collection of open rectangles is either empty or an open rectangle; this is proved by induction starting with the case of two rectangles.

The definition of an open set \(S \subset \mathbb{R}^2\) as a union of basic open sets is exactly as in Definition 2.1, as is the definition of closed subset \(C = \mathbb{R}^2 \setminus S\) as the complement of an open subset. The only subsets of \(\mathbb{R}^2\) which are both open and closed are the empty set \(\emptyset\) and \(\mathbb{R}^2\) itself; see Proposition 9.13 and Theorem 9.14 below.

We adapt Lemma 2.5 to the current situation; the proof is identical.

**Lemma 9.1.** A set \(S \subset \mathbb{R}^2\) is open if and only if for every \(v \in S\) there is an open rectangle \(R\) such that \(v \in R \subset S\).

In fact, when verifying this criterion, one can restrict one’s attention to squares, or even to squares or rectangles whose side lengths are sufficiently small. Let us look at some simple examples.
Example 9.2. The right half plane \( S = \{(x, y) | x > 0 \} \) is open because it is a rectangle with \( a = 0 \), \( b = \infty \), \( c = -\infty \), \( d = \infty \), so that \( S = \mathbb{R}_{0,\infty} \). On the other hand, \( \overline{S} = \{(x, y) | x \geq 0 \} \) is not open because any open rectangle \( R \) containing a point \((0, y) \in \overline{S}\) on the \( y \) axis must also contains points with negative \( x \) coordinate, and so \( R \not\subset \overline{S} \). Indeed, \( \overline{S} \) is closed since its complement \( \mathbb{R}^2 \setminus \overline{S} = \{(x, y) | x < 0 \} \), the left half plane, is an open rectangle, namely \( \mathbb{R}^{0,\infty}_{-\infty,-\infty} \).

Example 9.3. Here we show that the unit disk, meaning the interior of the unit circle \( D = \{(x, y) | x^2 + y^2 < 1 \} \) (9.2) is open. (A simpler but non-direct proof can be found below.) Let \( \mathbf{v} = (x, y) \in D \) and let \( s = \sqrt{1 - x^2 - y^2} > 0 \) denote the distance from \( \mathbf{v} \) to the closest point on the unit circle \( C = \{x^2 + y^2 = 1 \} \). Then we assert that the open square \( Q \) centered at \( \mathbf{v} \) with side length \( \sqrt{2} s \), namely \( Q = R_{a,c}^{b,d} \) with \( a = x - s/\sqrt{2} \), \( b = x + s/\sqrt{2} \), \( c = y - s/\sqrt{2} \), \( d = y + s/\sqrt{2} \), is contained in \( D \). Indeed, its corner points lie a distance \( s \) from the center (and so one of them is the closest point on the unit circle) and every point in \( Q \) is closer to its center, and hence must be contained in the disk; see Figure 6. Since every point in \( D \) belongs to such an open square, we conclude that \( D \) is open. A similar argument proves that the exterior of the unit circle \( E = \{(x, y) | x^2 + y^2 > 1 \} \) is also open. The unit circle itself is closed since it is the complement \( C = \mathbb{R}^2 \setminus (D \cup E) \) of the open set obtained as the union of its interior and exterior.

Remark: The analog of Proposition 2.6 is not valid in \( \mathbb{R}^2 \). For example, the interior of the unit circle cannot be written as the disjoint union of open rectangles.

Lemma 9.4. If \( S \subset \mathbb{R} \) is open, then \( U = \{(x, y) | x \in S \} \) and \( V = \{(x, y) | y \in S \} \) are open subsets of \( \mathbb{R}^2 \).
Proof: Given \((x, y) \in U\), let \((a, b) \subset S\) be an open interval containing \(x\). Then the unbounded rectangle \(R_{a, -\infty} = \{(x, y) | a < x < b\} \subset U\). Thus, every point \((x, y) \in U\) is contained in an open rectangle \(R \subset U\), proving that \(U\) is open. The proof for \(V\) is identical. \(Q.E.D.\)

**Lemma 9.5.** Let \(S \subset \mathbb{R}^2\) be open. Let \(z \in \mathbb{R}\). Then \(U_z = \{x \in \mathbb{R} | (x, z) \in S\}\) and \(V_z = \{y \in \mathbb{R} | (z, y) \in S\}\) are open subsets of \(\mathbb{R}\).

**Proof:** Let \(x \in U_z\). Let \(R_{a, c} \subset S\) be an open rectangle containing \((x, z)\). Then \((a, b) \subset U_z\) is an open interval containing \(x\). Since this occurs for every point in \(U_z\), we conclude that \(U_z\) is open. The proof for \(V_z\) is identical. \(Q.E.D.\)

Define the *Cartesian product* of subsets \(A, B \subset \mathbb{R}\) to be the subset

\[
A \times B = \{(x, y) | x \in A, y \in B \} \subset \mathbb{R}^2.
\]

For example, the Cartesian product of a pair of open intervals \((a, b), (c, d) \subset \mathbb{R}\) is the open rectangle (9.1):

\[
R_{a, c}^{b, d} = (a, b) \times (c, d).
\]

**Lemma 9.6.** If \(S, T \subset \mathbb{R}\) are open, then their Cartesian product \(S \times T \subset \mathbb{R}^2\) is open.

**Proof:** It suffices to note that every point \(x \in S\) is contained in an open interval, \(x \in I \subset S\), and every point \(y \in T\) is contained in an open interval \(y \in J \subset T\). Thus, every point \((x, y) \in S \times T\) is contained in the corresponding Cartesian product rectangle: \((x, y) \in R = I \times J \subset S \times T\). \(Q.E.D.\)

**Proposition 9.7.** If \(f: \mathbb{R} \to \mathbb{R}\) is a continuous function, then the sets

\[
S^+ = \{(x, y) | y > f(x)\}, \quad S^- = \{(x, y) | y < f(x)\},
\]

lying above and below its graph are open.

**Proof:** We prove \(S^+\) is open. Suppose \((x, y) \in S^+\) so \(y > f(x)\). Let \(c < f(x) < d < y < e\). Then \(x \in f^{-1}(c, d)\), which is open. Let \(a < x < b\) be such that \((a, b) \subset f^{-1}(c, d)\). We claim that the rectangle \(R^{b, e}_{a, d} \subset S^+\). Indeed, given \((z, w) \in R^{b, e}_{a, d}\), so \(z \in (a, b) \subset f^{-1}(c, d)\), we have \(f(z) < d < w\), and hence \((z, w) \in S^+\). Thus, each point \((x, y) \in S^+\) is contained in a rectangle \(R^{b, e}_{a, d} \subset S^+\), which implies that \(S^+\) is open. The analogous proof for \(S^-\) is left to the reader. \(Q.E.D.\)

If \(f(x)\) is discontinuous, this result may or may not be true. For the sign function (4.1), the sets

\[
S^+ = \{(x, y) | y > \text{sign } x\}, \quad S^- = \{(x, y) | y < \text{sign } x\},
\]

are not open, whereas

\[
S^+ = \{(x, y) | y > 1/x^2, \ x \neq 0\}, \quad S^- = \{(x, y) | y < 1/x^2, \ x \neq 0\},
\]
are open. We can even add in the $y$ axis to the latter set, whereby
\[ \hat{S} = \{ (x,y) \mid x = 0 \text{ or } y < 1/x^2 \} \]
is also open. This can be proved using the method in Example 9.10 below.

**Corollary 9.8.** If $f, g: \mathbb{R} \to \mathbb{R}$ are continuous functions, then the strip
\[ S = \{ (x,y) \mid f(x) < y < g(x) \} = \{ y > f(x) \} \cap \{ y < g(x) \} \]  
(9.5)
between their graphs is open.

**Remark:** Corollary 9.8 does not require that $f(x) < g(x)$ everywhere, and so their graphs are allowed to cross. Of course if $f(x) \geq g(x)$ everywhere, then $S = \emptyset$.

The theorem and corollary are also valid if the functions $f, g$ are only defined on an open set. In particular, given an open interval $I \subset \mathbb{R}$, either bounded or unbounded,
\[ S = \{ (x,y) \mid x \in I, f(x) < y < g(x) \} \]  
(9.6)
is open.

**Example 9.9.** The open unit disk (9.2) can be characterized as the “strip”
\[ D = \{ (x,y) \mid x^2 + y^2 < 1 \} = \{ (x,y) \mid -1 < x < 1, -\sqrt{1-x^2} < y < \sqrt{1-x^2} \} \]
lying between the graphs of the two indicated continuous square root functions. Thus, the fact that it is open follows immediately from (9.6).

**Example 9.10.** Let $a < b$. Let us prove that the hyperbolic strip
\[ S = \{ (x,y) \mid a < xy < b \} \]  
(9.7)
lying between the hyperbolas $y = a/x$ and $y = b/x$ is open. Openness of a strip (9.6) implies that
\[ S^+ = \{ (x,y) \mid x > 0, a/x < y < b/x \} \quad \text{and} \quad S^- = \{ (x,y) \mid x < 0, b/x < y < a/x \} \]
are both open. If $0 \leq a < b$ or $a < b \leq 0$ then $S = S^+ \cup S^-$ and hence is open. Otherwise, when $a < 0 < b$, the $y$ axis $Y = \{ x = 0 \}$ also belongs to $S$, so $S = S^+ \cup S^- \cup Y$. Now since $S^+$ is open, each point $(x,y) \in S^+$ is contained in an open rectangle $R \subset S^+$. The same holds for $S^-$. Let $m = \min \{ -a, b \} > 0$. Given $(0,y) \in Y$, we easily see that $(0,y) \in R_{-c,-m/c} e^{m/c} \subset S$ provided $0 < c < 1/|y|$. (If $y = 0$ then any value of $c \in \mathbb{R}^+$ works.) Thus every point in $S$ belongs to an open rectangle contained in $S$, proving that $S$ is open.

As before, the complement of an open set is a closed set; in other words, $C \subset \mathbb{R}^2$ is closed if and only if $S = \mathbb{R}^2 \setminus C$ is open. An example is the closed rectangle
\[ \overline{R}_{a,b}^{c,d} = \{ (x,y) \mid a \leq x \leq b, c \leq y \leq d \}. \]  
(9.8)
The closure $\overline{A}$ of a set $A \subset \mathbb{R}^2$ is the intersection of all closed subsets containing $A$, which is also the smallest closed subset containing it.
If $S$ is open with closure $\overline{S}$, we define the boundary of $S$ to be the subset
\[ \partial S = \overline{S} \setminus S. \]  \tag{9.9} \]
We note that $\partial S$ is closed because its complement is the union of two open sets:
\[ \mathbb{R}^2 \setminus \partial S = S \cup (\mathbb{R}^2 \setminus \overline{S}). \]
For example, the closure of the open unit disk (9.2) is the closed unit disk
\[ \overline{D} = \{(x, y) \mid x^2 + y^2 \leq 1\}, \]
and its boundary is the unit circle
\[ C = \partial D = \{(x, y) \mid x^2 + y^2 = 1\}. \]  \tag{9.10} \]
Similarly, the closed rectangle (9.8) is the closure of the open rectangle (9.1). The boundary is the rectangular curve consisting of the four line segments joined at their endpoints:
\[ \partial R_{a,c}^{b,d} = \overline{R}_{a,c}^{b,d} \setminus R_{a,c}^{b,d} = \{(x, c) \mid a \leq x \leq b\} \cup \{(x, d) \mid a \leq x \leq b\} \]
\[ \cup \{(a, y) \mid c \leq y \leq d\} \cup \{(b, y) \mid c \leq y \leq d\}. \]  \tag{9.11} \]
By a half open rectangle we mean a subset of the form
\[ \tilde{R} = R_{a,c}^{b,d} \cup B, \quad \text{where} \quad \emptyset \neq B \subset \partial R_{a,c}^{b,d}. \]
The more common cases are when $B$ consists of 1, 2, or 3 of the line segments (9.11).

The definition of limit point for subsets $A \subset \mathbb{R}^2$ is modeled directly on Definition 5.7. There is a direct analog of the Limit Point Theorem 5.10. First, a subset $A \subset \mathbb{R}^2$ is bounded if it is contained in a bounded open rectangle:
\[ D \subset R_{a,c}^{b,d} \quad \text{for some} \quad a, b, c, d \in \mathbb{R}, \quad a < b, \ c < d. \]  \tag{9.12} \]
A set that is both closed and bounded is called compact.

**Theorem 9.11.** Let $A \subset \mathbb{R}^2$ be a bounded infinite subset. Then the set of limit points of $A$ is not empty.

**Proof:** First, suppose that the subset is contained in the unit square, so $A \subset Q = \{0 \leq x, y \leq 1\}$. We will construct the binary expansions of the $x$ and $y$ coordinates of a limit point
\[ \ell = (a, b) = (a_1 a_2 a_3 \ldots, b_1 b_2 b_3 \ldots) \in Q. \]  \tag{9.13} \]
First, we divide the unit square into four half size subsquares:
\[ Q_1 = \{0 \leq x, y \leq \frac{1}{2}\}, \quad Q_2 = \{\frac{1}{2} \leq x \leq 1, \ 0 \leq x \leq \frac{1}{2}\}, \]
\[ Q_3 = \{0 \leq x \leq \frac{1}{2}, \ \frac{1}{2} \leq y \leq 1\}, \quad Q_4 = \{\frac{1}{2} \leq x, y \leq 1\}, \]
which intersect along the common segment of their boundaries; see Figure 7 At least one of them contains infinitely many points in $A$, and we select that square, or any one of those squares when there is more than one. If the selected square is the bottom left hand one $Q_1$, record $a_1 = b_1 = 0$; if it is the bottom right hand one $Q_2$, record $a_1 = 1, \ b_1 = 0$;
it is the top left hand one $Q_3$, record $a_1 = 0$, $b_1 = 1$; if it is the top right hand one $Q_4$, record $a_1 = b_1 = 1$. Note that any point whose first binary digit of each of its coordinates are the recorded ones belongs to the corresponding square. One then similarly divides the selected square into four half sized subsquares and uses the same rule to determine the second binary digits $a_2, b_2$ of the eventual limit point. Repeating the process ad infinitum produces successive binary digits of a point (9.13) belongs to all the chosen subsquares. The proof that the resulting point (9.13) is a limit point proceeds as in the one-dimensional case, noting that, for any $r > 0$, whenever $2^{-k} < r$ for $k \in \mathbb{N}$, the square \{ $a-r < x < a+r$, $b-r < y < b$ \} centered at $\ell = (a,b)$ contains all the selected squares of size $2^{-k}$, and hence contains infinitely many points in $A$, thereby proving the theorem when $A \subset Q$.

More generally, given that $A$ is bounded, there is at least one unit square $Q_{m,n} = \{ m \leq x \leq m+1, \ n \leq y \leq n+1 \}$, for some $m, n \in \mathbb{N}$, that contains infinitely many points in $A$. Let $B = \{ (x-m, y-n) \mid (x,y) \in A \cap Q_{m,n} \} \subset Q_{0,0} = Q$. Since $B$ is infinite, it contains a limit point $\ell = (a,b) \in Q$ for $B$, and it is easily checked that the corresponding point $l = (a+m, b+n) \in Q_{m,n}$ is a limit point for $A$. Q.E.D.

As before, we construct a topology on any subset $A \subset \mathbb{R}^2$ by defining a \textit{relatively open subset} to be the intersection, $S \cap A$, with an open subset $S \subset \mathbb{R}^2$. In particular, the \textit{basic open subsets} are the intersections of $A$ with open rectangles: $R_{a,c}^{b,d} \cap A$. Any relatively open subset is a union of basic open subsets. If $A$ itself is open, every relatively open subset $S \cap A$ is also an open subset of $\mathbb{R}^2$.

For example, suppose $a < b$ and $c < d$, so that

$$A = \{ (x,y) \mid a \leq x \leq b, \ c \leq y \leq d \} \subset \mathbb{R}^2$$

is a closed rectangle. Then the basic open subsets of $A$ are the intersections $R \cap A$, where $R$ is an open rectangle, which are either open rectangles, or half open rectangles, such as \{ $a \leq x < e$, $f < y < g$ \} for $a < e \leq b$ and $c \leq f < g \leq d$, or \{ $a \leq x < e$, $f < y \leq d$ \} for

\begin{figure}
\centering
\includegraphics[width=\textwidth]{subdividing_square.png}
\caption{Subdividing a Square.}
\end{figure}
a < e < b and c ≤ f < d, and so on, as well as the closed rectangle A itself.

**Definition 9.12.** A set $S$ is called *disconnected* if it can be written as the disjoint union of two nonempty open subsets. In other words there exist relatively open sets $S_1, S_2 ⊆ S$ such that

$$S_1 \cup S_2 = S, \quad S_1 \cap S_2 = \emptyset, \quad S_1 \neq \emptyset, \quad S_2 \neq \emptyset. \tag{9.14}$$

Vice versa, a set is *connected* if it is not disconnected.

**Proposition 9.13.** If $S$ is connected, the only relatively open and closed subsets are the set $S$ itself and the empty set.

**Proof:** Indeed, if $\emptyset \neq S_1 ⊆ S$ is open and closed, its complement $S_2 = S \setminus S_1 \neq \emptyset$ is also both open and closed, and hence $S = S_1 \cup S_2$ would be the union of disjoint nonempty open subsets and thus disconnected. \( Q.E.D. \)

In $\mathbb{R}$ the only nonempty connected sets are the intervals; see Theorem 2.7. In $\mathbb{R}^2$ where there are many connected sets, including a rectangle (open or closed or half-open), a rectangular boundary, a circle, as well as its interior and exterior, etc. The most basic example is the entire plane.

**Theorem 9.14.** The space $\mathbb{R}^2$ is connected.

**Proof:** Suppose not, so that there exist disjoint nonempty open subsets $S_1, S_2 \subseteq \mathbb{R}^2$ such that $S_1 \cup S_2 = \mathbb{R}^2$. Suppose $(0,0) \in S_1$. Given $(a,b) \neq (0,0)$, consider the line $L_{a,b} = \{(ta, tb) \mid t \in \mathbb{R}\}$ passing through both points. We claim that the disjoint subsets $T_i = \{t \in \mathbb{R} \mid (ta, tb) \in S_i\} \subset \mathbb{R}$ for $i = 1, 2$ are both open, and $T_1 \cup T_2 = \mathbb{R}$. Indeed, since $S_i$ is open, any point $(ta, tb) \in S_i$ is contained in an open rectangle $R \subset S_i$, but then $\{t \in \mathbb{R} \mid (ta, tb) \in R\} \subset T_i$ forms an open interval, proving that $T_i$ is open. Since $0 \in T_1 \neq \emptyset$, Theorem 2.7 implies $T_2 = \emptyset$ and hence $(a, b) \in L_{a, b} \subset S_1$. Since this holds for every point $(a, b)$, we conclude that $\mathbb{R}^2 = S_1$ and $S_2 = \emptyset$, thus proving connectivity. \( Q.E.D. \)

A similar proof can be used to prove the connectivity of rectangles, triangles, disks, etc. Indeed, the proof applies to prove connectivity of any *star shaped region*, by which we mean a region $D$ such that if $(a, b) \in D$, then so is every point on the line segment connecting it to the origin, i.e., $(ta, tb) \in D$ for all $0 \leq t \leq 1$. One can also replace the origin by any other point. Unions of connected regions with nonempty overlap are also connected; the proof is left as an exercise.

**Lemma 9.15.** If $S$ and $T$ are connected, and $S \cap T \neq \emptyset$, then $S \cup T$ is connected.

The following result will be used later, to prove Theorem 10.13.
Lemma 9.16. Let \( a < b \) be finite. Let \( D = \{ (x, y) \mid a \leq x, y \leq b \} \) be the closed square with opposite corners \((a, a)\) and \((b, b)\). Let

\[
C = \{(x, x) \mid a \leq x \leq b\} = \Delta \cap D
\]

be its diagonal. Let \( S \subset D \) be a relatively open subset such that \( C \subset S \). Then there exists \( r > 0 \) such that \( S \) contains the diagonal strip of width 2:

\[
C \subset U_r = \{ (x, y) \in D \mid |x - y| < r \} \subset S. \tag{9.15}
\]

Proof: See Figure 8; the open set is in blue, and contains the pink diagonal strip. Given any \( a \leq x \leq b \) and \( r > 0 \), let \( Q_{r,x} = R^{x+r,x+r}_{x-r,x-r} \) be the relatively open square of width 2 centered at \((x, x) \in C\), and let \( \widehat{Q}_{r,x} = Q_{r,x} \cap D \) be the corresponding relatively open subset of \( D \). Note that if \( a + r \leq x \leq b - r \), then \( \widehat{Q}_{r,x} = Q_{r,x} \); otherwise it is a half open square of smaller size. Moreover,

\[
U_r = \bigcup_{a \leq x \leq b} \widehat{Q}_{r,x}.
\]

Thus, if we can find \( r > 0 \) such that every such square \( \widehat{Q}_{r,x} \subset S \), then this implies \( C \subset U_r \subset S \). Note that \( \widehat{Q}_{s,x} \subset \widehat{Q}_{r,x} \) whenever \( 0 < s < r \).

Since \( S \subset D \) is open, every point \( (x, x) \in C \subset S \) is contained in such a square; i.e., there exists \( s > 0 \) such that

\[
(x, x) \in \widehat{Q}_{s,x} = R^{x+s,x+s}_{x-s,x-s} \cap A \subset S.
\]

Let \( s_x = \sup \{ s > 0 \mid \widehat{Q}_{s,x} \subset S \} \). Let \( r_* = \inf \{ s_x \mid a \leq x \leq b \} \). If \( r_* > 0 \) and \( 0 < r < r_* \), then every \( \widehat{Q}_{r,x} \subset S \), and the result is proved. Thus, we need to eliminate the possibility that \( r_* = 0 \). This means there exist points \( a \leq x_n \leq b \) for \( n \in \mathbb{N} \) such that \( s_{x_n} \to 0 \) as \( n \to \infty \). Let \( x_* \) be a limit point of \( \{ x_n \mid n \in \mathbb{N} \} \subset [a, b] \), whose existence is guaranteed.
by Theorem 5.10. Let \( s_* = s_{x_*} \), so that \( Q_{s_*} \subset S \) whenever \( s < s_* \). If \( |y - x_*| < \frac{1}{2} s \), then \( Q_{s/2,y} \subset Q_{s,x_*} \subset S \), and hence \( s_y \geq \frac{1}{2} s \). On the other hand, by the definition of limit point, there exist infinitely points \( x_n \) satisfying \( |x_n - x_*| < \frac{1}{2} s \), and hence \( s_{x_n} \geq \frac{1}{2} s \), which contradicts our specification of \( x_n \).

Remark: The result is not true if the diagonal subset is unbounded. For example

\[
\Delta \subset S = \{ (x, y) \mid x = 0 \quad \text{or} \quad x - 1/|x| < y < x + 1/|x| \}
\]

is open, but there is no strip of the form

\[
U_r = \{ (x, y) \mid |x - y| < r \} \subset S \quad \text{for} \quad r > 0.
\]

10. Continuous Functions on the Plane.

Let us now investigate the continuity properties of functions \( F: \mathbb{R}^2 \to \mathbb{R} \). Here, the value of the function at a point \((x, y) \in \mathbb{R}^2 \) is the real number \( F(x, y) \in \mathbb{R} \). The requirement that \( F \) be continuous is word for word the same as above. Specifically, \( F \) is continuous if and only if \( F^{-1}(S) \subset \mathbb{R}^2 \) is open whenever \( S \subset \mathbb{R} \) is open. As before, we only need to check the inverse images of basic open sets, i.e., the open intervals. Thus, \( F \) is continuous if and only if, whenever \( I \subset \mathbb{R} \) is an open interval, then \( F^{-1}(I) \) is an open subset of \( \mathbb{R}^2 \), or, equivalently, for every \((x, y) \in F^{-1}(I) \), there is an open rectangle \( R \) such that \((x, y) \in R \subset F^{-1}(I) \), or, alternatively, \((x, y) \in R \) and \( F(R) \subset I \).

Example 10.1.

(a) The addition function \( A(x, y) = x + y \) from \( \mathbb{R}^2 \) to \( \mathbb{R} \) is continuous. Indeed, if \((a, b) \subset \mathbb{R} \) is any interval, then \( A^{-1}(a, b) = \{ (x, y) \mid a < x + y < b \} \) is the strip lying between the straight lines \( y = a - x \) and \( y = b - x \), which, as a consequence of Corollary 9.8, is open.

(b) The subtraction function \( S(x, y) = x - y \) from \( \mathbb{R}^2 \) to \( \mathbb{R} \) is continuous. Indeed, if \((a, b) \subset \mathbb{R} \) is any interval, then \( S^{-1}(a, b) = \{ (x, y) \mid a < x - y < b \} \) is the open strip lying between the straight lines \( y = a + x \) and \( y = b + x \).

(c) The multiplication function \( M(x, y) = xy \) from \( \mathbb{R}^2 \) to \( \mathbb{R} \) is continuous. Indeed, if \((a, b) \subset \mathbb{R} \) is any interval, then \( M^{-1}(a, b) = \{ (x, y) \mid a < xy < b \} \) is a hyperbolic strip whose openness was established in Example 9.10.

(d) The division function \( D(x, y) = x/y \) from \( \mathbb{R}^2 \setminus \{y = 0\} \) to \( \mathbb{R} \) is continuous. Indeed, if \((a, b) \subset \mathbb{R} \) is any interval, then \( D^{-1}(a, b) = \{ (x, y) \mid x > 0, \; a y < x < b y \} \cup \{ (x, y) \mid x < 0, \; b y < x < a y \} \) is the union of two open wedges lying between the straight lines \( x = a y \) and \( x = b y \).

(e) The maximum function \( X(x, y) = \max \{x, y\} \) from \( \mathbb{R}^2 \) to \( \mathbb{R} \) is continuous. Indeed, if \((a, b) \subset \mathbb{R} \) is any interval, then \( X^{-1}(a, b) = \{ (x, y) \mid a < \max \{x, y\} < b \} \) is the union of the two overlapping unbounded rectangles \( R^{h,b}_{-\infty, a} \) and \( R^{h,b}_{a, -\infty} \). Similarly, the minimum function \( N(x, y) = \min \{x, y\} \) is continuous since \( N^{-1}(a, b) = \{ (x, y) \mid a < \min \{x, y\} < b \} \) is the union of \( R^{\infty,b}_{a, a} \) and \( R^{h,b}_{a, a} \).
The next result provides an easy way to build continuous functions of two variables from a continuous function of one variable. More interesting combinations will appear later.

**Lemma 10.2.** If $f: \mathbb{R} \to \mathbb{R}$ is continuous, then the functions $F, G: \mathbb{R}^2 \to \mathbb{R}$ given by $F(x, y) = f(x)$ and $G(x, y) = f(y)$ are both continuous.

**Proof:** Given an open set $S \subset \mathbb{R}$, we have

$$F^{-1}(S) = \{ (x, y) \mid f(x) \in S \} = \{ (x, y) \mid x \in f^{-1}(S) \},$$

$$G^{-1}(S) = \{ (x, y) \mid f(y) \in S \} = \{ (x, y) \mid y \in f^{-1}(S) \}.$$

Since $f^{-1}(S) \subset \mathbb{R}$ is open as a consequence of the continuity of $f$, Lemma 9.4 implies both $F^{-1}(S)$ and $G^{-1}(S)$ are open subsets of $\mathbb{R}^2$. \[Q.E.D.\]

Conversely, given a continuous function of two variables, we can fix the value of one of its variables to produce a continuous function of one variable, sometimes called a “partial function”.

**Proposition 10.3.** Let $F: \mathbb{R}^2 \to \mathbb{R}$ be continuous. Fix $a, b \in \mathbb{R}$ and set $f(x) = F(x, b)$ and $g(y) = F(a, y)$. Then the scalar functions $f, g: \mathbb{R} \to \mathbb{R}$ are both continuous.

**Proof:** Let $S \subset \mathbb{R}$ be open. Then

$$f^{-1}(S) = \{ x \mid (x, b) \in F^{-1}(S) \}.$$

But continuity of $F$ implies $F^{-1}(S) \subset \mathbb{R}^2$ is open. Lemma 9.5 implies $f^{-1}(S) \subset \mathbb{R}$ is open. A similar argument works for $g$. \[Q.E.D.\]

**Example 10.4.** Consider the function

$$F(x, y) = \begin{cases} \frac{xy}{x^2 + y^2}, & (x, y) \neq (0, 0), \\ 0, & (x, y) = (0, 0). \end{cases}$$

Then, given $a, b \neq 0$, the functions

$$f(x) = F(x, b) = \frac{bx}{x^2 + b^2}, \quad g(y) = F(a, y) = \frac{ay}{y^2 + a^2},$$

are both continuous. Also, when $a = b = 0$,

$$f(x) = F(x, 0) \equiv 0, \quad g(y) = F(0, y) \equiv 0,$$

are also trivially continuous. Surprisingly, the function $F(x, y)$ is *not* continuous! Indeed, consider the open interval $\left(-\frac{1}{2}, \frac{1}{2}\right)$. Its inverse image is

$$A = F^{-1}\left(-\frac{1}{2}, \frac{1}{2}\right) = \{(0, 0)\} \cup \{(x, y) \mid xy < \frac{1}{2}\}$$

$$= \{(0, 0)\} \cup \{(x, y) \mid x \neq y \}.$$

The second equality follows from the fact that

$$\frac{1}{2}(x^2 + y^2) > \pm xy \quad \text{if and only if} \quad (x \pm y)^2 = x^2 \pm 2xy + y^2 \neq 0.$$
The latter set, consisting of all points in \( \mathbb{R}^2 \) not on the diagonal \( x = y \), is open, but including the origin makes \( A \) into a non-open subset. Indeed, since \( F(x,x) = \frac{1}{2} \) for all \( 0 \neq x \in \mathbb{R} \), there is no open rectangle \( R \) containing \((0,0) \in A \) such that \( R \subset A \) because all points in \( R \) of the form \((x,x) \) with \( x \neq 0 \) do not belong to \( A \).

We conclude from the above example that the converse of Proposition 10.3 is false. Continuity of the two “partial functions” \( f(x) = F(x,b) \) and \( g(y) = F(a,y) \) is not sufficient to ensure the continuity of the full function \( F(x,y) \).

**Proposition 10.5.** Let \( F: \mathbb{R}^2 \to \mathbb{R} \) be continuous. If \( S \subset \mathbb{R}^2 \) is connected, then \( F(S) \subset \mathbb{R} \) is connected, and hence an interval.

**Proof:** Suppose, to the contrary, that \( F(S) \subset U_1 \cup U_2 \) where \( U_1, U_2 \subset \mathbb{R} \) are disjoint open subsets with \( U_i \cap F(S) \neq \emptyset \) for \( i = 1,2 \). Then \( S \subset F^{-1}(U_1) \cup F^{-1}(U_2) \) where \( F^{-1}(U_1), F^{-1}(U_2) \) are also nonempty and disjoint, and, because \( F \) is continuous, open. But this contradicts the assumed connectivity of \( S \). \( \square \)

**Remark:** We have stated Proposition 10.5 for functions from \( \mathbb{R}^2 \) to \( \mathbb{R} \), but the argument applies in complete generality.

**Warning:** It is not true that if \( S \subset \mathbb{R} \) is connected, then so is \( F^{-1}(S) \). For example, if \( F(x,y) = x^2 \), then \( F^{-1}(1,4) = \{ (x,y) \mid 1 < x < 2 \} \cup \{ (x,y) \mid -2 < x < -1 \} \).

Proposition 10.5 provides an immediate proof of the generalization of the Intermediate Value Theorem 4.6.

**Theorem 10.6.** Let \( F: \mathbb{R}^2 \to \mathbb{R} \) be continuous. Suppose \( S \subset \mathbb{R}^2 \) is connected, and \( F(a,b) = \alpha < F(c,d) = \beta \) for \((a,b),(c,d) \in S \). Then, given any \( \alpha < z < \beta \), there exists \((x,y) \in S \) such that \( F(x,y) = z \).

**Proof:** Suppose not. Then we can write \( S \subset S_1 \cup S_2 \), where \( S_1 = F^{-1}(-\infty,z) \), \( S_2 = F^{-1}(z,\infty) \), which are disjoint, open subsets, and also nonempty since \((a,b) \in S_1 \) and \((c,d) \in S_2 \). But this contradicts the assumed connectivity of \( S \). \( \square \)

As before, we can restrict the function to a subdomain \( A \subset \mathbb{R}^2 \). If \( A \) is open, the continuity condition does not change because a relatively open subset \( S \subset A \) is just an open subset of \( \mathbb{R}^2 \) that is contained in \( A \). On the other hand, for more general subsets \( A \) one must analyze the structure of relatively open subsets of \( A \) in order to understand continuity of functions defined on \( A \). Any continuous \( F: \mathbb{R}^2 \to \mathbb{R} \) automatically restricts to a continuous function \( F: A \to \mathbb{R} \) for any \( A \) contained in its domain. But, as before, it may not be possible to reverse the process.

The Extreme Value Theorem for functions of two variables states that a continuous function on a closed, bounded (compact) subset is bounded from above and below and achieves its maximum and minimum values. Both conditions — bounded and closed — are essential for its validity. If we also assume connectedness, the result can be strengthened as follows:
Theorem 10.7. Let \( S \subset \mathbb{R}^2 \) be a connected compact set. If \( F: S \to \mathbb{R} \) is continuous, then its range \( F(S) = \{ F(x,y) \mid (x,y) \in S \} \subset \mathbb{R} \) is a bounded closed interval. Thus, \( F(S) = [m,M] \) where \( M = \sup F(S) \) and \( m = \inf F(S) \). Moreover, given any \( m \leq z \leq M \), there exists at least one point \( (x,y) \in S \) such that \( F(x,y) = z \).

Proof: The proofs of the various parts are evident adaptations of the proofs of their one-dimensional versions. One first establishes boundedness of \( F \) as in Theorem 5.13. Then the two-dimensional version of the Extreme Value Theorem follows the method for Theorem 5.14. The final statement is an immediate consequence of the two variable Intermediate Value Theorem 10.6.

Q.E.D.

Similarly, a vector-valued function \( f: \mathbb{R} \to \mathbb{R}^2 \) is called continuous if \( f^{-1}(S) \subset \mathbb{R} \) is open whenever \( S \subset \mathbb{R}^2 \) is open. As always, we need only check when \( S \) is a basic open subset, i.e., an open rectangle. We can write \( f(t) = (f(t),g(t)) \), where \( f,g: \mathbb{R} \to \mathbb{R} \). We will mostly (but not always) use \( t \in \mathbb{R} \) as the argument of \( f \) so as to distinguish it from points \( (x,y) \in \mathbb{R}^2 \).

Lemma 10.8. If \( f,g: \mathbb{R} \to \mathbb{R} \) are continuous, then \( f(t) = (f(t),g(t)) \) defines a continuous function \( f: \mathbb{R} \to \mathbb{R}^2 \).

Proof: Given a rectangle \( R_{a,c}^{b,d} \), we see that \( f(t) \in R_{a,c}^{b,d} \) if and only if \( a < f(t) < b \) and \( c < g(t) < d \). Thus \( f^{-1}(R_{a,c}^{b,d}) = \{ t \in \mathbb{R} \mid a < f(t) < b \} \cap \{ x \in \mathbb{R} \mid c < g(t) < d \} = f^{-1}(a,b) \cap g^{-1}(c,d) \), which, being the intersection of two open sets, is open. Since this holds for every open rectangle, we conclude that \( f \) is continuous.

Q.E.D.

The proof that the composition of continuous functions is, in general, continuous proceeds exactly as before; see the earlier proof of the last statement of Theorem 4.3. With this in hand, we are finally in a position to give a short proof of the remaining statements in Theorem 4.3, on the continuity of algebraic combinations of continuous functions. Given continuous functions \( f,g: \mathbb{R} \to \mathbb{R} \), let \( f(x) = (f(x),g(x)) \) be the vector-valued function, \( \mathbb{R} \to \mathbb{R}^2 \), which is itself continuous by Lemma 10.8. Let \( A,S,M,D,X,N: \mathbb{R}^2 \to \mathbb{R} \) be the continuous functions in Example 10.1. Then, we can express

\[
\begin{align*}
f(x) + g(x) &= A \circ f(x), \\
f(x)g(x) &= M \circ f(x), \\
\max \{ f(x),g(x) \} &= X \circ f(x), \\
f(x) - g(x) &= S \circ f(x), \\
f(x)/g(x) &= D \circ f(x), \\
\min \{ f(x),g(x) \} &= N \circ f(x).
\end{align*}
\]

Each of these combinations can thus be expressed as the composition of two continuous functions, and hence is also a continuous function, thus completing the proofs of the heretofore unproved statements in Theorem 4.3.

Proposition 10.9. If \( f: [a,b] \to \mathbb{R} \) is continuous on a closed interval, then its image \( C = f[a,b] \subset \mathbb{R}^2 \) is a closed subset.
Proof: Let \( S = \mathbb{R}^2 \setminus C \). Given \((c, d) \in S\), consider the function
\[
h(t) = \max\{ |f(t) - c|, |g(t) - d| \}.
\]
Since \( h: [a, b] \to \mathbb{R} \) is built up from combinations of continuous functions, we can use Theorem 4.3 to verify that it is continuous. Let \( \alpha \in \mathbb{R} \) be a lower bound for \( h[a, b] \), so \( h(t) \geq \alpha \) for all \( a \leq t \leq b \). We claim that the open square \( R_{c-\alpha,d-\alpha}^{c+\alpha,d+\alpha} \subset C \). Indeed, if \((x, y) \in R_{c-\alpha,d-\alpha}^{c+\alpha,d+\alpha}\) then both \(|x - c|\) and \(|y - d|\) are < \( \alpha \), but this implies \((x, y) \in S\) since if \( x = f(t), y = g(t) \) for some \( a \leq t \leq b \), then \( h(t) < \alpha \). Since every point \((c, d) \in S\) belongs to an open square contained in \( S \), this proves that \( S \) is open, and hence its complement \( C = \mathbb{R}^2 \setminus S \) is closed. \( Q.E.D. \)

Given a continuous vector-valued function \( f: I \to \mathbb{R}^2 \), where \( I \subset \mathbb{R} \) is an interval (open, closed, half open, bounded or unbounded) we can identify its range with a parametrized plane curve:
\[
C = f(I) = \{(f(t), g(t)) \mid t \in I\} \subset \mathbb{R}^2.
\]
In the case \( I = [a, b] \) is a closed interval, and hence the curve \( C = f[a, b] \) is a closed subset, we call \( a = f(a) \) and \( b = f(b) \) the endpoints of \( C \).

The curve \( C \) is called simple if \( f \) is one-to-one, meaning that if \( s \neq t \) then \( f(s) \neq f(t) \).

\textbf{Theorem 10.10.} If \( C = f[a, b] \) is a simple curve, then its complement \( \mathbb{R}^2 \setminus C \) is a connected open set.

This result is, perhaps surprisingly, not easy to prove, and requires the more sophisticated techniques developed to prove the Jordan Curve Theorem 10.11 below.

A closed curve is the image of a continuous function \( f: [a, b] \to \mathbb{R}^2 \) on a closed interval that satisfies \( f(a) = f(b) \). Examples include the unit circle (9.10) and the rectangular boundary of an open rectangle (9.11). The closed curve is simple if \( f: [a, b] \to \mathbb{R}^2 \) is one-to-one; equivalently, the only values \( a \leq s < t \leq b \) for which \( f(s) = f(t) \) are \( s = a \) and \( t = b \). A simple closed curve is also known as a Jordan curve, named after the nineteenth French mathematician Camille Jordan, who first proved the following theorem.

\textbf{Warning:} The term “closed” could be misleading here. Proposition 10.9 says that the image of \( f \) is a closed subset of \( \mathbb{R}^2 \) independent of the extra condition, but that is not what is usually implied by the term “closed curve”.

\textbf{Theorem 10.11.} Let \( C \subset \mathbb{R}^2 \) be a simple closed curve. The complement \( \mathbb{R} \setminus C \) is the union of two disjoint nonempty open connected sets, one bounded — the curve’s interior — and one unbounded — the curve’s exterior.

Figure 9 illustrates the theorem with a moderately complicated Jordan curve, whose interior is shaded and whose exterior is white. The Jordan Curve Theorem 10.11 seems intuitively obvious, but is surprisingly difficult to prove rigorously. One of the reasons for
this difficulty is that the curve in question could be a complicated fractal. Jordan’s famous proof, cf. [7], served to inspire the modern field of algebraic topology, [25].

Finally, we consider vector-valued functions of two variables, that is $F: \mathbb{R}^2 \to \mathbb{R}^2$, which is given by $F(x, y) = (F(x, y), G(x, y))$, where its components $F, G: \mathbb{R}^2 \to \mathbb{R}$ are scalar-valued. We say that $F$ is continuous if $F^{-1}(S) \subset \mathbb{R}^2$ is open whenever $S \subset \mathbb{R}^2$ is open. As always, we need only check the condition when $S$ is a basic open subset, i.e., an open rectangle.

The following analog of Lemma 10.8 is proved similarly, and thus left to the reader.

**Lemma 10.12.** If $F, G: \mathbb{R}^2 \to \mathbb{R}$ are continuous, then $F(x, y) = (F(x, y), G(x, y))$ defines a continuous function $F: \mathbb{R}^2 \to \mathbb{R}^2$.

As before, the composition of continuous functions is continuous, with an identical proof. Thus, by combining Lemma 10.12 with Example 10.1, we deduce that algebraic combinations of continuous functions of two variables, e.g.,

\[
F(x, y) + G(x, y), \quad F(x, y) - G(x, y), \quad F(x, y)G(x, y),
\]

\[
\max \{F(x, y), G(x, y)\}, \quad \min \{F(x, y), G(x, y)\},
\]

are all continuous. The quotient function $F(x, y)/G(x, y)$ is continuous on the subset $D = \{(x, y) | G(x, y) \neq 0\}$. Note that since $D = G^{-1}(\mathbb{R} \setminus \{0\})$, continuity of $G$ implies $D \subset \mathbb{R}^2$ is open, and nonempty unless $G(x, y) \equiv 0$ is the zero function.

We can specialize these combinations using Lemma 10.2. A particularly interesting combination is the function

\[ F(x, y) = f(x) - f(y) \]  \hspace{1cm} (10.2)

which is continuous whenever $f: \mathbb{R} \to \mathbb{R}$ is continuous. Moreover, $F$ vanishes on the diagonal

\[ \Delta = \{x = y\} = \{(x, x) | x \in \mathbb{R}\} \subset \mathbb{R}^2, \]  \hspace{1cm} (10.3)
meaning \( F(x, x) = 0 \) for all \( x \in \mathbb{R} \).

The next result establishes the \textit{uniform continuity} of continuous functions defined on closed intervals.

\textbf{Theorem 10.13.} Let \( f: [a, b] \to \mathbb{R} \) be continuous. Given \( s > 0 \), there exists \( r > 0 \) such that \( |f(x) - f(y)| < s \) whenever \( x, y \in [a, b] \) and \( |x - y| < r \).

\textit{Proof:} Let \( F(x, y) = f(x) - f(y) \), which is continuous. Given that \( S = F^{-1}(-s, s) \) is open and contains the diagonal \( \{(x, x) | x \in [a, b] \} \), let \( U_r \subset S \) be the diagonal strip produced by Lemma 9.16. Then \( |x - y| < r \) implies \((x, y) \in U_r \subset F^{-1}(-s, s)\) which in turn implies \( |f(x) - f(y)| < s \). \( Q.E.D. \)

There is also a two variable analog of the Uniform Continuity Theorem 10.13, which can be stated as follows.

\textbf{Theorem 10.14.} Let \( \overline{D} \subset \mathbb{R}^2 \) be a compact domain, and \( F: \overline{D} \to \mathbb{R} \) a continuous function. Given \( s > 0 \), there exists \( r > 0 \) such that \( |F(x, y) - F(z, w)| < s \) whenever \((x, y), (z, w) \in \overline{D}\) with \( |x - z| < r \) and \( |y - w| < r \).

The proof relies on a similar analysis of the function \( H(x, y, z, w) = F(x, y) - F(z, w) \) depending on four variables, and we skip the details. For this and many other reasons, e.g., the fact that we live in a three-dimensional universe, or, relativistically speaking, four-dimensional space-time, it would be worth developing the calculus of functions of \( n \) variables all at once, but we have chosen to retain our focus on the two variable case to avoid excessive notational clutter. Moreover, adapting the results and proofs from functions of two variables to functions of more than two variables is almost always straightforward.

The two dimensional analog of Theorem 4.8 is true, but much harder to prove. It was finally established, in all dimensions, by the Dutch mathematician L.E.J. Brouwer in 1912, cf. \([8]\). This celebrated result is known as the Theorem on Invariance of Domain.

\textbf{Theorem 10.15.} Let \( U \subset \mathbb{R}^2 \) be open. If \( F: U \to \mathbb{R}^2 \) is continuous and one-to-one, then its range \( V = F(U) \) is open and its inverse \( F^{-1}: V \to U \) is continuous.

Functions that are continuous and one-to-one, hence with continuous inverse, are known as \textit{homeomorphisms}. Theorem 10.15 and its generalizations imply that the dimension of a set cannot change under a homeomorphism. This intuitively “obvious” result stands in contrast to Cantor’s surprising construction of a highly discontinuous function that maps the one-dimensional unit interval \( I = [0, 1] \) in a one-to-one manner to the two-dimensional unit square \( Q = \{(x, y) | 0 \leq x, y \leq 1\} \). Cantor’s function\(^\dagger\) can be defined as follows.

\(^\dagger\) Actually, Cantor had a different construction of such a function involving continued fractions.
Given the binary expansion \( .b_1b_2b_3b_4 \ldots \) of any \( t \in [0,1] \), so that each \( b_i = 0 \) or 1, one constructs a unique point \((x,y) = f(t) \in Q\) with binary expansions
\[
x = .b_1b_3b_5 \ldots , \quad y = .b_2b_4b_6 \ldots .
\]
To define the map unambiguously, we do not allow binary expansions with all \( b_i = 1 \) for all \( i \geq n \) for some \( n \in \mathbb{N} \), with one exception: the endpoint \( 1 = .1111111 \ldots \). For example, we only allow the binary representation \(.10000 \ldots \) of \( \frac{1}{2} \), and not the alternative \(.01111 \ldots \).
The same choice clearly applies to the image points \( x, y \). The inverse of \( f : I \to Q \), maps \( Q \) back to \( I \) in the evident manner, shuffling the binary digits of \( x \) and \( y \) to produce that of \( t \).
The Invariance of Domain Theorem 10.15 implies that there is no continuous one-to-one function with this behavior.

We end this section with a detailed analysis of a particularly interesting example of the restriction of a continuous function to a subset.

**Example 10.16.** Let us return to the countably infinite subset
\[
A = \{ 0 \} \cup \tilde{A}, \quad \text{where} \quad \tilde{A} = \{ 1/n \mid n \in \mathbb{N} \}
\]
considered in Example 5.6. As we saw, the function \( f : A \to \mathbb{R} \) is continuous if and only if the corresponding sequence \( f_n = f(1/n) \) has a limit, namely \( \lim_{n \to \infty} f_n = f(0) \).

Consider the Cartesian product sets
\[
B = A \times A, \quad \tilde{B} = \tilde{A} \times \tilde{A}.
\]
Note first that
\[
B = \{ (0,0) \} \cup (\tilde{A} \times \{ 0 \}) \cup (\{ 0 \} \times \tilde{A}) \cup \tilde{B}.
\]
The relatively open subsets \( S \) are obtained by intersecting \( B \) or \( \tilde{B} \) with open rectangles. An analysis similar to the one-dimensional version in Example 5.6 shows that any subset \( S \subset \tilde{B} \) is open. If the point \((0,1/n) \in S \subset B\), then \( S \) must also contain a countably infinite subset of the form \( \{ (1/m,1/n) \mid M \leq m \in \mathbb{N} \} \subset \tilde{B} \) for some \( M \in \mathbb{N} \); similarly if \((1/m,0) \in S \subset B\), then \( S \) must contain a subset of the form \( \{ (1/m,1/n) \mid N \leq n \in \mathbb{N} \} \subset \tilde{B} \) for some \( N \in \mathbb{N} \). Finally, if \((0,0) \in S \subset B\), then \( S \) must also contain a subset of the form \( \{ (1/m,1/n) \mid M \leq m \in \mathbb{N}, N \leq n \in \mathbb{N} \} \subset \tilde{B} \) for some \( M, N \in \mathbb{N} \). Replacing \( M, N \) by their maximum, we simplify the latter criterion to just be that \( S \) contain a subset of the form
\[
U_M = \{ (1/m,1/n) \mid M \leq m, n \in \mathbb{N} \} \subset \tilde{B}.
\]
Any subset satisfying these three requirements is open. The limit points of \( B \) are the points on the \( x \) and \( y \) axis, namely \((0,0), (1/m,0), (0,1/n)\) for \( m, n \in \mathbb{N} \).

A function \( \tilde{F} : \tilde{B} \to \mathbb{R} \) can be identified with a doubly infinite sequence:
\[
f_{m,n} = \tilde{F} \left( \frac{1}{m}, \frac{1}{n} \right) \in \mathbb{R} \quad \text{for all} \quad m, n \in \mathbb{N}.
\]
Any such function is continuous since all subsets of $\tilde{B}$ are open. What about a function $F: B \to \mathbb{R}$? In this case, the points in $B$ with one or both coordinates equal to 0 serve to define limits of the doubly infinite sequence. Let us set

$$a = F(0,0), \quad b_m = F\left(\frac{1}{m},0\right), \quad c_n = F\left(0,\frac{1}{n}\right), \quad f_{m,n} = F\left(\frac{1}{m},\frac{1}{n}\right).$$

Then continuity of $F$ implies that all the limits of the doubly infinite sequence $f_{m,n}$ exist:

$$c_n = \lim_{m \to \infty} f_{m,n} = \lim_{1/m \to 0} F\left(\frac{1}{m},\frac{1}{n}\right) = F\left(0,\frac{1}{n}\right),$$

$$b_m = \lim_{n \to \infty} f_{m,n} = \lim_{1/n \to 0} F\left(\frac{1}{m},\frac{1}{n}\right) = F\left(\frac{1}{m},0\right),$$

$$a = \lim_{m,n \to \infty} f_{m,n} = \lim_{(1/m,1/n) \to (0,0)} F\left(\frac{1}{m},\frac{1}{n}\right),$$

$$= \lim_{m \to \infty} b_m = \lim_{1/m \to 0} F\left(\frac{1}{m},0\right) = \lim_{n \to \infty} c_n = \lim_{1/n \to 0} F\left(0,\frac{1}{n}\right).$$

As usual, if $F: \mathbb{R}^2 \to \mathbb{R}$ is continuous, its restriction $F: B \to \mathbb{R}$ is also continuous, and hence obeys the above limiting conditions. Vice versa, given a continuous $F: B \to \mathbb{R}$, we can reconstruct a continuous $F: \mathbb{R}^2 \to \mathbb{R}$ that agrees with it when restricted to $B$ by using a suitable interpolation method, e.g., by using piecewise linear functions on triangles. We leave it to the motivated reader to fill in the details.

Now, let us look at the following subset:

$$\hat{B} = \{(0,0)\} \cup \tilde{B}, \quad (10.7)$$

i.e., we delete all the limit points of $B \supset \tilde{B}$ except for the origin. As before, any subset of $\hat{B}$ is open in $\hat{B}$. On the other hand, if $(0,0) \in S \subset \hat{B}$ is open, then, as above, $S$ must contain a subset of the form (10.6).

Now, let us return to our one-dimensional discrete set (10.4). Let

$$\tilde{f}: \tilde{A} \to \mathbb{R}, \quad \text{with} \quad f_n = \tilde{f}(1/n)$$

be any function, or, equivalently, any sequence $\{f_1, f_2, f_3, \ldots\}$. We are interested in whether or not the sequence $\{f_n\}$ converges or, equivalently, whether there exists a continuous extension

$$f: A \to \mathbb{R} \quad \text{such that} \quad f(1/n) = \tilde{f}(1/n) \quad \text{for all} \quad n \in \mathbb{N},$$

in which case

$$\lim_{n \to \infty} f_n = \lim_{1/n \to 0} f(1/n) = f(0).$$

**Definition 10.17.** The function $\tilde{f}: \tilde{A} \to \mathbb{R}$, or corresponding sequence $f_n = \tilde{f}(1/n)$, satisfies the **Cauchy criterion** if

$$\hat{F}\left(\frac{1}{m},\frac{1}{n}\right) = \hat{f}\left(\frac{1}{m}\right) - \hat{f}\left(\frac{1}{n}\right) = f_m - f_n, \quad \hat{F}(0,0) = 0, \quad (10.8)$$
defines a continuous function \( \hat{F} : \hat{B} \to \mathbb{R} \).

The function \( \hat{F} \) in (10.8) is constructed in analogy with the function (10.2), whose continuity was noted above. Note that it vanishes on the diagonal \( \hat{B} \cap \Delta \), cf. (10.3), so \( \hat{F}(0,0) = \hat{F}(1/m,1/m) = 0 \) for all \( m \in \mathbb{N} \). It An important result, which is named after the nineteenth century French mathematician Augustin-Louis Cauchy, a key founder of modern analysis, states that a sequence that satisfies the Cauchy criterion has a unique limit point.

**Theorem 10.18.** If the function \( \tilde{f} : \tilde{A} \to \mathbb{R} \) or, equivalently, the infinite sequence \( f_n = \tilde{f}(1/n) \) satisfies the Cauchy criterion, meaning the function \( \hat{F} : \hat{B} \to \mathbb{R} \) defined in (10.8), is continuous, then there exists a continuous extension \( f : A \to \mathbb{R} \) such that \( f(1/n) = \tilde{f}(1/n) \) for all \( n \in \mathbb{N} \), and hence the corresponding sequence converges with limiting value \( f(0) \).

Moreover, one can extend \( \hat{F} \) to a continuous function \( F : B \to \mathbb{R} \) by setting \( F(x,y) = f(x) - f(y) \) for all \( x, y \in A \).

**Proof:** Given any \( r > 0 \), consider the open interval \((-r, r)\), which contains the origin. Thus,

\[
\hat{F}^{-1}(-r, r) = \{0,0\} \cup \left\{ \left( \frac{1}{m}, \frac{1}{n} \right) \mid \left| \hat{F}\left( \frac{1}{m}, \frac{1}{n} \right) \right| < r \right\}
\]

is open in \( \hat{B} \), meaning that it contains a subset of the form (10.6) for some \( M \in \mathbb{N} \). In other words, the sequence \( \{f_n\} \) satisfies the Cauchy criterion if and only if for every \( r > 0 \) there exists \( M \in N \) such that

\[
|f_m - f_n| < r \quad \text{whenever} \quad m, n \geq M. \tag{10.9}
\]

The first thing to note is that (10.9) implies that the sequence is bounded. To see this, setting \( m = M \) implies

\[
r - f_M \leq f_n \leq f_M + r \quad \text{whenever} \quad n \geq M.
\]

Thus, by including the finitely many points \( f_1, \ldots , f_{M-1} \) not accounted for as yet, we obtain overall bounds for the entire sequence:

\[
\min \{ f_1, \ldots , f_{M-1}, r - f_M \} \leq f_n \leq \max \{ f_1, \ldots , f_{M-1}, r + f_M \} \quad \text{for all} \quad n \in \mathbb{N}.
\]

According to the Limit Point Theorem 5.10, the bounded infinite set defined by the sequence \( \{f_n\} \) has at least one limit point, say \( \ell \). We claim that the extended function \( f : A \to \mathbb{R} \) given by

\[
f(x) = \begin{cases} 
  f_n = \tilde{f}(1/n), & x = 1/n \in \tilde{A}, \\
  \ell, & x = 0,
\end{cases} \tag{10.10}
\]

Experts will now recognize the standard Cauchy criterion for convergence of sequences.

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\( \dagger \) Experts will now recognize the standard Cauchy criterion for convergence of sequences.
is continuous. This will establish the result, demonstrating that

$$\lim_{n \to \infty} f_n = \ell.$$  

To prove continuity, Lemma 5.9 implies that $\ell$ is also a limit point of the subsequence $\{ f_n \mid M \leq n \in \mathbb{N} \}$ obtained by omitting the first $M - 1$ terms. Thus, given $r > 0$, the limit point criterion implies that there exists $m \geq M$ such that $f_m \in (L - r, L + r) \setminus \{ \ell \}$. But then

$$|f_n - \ell| \leq |f_n - f_m| + |f_m - \ell| < 2r \quad \text{for all} \quad n \geq M.$$  

We conclude that

$$f^{-1}(\ell - 2r, \ell + 2r) \supset \{0\} \cup \{1/n \mid M \leq n \in \mathbb{N}\}, \quad (10.11)$$

which implies that the left hand side is an open subset of $A$. Since this holds for all $r > 0$, we deduce the continuity of the extended function (10.10). The final statement of the Theorem is then easily established. \[Q.E.D.\]

One can extend the Cauchy criterion to treat removable discontinuities of continuous functions. Given $a \in \mathbb{R}$, consider the subset

$$C_a = \{(a, a)\} \cup W_a, \quad \text{where} \quad W_a = \{(x, y) \mid (x - a)(y - a) \neq 0\}$$

$$= \mathbb{R}^2 \setminus \{(x = a) \cup \{y = a\}\},$$

is the open subset obtained by deleting the horizontal and vertical lines passing through $(a, a)$. The basic open subsets of $C_a$ consist of all open rectangles that do not contain $(a, a)$ along with sets of the form $\{(a, a)\} \cup (R \cap W_a)$ where $R \subset \mathbb{R}^2$ is an open rectangle containing $(a, a)$. A continuous function $f: \mathbb{R} \to \mathbb{R}$ defines a continuous function $F: \mathbb{R}^2 \to \mathbb{R}$ given by $F(x, y) = f(x) - f(y)$, which remains continuous when restricted to $C_a$. The following continuous Cauchy criterion allows us to test for removable discontinuities in a function without knowing what the limiting value at the discontinuity is.

**Theorem 10.19.** Suppose $\tilde{f}: \mathbb{R} \setminus \{a\} \to \mathbb{R}$ is continuous. Define $\tilde{F}: C_a \to \mathbb{R}$ by

$$\tilde{F}(x, y) = \begin{cases} 
\tilde{f}(x) - \tilde{f}(y), & x \neq a \quad \text{and} \quad y \neq a, \\
0 & x = y = a.
\end{cases}$$

Then $\tilde{f}$ has a removable discontinuity at $x = a$ if and only if $\tilde{F}$ is continuous.

**Proof:** The statement that $\tilde{f}$ has a removable discontinuity at $x = a$ means that there exists a continuous function $f: \mathbb{R} \to \mathbb{R}$ such that $f(x) = \tilde{f}(x)$ for all $x \neq a$. In this case, the function $F: \mathbb{R}^2 \to \mathbb{R}$ given by $F(x, y) = f(x) - f(y)$ is continuous. Moreover, $F(x, y) = \tilde{F}(x, y)$ when $(x, y) \in C_a$, and hence $\tilde{F}: C_a \to \mathbb{R}$ is continuous.

To prove the converse, let us assume that $a = 0$ without loss of generality, and thus the set considered in the preceding example satisfies $\tilde{B} \subset C_0$. Let $\tilde{F}: \tilde{B} \to \mathbb{R}$ be the restriction of $\tilde{F}$, so $\tilde{F}(x, y) = \tilde{F}(x, y)$ whenever $(x, y) \in \tilde{B}$. Continuity of $\tilde{F}$ implies continuity of $\tilde{F}$,
which, by Theorem 10.18, implies that there is a continuous extension \( f: A \to \mathbb{R} \) of \( \tilde{f} \) with limiting value \( \ell = \lim_{n \to \infty} \tilde{f}(1/n) = f(0) \). We claim that the further extension

\[
 f(x) = \begin{cases} 
 \tilde{f}(x), & x \neq 0, \\
 \ell, & x = 0, 
\end{cases}
\]

is also continuous, which serves to prove the result.

Let \( s > 0 \). Continuity of \( \tilde{F} \) implies \( \tilde{F}^{-1}(-s, s) \subset C_0 \) is open and contains the origin, and hence contains a subset of the form

\[
 W_0 \cap Q_r = \left\{ (0, 0) \right\} \cup \left\{ (x, y) \mid 0 < |x|, |y| < r \right\} \subset \tilde{F}^{-1}(-s, s),
\]

where \( Q_r = R_{-r,-r}^{r,r} \) is the square of width \( 2r \) centered at the origin. In other words, if \( 0 < |x|, |y| < r \) then \( |F(x,y)| = |f(x)-f(y)| < s \). Moreover, since \( \ell = \lim_{n \to \infty} f(1/n) \), there exists \( n \in \mathbb{N} \) such that \( 1/n < r \) and \( |f(1/n) - \ell| < s \). But then, whenever \( |x| < r \),

\[
 |f(x) - \ell| \leq |f(x) - f(1/n)| + |f(1/n) - \ell| < 2s
\]

Thus, \((-r, r) \subset f^{-1}(\ell-2s, \ell+2s)\). The fact that this holds for all \( 0 < s \in \mathbb{R} \) proves that \( f \) is continuous.

Q.E.D.

11. Differentiating Functions of Two Variables.

The next step is to define the derivative of a function of two variables and investigate its properties. In this case, we can no longer form a single difference quotient to define its derivative. Instead, we introduce an analog of the Carathéodory formulation (7.4).

**Definition 11.1.** Let \( F: \mathbb{R}^2 \to \mathbb{R} \) be continuous. We say that \( F \) is *differentiable* at the point \((a, b) \in \mathbb{R}^2\) if there exist continuous functions \( Q, R: \mathbb{R}^2 \to \mathbb{R} \) such that

\[
 F(x,y) = F(a,b) + Q(x,y) (x-a) + R(x,y) (y-b). \tag{11.1}
\]

The functions \( Q(x, y), R(x, y) \) are *not* uniquely determined by the equation (11.1). Indeed, if \( S(x, y) \) is any continuous function then

\[
 \tilde{Q}(x,y) = Q(x,y) + S(x,y)(y-b), \quad \tilde{R}(x,y) = R(x,y) - S(x,y)(x-a),
\]

will also satisfy (11.1). Nevertheless, as we will soon see, their values \( Q(a,b) \) and \( R(a,b) \) are uniquely determined, and serve to specify the two components of the *derivative* \( F'(a,b) \). More specifically, these values are said to define the *partial derivatives* of the differentiable function \( F \) at the point \((a, b) \), and we will use the following subscript notation:

\[
 F_x(a,b) = Q(a,b), \quad F_y(a,b) = R(a,b).
\]

The reason for the term “partial derivative” is partly because they each form “part” of the derivative, which is a pair of real numbers or, equivalently, a vector commonly known as the *gradient* of \( F \), and often denoted by the symbol

\[
 \nabla F(a,b) = F'(a,b) = (F_x(a,b), F_y(a,b)). \tag{11.2}
\]
Another reason for the name is that the partial derivatives are obtained by differentiating “part” of the function $F(x, y)$. Namely if we, say, fix $y = b$ then the result is a scalar function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = F(x, b)$, which, according to Proposition 10.3, is continuous. Setting $y = b$ in (11.1) produces

$$F(x, b) = F(a, b) + Q(x, b) (x - a)$$

or, equivalently,

$$f(x) = f(a) + (x - a) q(x) \quad \text{where} \quad q(x) = Q(x, b).$$

This is the Carathéodory formula (7.4) for the function $f(x)$, and hence

$$q(a) = Q(a, b) = F_x(a, b) = f'(a),$$

which in turn implies $F_x(a, b) = f'(a)$ is uniquely determined. Similarly, fixing $x = a$ leads to the partial function $g(y) = F(a, y)$, and now (11.1) reduces to

$$F(a, y) = F(a, b) + R(a, y) (y - b)$$

or, equivalently,

$$g(y) = g(b) + (y - b) r(y) \quad \text{where} \quad r(y) = R(a, y),$$

and hence

$$r(b) = R(a, b) = F_y(a, b) = g'(b).$$

**Example 11.2.** Consider the function

$$F(x, y) = x^2 y^3 + 5 y^2 + 2 x - 1.$$ 

We easily compute its partial derivatives by just regarding one variable as a fixed constant and differentiating with respect to the other using all the usual rules of differentiation for a function of the single variable. Thus, the $x$ and $y$ partial derivatives are, respectively,

$$F_x(x, y) = 2 x y^3 + 2, \quad F_y(x, y) = 3 x^2 y^2 + 10 y.$$ 

Differentiability of $F$ at the origin $(x, y) = (0, 0)$ follows from the identity

$$F(x, y) = -1 + x Q(x, y) + y R(x, y),$$

where, for instance,

$$Q(x, y) = x y^3 + 3 y^2 + 2, \quad R(x, y) = 3 y.$$ 

As above $Q$ and $R$ are not uniquely determined. Thus, $F_x(0, 0) = Q(0, 0) = 2$, $F_y(0, 0) = R(0, 0) = 0$, in accordance with the preceding calculation. Differentiability at a general point $(a, b)$ follows similarly using a slightly more complicated algebraic manipulation. However, one can use the general rules for differentiation to verify the differentiability of the polynomial function $F(x, y)$ without having to perform the explicit manipulations.
**Example 11.3.** *Warning:* The existence of partial derivatives does not necessarily imply that the function is differentiable. For example, consider the function

\[ F(x, y) = \begin{cases} \frac{xy}{x^2 + y^2}, & (x, y) \neq (0, 0), \\ 0, & (x, y) = (0, 0), \end{cases} \]

from Example 10.4. Since

\[ f(x) = F(x, 0) \equiv 0, \quad g(y) = F(0, y) \equiv 0, \]

we clearly have

\[ f'(0) = 0, \quad g'(0) = 0, \]

and hence both partial derivatives of \( F \) at \((0, 0)\) exist and are equal to \( 0 = F_x(0, 0) = F_y(0, 0) \). But the function \( F(x, y) \) is not continuous, and hence there are no continuous functions \( Q, R \) such that (11.1) holds. Thus, despite possessing both partial derivatives, \( F \) is *not* differentiable at the origin.

Algebraic combinations, including sums and differences, products, quotients (when the denominator doesn’t vanish), maxima, and minima of differentiable functions are differentiable. We leave their verification to the interested reader. Furthermore, since partial derivatives can be computed by taking an ordinary derivative while fixing one of the two variables, they obey all the algebraic rules we established in Section 7. In particular, the partial derivative of the sum and the difference of functions is the sum and difference of their partial derivatives. The product rule for the partial derivatives of

\[ H(x, y) = F(x, y) G(x, y) \]

takes the form

\[ H_x(a, b) = F_x(a, b) G(a, b) + F(a, b) G_x(a, b), \]

\[ H_y(a, b) = F_y(a, b) G(a, b) + F(a, b) G_y(a, b). \]

As for the reciprocal \( R(x, y) = 1/F(x, y) \) provided \( F(x, y) \neq 0 \), we can differentiate the product

\[ 1 = F(x, y) R(x, y), \]

producing

\[ 0 = F_x(a, b) R(a, b) + F(a, b) R_x(a, b) = \frac{F_x(a, b)}{F(a, b)} + F(a, b) R_x(a, b), \]

\[ 0 = F_y(a, b) R(a, b) + F(a, b) R_y(a, b) = \frac{F_y(a, b)}{F(a, b)} + F(a, b) R_y(a, b), \]

and hence

\[ R_x(a, b) = -\frac{F_x(a, b)}{F(a, b)^2}, \quad R_y(a, b) = -\frac{F_y(a, b)}{F(a, b)^2}. \]

We leave it to the reader to write out the quotient rule, as well as fill in the omitted proofs.
As for the Chain Rule, there are several versions depending upon which type of functions are being composed. Consider first the case $h(t) = F(f(t), g(t))$ where $f, g: \mathbb{R} \to \mathbb{R}$ are differentiable at $t = a$, while $F: \mathbb{R} \to \mathbb{R}$ is differentiable at $(x, y) = (f(a), g(a))$. Then the composite function $h(t)$ is differentiable at $t = a$, and

$$h'(a) = F_x(f(a), g(a)) f'(a) + F_y(f(a), g(a)) g'(a). \quad (11.3)$$

To prove this, we use (11.1) and the equations

$$f(t) = f(a) + (t - a) q(t), \quad g(t) = g(a) + (t - a) r(t), \quad \text{where} \quad f'(a) = q(a), \quad g'(a) = r(a),$$

to write

$$h(t) = F(f(t), g(t)) = F(f(a), g(a)) + Q(f(t), g(t)) \left[ f(t) - f(a) \right] + R(f(t), g(t)) \left[ g(t) - g(a) \right] = h(a) + (t - a) \left[ Q(f(t), g(t)) q(t) + R(f(t), g(t)) r(t) \right].$$

The function multiplying $t - a$ is continuous, and hence

$$h'(a) = Q(a, b) q(a) + R(a, b) r(a) = F_x(f(a), g(a)) f'(a) + F_y(f(a), g(a)) g'(a).$$

The other two versions of the chain rule are proved similarly. The composition $G(x, y) = f(F(x, y))$ at the point $(a, b)$ has derivatives

$$G_x(a, b) = f'(F(a, b)) F_x(a, b), \quad G_y(a, b) = f'(F(a, b)) F_y(a, b). \quad (11.4)$$

Finally, if $G: \mathbb{R}^2 \to \mathbb{R}^2$ is given by $G(x, y) = (I(x, y), J(x, y))$, then the partial derivatives of their composition

$$K(x, y) = F \circ G(x, y) = F(I(x, y), J(x, y))$$

at the point $(a, b)$ are

$$K_x(a, b) = F_x(a, b) I_x(a, b) + F_y(a, b) J_x(a, b),$$
$$K_y(a, b) = F_x(a, b) I_y(a, b) + F_y(a, b) J_y(a, b). \quad (11.5)$$

The formulas (11.3–5) can all be subsumed under a general Chain Rule formula involving Jacobian matrices for the derivative of all possible compositions, $[1, 13]$.

We can proceed on to higher order partial derivatives by playing the same game as in the univariate case. Starting with (11.1), suppose both $Q(x, y)$ and $R(x, y)$ are themselves differentiable at $(a, b)$, so that

$$Q(x, y) = Q(a, b) + S(x, y) (x - a) + T(x, y) (y - b),$$
$$R(x, y) = R(a, b) + V(x, y) (x - a) + W(x, y) (y - b), \quad (11.6)$$

for $S, T, V, W$ continuous, with

$$Q_x(a, b) = S(a, b), \quad Q_y(a, b) = T(a, b), \quad R_x(a, b) = V(a, b), \quad R_y(a, b) = W(a, b).$$
On the other hand, differentiating (11.1) and substituting using (11.6), we find
\[ F_x(x, y) = Q(x, y) + Q_x(x, y)(x - a) + R_x(x, y)(y - b) \]
\[ = Q(a, b) + \left[ S(x, y) + Q_x(x, y) \right] (x - a) + \left[ T(x, y) + R_x(x, y) \right] (y - b), \]
\[ F_y(x, y) = R(x, y) + Q_y(x, y)(x - a) + R_y(x, y)(y - b) \]
\[ = R(a, b) + \left[ V(x, y) + Q_y(x, y) \right] (x - a) + \left[ W(x, y) + R_y(x, y) \right] (y - b), \]
and hence
\[ F_{xx}(a, b) = S(a, b) + Q_x(a, b) = 2S(a, b), \]
\[ F_{xy}(a, b) = T(a, b) + R_x(a, b) = T(a, b) + V(a, b), \]
\[ F_{yx}(a, b) = V(a, b) + Q_y(a, b) = T(a, b) + V(a, b), \]
\[ F_{yy}(a, b) = W(a, b) + R_y(a, b) = 2W(a, b). \]
The second expression is the result of first applying the \( x \) partial derivative to \( F \) and then applying the \( y \) partial derivative to \( F_x \), whereas the third expression is in the opposite order: first apply the \( y \) partial derivative to \( F \) and then apply the \( x \) partial derivative to \( F_y \). Interestingly, their right hand sides are exactly the same. Combining (11.1, 6), we deduce
\[ F(x, y) = F(a, b) + F_x(a, b)(x - a) + F_y(a, b)(y - b) + \]
\[ + S(x, y)(x - a)^2 + U(x, y)(x - a)(y - b) + W(x, y)(y - b)^2, \]
where \( U(x, y) = T(x, y) + V(x, y) \). If (11.7) holds where \( S, U, W \) are continuous, we say that \( F \) is \textit{twice differentiable} at \((a, b)\). As before, these functions are not uniquely determined; however, their values at the point \((a, b)\) are, and equate to the second order partial derivatives of \( F \) there:
\[ F_{xx}(a, b) = 2S(a, b), \quad F_{xy}(a, b) = F_{yx}(a, b) = U(a, b), \quad F_{yy}(a, b) = 2W(a, b). \]
The fact that the two “mixed derivatives” have the same formula is an important result, known as the Equality of Mixed Partials.

\textbf{Theorem 11.4.} Suppose \( F(x, y) \) is twice differentiable at the point \((a, b)\). Then
\[ F_{xy}(a, b) = F_{yx}(a, b). \]

It is not hard to show that algebraic combinations of twice differentiable functions remain twice differentiable. Thus, all polynomial functions are twice differentiable as are all rational functions as long as the denominator doesn’t vanish.

\textbf{Example 11.5.} Returning to the function
\[ F(x, y) = x^2y^3 + 5y^2 + 2x - 1 \]
considered in Example 11.2, with
\[ F_x(x, y) = 2xy^3 + 2, \quad F_y(x, y) = 3x^2y^2 + 10y, \]
we readily compute its second order partial derivatives:

\[ F_{xx}(x, y) = 2y^3, \quad F_{xy}(x, y) = F_{yx}(x, y) = 6xy^2, \quad F_{yy}(x, y) = 6x^2y + 10, \]

where the two mixed partial derivatives are equal because \( F \) is twice differentiable.

**Example 11.6.** Consider the function

\[
F(x, y) = \begin{cases} 
y \frac{x^2 - y^2}{x^2 + y^2}, & (x, y) \neq (0, 0), \\
0, & (x, y) = (0, 0). 
\end{cases}
\]

A direct computation shows that

\[
F_x(x, y) = \frac{y(x^4 + 4x^2y^2 - y^4)}{(x^2 + y^2)^2}, \quad F_y(x, y) = \frac{x(x^4 - 4x^2y^2 - y^4)}{(x^2 + y^2)^2}, \quad (x, y) \neq (0, 0).
\]

On the other hand,

\[
F(x, y) = xQ(x, y) \quad \text{where} \quad Q(x, y) = \begin{cases} 
y \frac{x^2 - y^2}{x^2 + y^2}, & (x, y) \neq (0, 0), \\
0, & (x, y) = (0, 0). 
\end{cases}
\]

It can be shown that \( Q \) is continuous, and hence \( F_x(0, 0) = 0 \). Now, set

\[
g(y) = F_x(0, y) = -y, \quad \text{and hence} \quad g'(0) = F_{xy}(0, 0) = -1,
\]

while

\[
h(x) = F_y(x, 0) = x, \quad \text{and hence} \quad h'(0) = F_{yx}(0, 0) = +1.
\]

Thus, both mixed partial derivatives of \( F \) exist at the origin, but are not equal. This is because there are no continuous functions \( S, U, W \) such that (11.7) holds, which in this case would be

\[
F(x, y) = x^2S(x, y) + xyU(x, y) + y^2W(x, y).
\]

For instance, if we choose \( S = W = 0 \), the function

\[
U(x, y) = \begin{cases} 
\frac{x^2 - y^2}{x^2 + y^2}, & (x, y) \neq (0, 0), \\
0, & (x, y) = (0, 0),
\end{cases}
\]

satisfies the preceding equation, but is not continuous. Indeed, its restrictions to the \( x \) and \( y \) axes

\[
U(x, 0) = \begin{cases} 
1, & x \neq 0, \\
0, & x = 0, 
\end{cases} \quad U(0, y) = \begin{cases} 
-1, & y \neq 0, \\
0, & y = 0,
\end{cases}
\]

are not continuous, in violation of Proposition 10.3.

By analogy with the one-dimensional version, we will call equation (11.7) is the first order Taylor expansion of the function \( F(x, y) \). The general version of Taylor’s Theorem for a function of two variables follows. Given \( n \in \mathbb{N} \), we say that \( F(x, y) \) is \( n \)-differentiable if all partial derivatives of \( F \) of order \( \leq n \) are continuous. We use the notation \( F_{ij} \) for
non-negative integers $i, j \geq 0$ to denote the partial derivative of order $k = i + j$ obtained by differentiating $i$ times with respect to $x$ and $j$ times with respect to $y$. For example, $F_{10} = F_x, F_{20} = F_{xx}, F_{11} = F_{xy}, F_{23} = F_{xxyy}$, and so on. If $F$ is at least $k$-differentiable, then a general version of the Equality of Mixed Partial Derivatives Theorem 11.4 implies that it does not matter in which order these derivatives are taken. The Taylor polynomial of degree $n$ for $F$ at the point $(a, b)$ is

$$P_n(x, y) = \sum_{0 \leq i + j \leq n} F_{ij}(a, b) \frac{(x-a)^i (y-b)^j}{i! j!}.$$  \hspace{1cm} (11.10)

**Theorem 11.7.** Let $F(x, y)$ be $n$-differentiable, and let $P_n(x, y)$ be its Taylor polynomial (11.10) of degree $n$ at the point $(a, b) \in \mathbb{R}^2$. Then $F(x, y)$ be $(n+1)$-differentiable at $(a, b)$ if and only if there exists continuous functions $R_{ij}(x, y)$ for all non-negative integers $0 \leq i, j \leq n+1$ satisfying $i + j = n + 1$, such that

$$F(x, y) = P_n(x, y) + \sum_{i+j=n+1} R_{ij}(x, y) \frac{(x-a)^i (y-b)^j}{i! j!},$$  \hspace{1cm} (11.11)

in which case $R_{ij}(a, b) = F_{ij}(a, b)$. Moreover, there exists a point $(z, w)$ in the open rectangle with opposite corners $(a, b)$ and $(x, y)$ such that

$$R_{ij}(x, y) = F_{ij}(z, w), \quad i + j = n + 1.$$  \hspace{1cm} (11.12)

12. Area.

We now turn to the problem that was the first immediately successful application of Newton’s calculus — computing the area of regions in the plane. The areas of simple shapes are well known from elementary geometry, but more complicated regions with curved boundaries require the full power of calculus – in particular the Fundamental Theorem 13.16.

Let us begin with the simplest possible region: a rectangle.

**Definition 12.1.** Let $a \leq b$ and $c \leq d$ be real numbers. The area of the bounded closed rectangle

$$\overline{R}_{a,c}^{b,d} = \{ (x, y) \mid a \leq x \leq b, \; c \leq y \leq d \}, \quad \text{area } \overline{R}_{a,c}^{b,d} = (d-c)(b-a).$$  \hspace{1cm} (12.1)

In other words, the area of a rectangle is the product of the lengths of its two sides. In particular, if $a = b$ and/or $c = d$, the rectangle degenerates into a line segment or a single point, both of which have 0 area.

We will take rectangles as our basic building blocks, and use them to define the area of general subsets $A \subset \mathbb{R}^2$. First, we postulate that the area of a disjoint union of rectangles is the sum of their individual areas:

$$\text{area } \left( \bigcup_{i=1}^{n} R_i \right) = \sum_{i=1}^{n} \text{area } R_i$$  \hspace{1cm} (12.2)
provided $R_i \cap R_j = \emptyset$ for all $i \neq j$. We can relax this requirement by allowing the rectangles to intersect along their boundaries, which have zero area. We thus call a collection of closed rectangles $R_1, \ldots, R_n$ essentially disjoint if the intersection of any pair $R_i \cap R_j$ has zero area, keeping in mind that the empty set also has zero area. The formula (12.2) continues to hold when the collection is essentially disjoint.

A fundamental property is that a subset necessarily has less area than the enclosing set:

$$\text{if } A \subset B \text{ then } \text{area } A \leq \text{area } B. \quad (12.3)$$

Thus, given a set $A \subset \mathbb{R}^2$, let $R = \{R_1, \ldots, R_n\}$ be an essentially disjoint collection of closed rectangles such that each $R_i \subset A$. Their union, denoted by $R^*$, is a subset of $A$, and hence

$$\text{area } A \geq \text{area } R^* = \text{area } \left( \bigcup_{i=1}^{n} R_i \right) = \sum_{i=1}^{n} \text{area } R_i,$$

where $R^* = \bigcup_{i=1}^{n} R_i$.

Note that if $A \neq \emptyset$, then there exist such collections since, at the very least, we could take $R_i = \{x_i\}$ where $\{x_1, \ldots, x_n\}$ is any finite set of points in $A$.

The Method of Exhaustion, dating back to Archimedes, relies on the intuition that, as we increase the number of rectangles so that they fill out (exhaust) more and more of their enclosing region, their total area forms a better and better approximation to its area; an example is sketched in Figure 10. Archimedes’ idea inspires our general definition of the area of a planar region.

**Definition 12.2.** Let $A \subset \mathbb{R}^2$. Let

$$E_A = \left\{ \sum_{i=1}^{n} \text{area } R_i \middle| \begin{array}{l}
 R_1, \ldots, R_n \text{ are essentially disjoint closed rectangles} \\
 \text{and } R_i \subset A \text{ for } i = 1, \ldots, n
 \end{array} \right\}, \quad (12.4)$$

which forms a subset of $[0, \infty) = \{ x \in \mathbb{R} \mid x \geq 0 \}$. Then we define the area of $A$ to be

$$\text{area } A = \begin{cases} 
 0, & \text{if } A = \emptyset, \\
 \sup E_A, & E_A \text{ is bounded}, \\
 \infty, & E_A \text{ is unbounded}. 
\end{cases} \quad (12.5)$$
Remark: There are more sophisticated definitions of area, such as Lebesgue measure, [21], that are better suited to advanced mathematical analysis and may produce different values for the area (measure) of more pathological subsets such as fractals, [12]. However, for our purposes this naïve but intuitive definition suffices.

Before proceeding, our first task is to check that the preceding definition of area is consistent, meaning that it gives the same answer when $A$ is itself a closed rectangle. In other words, we need to show that if $R = \mathbb{R}^n_{a,b}$ has area given by the original formula (12.1) and $R_1, \ldots, R_n$ are any collection of essentially disjoint closed rectangles with each $R_i \subset R$, then

$$\text{area } R^* = \sum_{i=1}^n \text{area } R_i \leq \text{area } R. \tag{12.6}$$

Since $R$ itself forms a possible collection, this implies consistency: area $R = \sup E_R$.

First suppose we chop up (or subdivide) $R$ into disjoint open rectangles by slicing it with a finite collection of horizontal and vertical lines, as illustrated in the first plot in Figure 11. In other words, we select numbers $a = x_0 < x_1 < x_2 < \cdots < x_{k-1} < x_k = b, \quad c = y_0 < y_1 < y_2 < \cdots < y_{l-1} < y_l = d,$

which, as before, we will refer to as partitions of the intervals $[a, b]$ and $[c, d]$, and consider the $n = kl$ closed rectangles

$$R_{i,j} = \mathbb{R}^{x_i, y_j}_{x_{i-1}, y_{j-1}} \subset R \quad \text{for } i = 1, \ldots, k, \quad j = 1, \ldots, l, \quad \text{such that } R = \bigcup_{i=1}^k \bigcup_{j=1}^l R_{i,j},$$

which we call a partition of $R$. Their boundaries are all contained the lines $x = x_i$ and $y = y_j$, which have zero area, and hence the $R_{i,j}$ are essentially disjoint. Their combined areas are

$$\text{area } \left( \bigcup_{i=1}^k \bigcup_{j=1}^l R_{i,j} \right) = \sum_{i=1}^k \sum_{j=1}^l \text{area } R_{i,j} = \sum_{i=1}^k \sum_{j=1}^l (x_i - x_{i-1}) (y_j - y_{j-1})$$

$$= \left[ \sum_{i=1}^k (x_i - x_{i-1}) \right] \left[ \sum_{j=1}^l (y_j - y_{j-1}) \right]$$

$$= (x_k - x_0) (y_l - y_0) = (d-c) (b-a) = \text{area } R.$$
Figure 11. Subdividing a Rectangle.

cut in half. For this purpose, we let
\[ X = \{a, a_1, \ldots, a_n, b_1, \ldots, b_n, b\} = \{x_0, \ldots, x_k\}, \]
\[ Y = \{c, c_1, \ldots, c_n, d_1, \ldots, d_n, d\} = \{y_0, \ldots, y_l\}, \]
where the \(x_i\)'s and \(y_j\)'s are of the form in the preceding paragraph. Here, the set notation for \(X, Y\) implies that we order the \(a_m\)'s and \(b_m\)'s and eliminate any repeats to obtain the \(x_i\)'s, and similarly for the \(c_m\)'s and \(d_m\)'s to obtain the \(y_j\)'s. The corresponding horizontal and vertical lines subdivide not only \(R\) into the collection of \(R_{i,j}\)'s but also each \(S_m\) is itself subdivided by a subset of the \(R_{i,j}\)'s, and hence, by the preceding argument
\[
\text{area } S_m = \sum_{R_{i,j} \subset S_m} \text{area } R_{i,j}. \tag{12.7}
\]
Since they are mutually essentially disjoint, each \(R_{i,j}\) is contained in at most one \(S_m\), although some may not appear at all if we have omitted any part of \(R\) in our original collection \(S_1, \ldots, S_n\). In this way,
\[
\text{area } R = \sum_{i=1}^{k} \sum_{j=1}^{l} \text{area } R_{i,j} \geq \sum_{m=1}^{n} \text{area } S_m,
\]
with equality if and only if every \(R_{i,j}\) is contained in one of the \(S_m\)'s. In other words, the area of a rectangle is the same no matter how we subdivide it into subrectangles. This thus establishes consistency of our definition of area.

Given the construction in the preceding paragraph, let \(Z = \{Z_1, \ldots, Z_q\}\) be the set of all rectangles \(R_{i,j}\) that are not contained in one of the \(R_m\)'s. Observe that
\[
Z^* = \bigcup_{p=1}^{q} Z_p = R \setminus S^* \text{ is the closure of the complement to } S^* = \bigcup_{m=1}^{n} S_m.
\]
Moreover, since every \(R_{i,j}\) appears in either \(S^*\) or in \(Z^*\),
\[
S^* \cup Z^* = R, \quad \text{and} \quad \text{area } S^* + \text{area } Z^* = \text{area } R. \tag{12.8}
\]
More generally, given a subset \(D \subset R\), we can use the proceeding collections of rectangles
to approximate its area:

$$\text{area } D = \sup \left\{ \sum_{R_{i,j} \subset D} \text{area } R_{i,j} \right\}.$$  \hfill (12.9)

Indeed, given any other essentially disjoint collection of rectangles $S_1, \ldots, S_n \subset D$, we subdivide them as above and use (12.7) to compute their total area in terms of the areas of the $R_{i,j}$’s.

We also note that an open rectangle

$$R_{a,c}^{b,d} = \{ (x, y) \mid a < x < b, \ c < y < d \}$$

has the same area as the corresponding closed rectangle

$$\text{area } R_{a,c}^{b,d} = \text{area } \overline{R}_{a,c}^{b,d} = (b - a)(d - c).$$  \hfill (12.10)

Indeed, if $0 < r \leq \min \{ \frac{1}{2}(b - a), \frac{1}{2}(d - c) \}$, the closed rectangle $\overline{R}_{a-r,c-r}^{b-r,d-r} \subset R_{a,c}^{b,d}$ and has area

$$\text{area } \overline{R}_{a-r,c-r}^{b-r,d-r} = (b - a - 2r)(c - d - 2r).$$

The supremum of these areas over all such $r$ equals (12.10). The same clearly holds for any half open rectangle that has the same closure.

There are many examples of subsets of $\mathbb{R}^2$ whose area is of interest. Some that come readily to mind include triangles, parallelograms, polygons, circles, ellipses, and so on. With some work, one establishes the well-known geometrical formulas for their areas. However, calculus provides a much easier and more powerful route to these formulas, as well as many others that are not covered in elementary geometry, e.g., the area between two intersecting parabolas.

Let us present a couple of general properties of area. First, prove the subset property (12.3), if $A$ is empty, the result is trivial. Otherwise, if $R_1, \ldots, R_n$ are essentially disjoint open rectangles contained in $A$, then they are also contained in $B$. Thus, recalling (12.4), $E_A \subset E_B$, and hence $\sup E_A \leq \sup E_B$.

In this exposition, we will concentrate on bounded regions, meaning that $A \subset R$ is contained in a bounded rectangle. Since area $A \leq \text{area } R < \infty$, any bounded region has finite area: $0 \leq \text{area } A < \infty$. On the other hand, many of our constructions can be extended straightforwardly to the unbounded case, and there are unbounded regions with finite area; for example, a straight line has zero area.

A second important property is how area behaves under unions of sets. We would intuitively expect that area $(A \cup B) \leq \text{area } A + \text{area } B$ with equality if and only if $A, B$ are disjoint, or more generally, when area $(A \cap B) = 0$. Surprisingly, this is not correct in general! For general sets, the best we can do is the opposite inequality:

$$\text{area } A + \text{area } B \leq \text{area } (A \cup B) \quad \text{provided} \quad \text{area } (A \cap B) = 0.$$  \hfill (12.11)
Indeed, if $R_1, \ldots, R_m$ are an essentially disjoint collection of closed rectangles contained in $A$ and $S_1, \ldots, S_n$ are an essentially disjoint collection of closed rectangles contained in $B$, then the combined collection $R_1, \ldots, R_m, S_1, \ldots, S_n$ is also essentially disjoint, since each $R_i \cap S_j \subset A \cap B$ and thus has area 0. Thus, taking the suprema over all such collections,

$$
\text{area } A + \text{area } B = \sup \left( \sum_{i=1}^{m} \text{area } R_i \right) + \sup \left( \sum_{j=1}^{n} \text{area } S_j \right)
$$

$$= \sup \left( \sum_{i=1}^{m} \text{area } R_i + \sum_{j=1}^{n} \text{area } S_j \right) \leq \text{area } (A \cup B).
$$

In general, we can only assert an inequality in (12.11) since there may well be other collections of rectangles contained in $A \cup B$ that cannot be split up into subrectangles of $A$ and $B$. Here is an example.

**Example 12.3.** Consider following two subsets of the unit square $Q = \mathbb{R}^{1,1}_{0,0}$:

$A = \{ (x, y) \mid 0 \leq x, y \leq 1, \ y < 1 \text{ if } x \in Q, \ y = 0 \text{ if } x \notin Q \}$,

$B = \{ (x, y) \mid 0 \leq x, y \leq 1, \ y = 1 \text{ if } x \in Q, \ y > 0 \text{ if } x \notin Q \}$.

First note that they are disjoint: $A \cap B = \emptyset$. Neither $A$ nor $B$ contains any open rectangle (why?), and hence area $A = \text{area } B = 0$. However, $A \cup B = Q$, with

$$1 = \text{area } Q = \text{area } (A \cup B) > \text{area } A + \text{area } B = 0.
$$

We can go further: setting $C = A \cup \mathbb{R}^{1,3/2}_{0,1}$ and $D = B \cup \mathbb{R}^{1,3/2}_{0,1}$, then

$$\text{area } C + \text{area } D = 2 \text{area } \mathbb{R}^{1,3/2}_{0,1} = 1 < \text{area } (C \cup D) = \text{area } \mathbb{R}^{1,3/2}_{0,0} = \frac{3}{2},
$$

even though $C \cap D = \mathbb{R}^{1,3/2}_{0,1}$ has area $\frac{1}{2}$.

Such pathological examples are counterintuitive, and it will be good to avoid them, which we do as follows. In general, we will call bounded subsets $A, B \subset \mathbb{R}^2$ area compatible if their intersection is empty or more generally has zero area, and the area of their union is the sum of their individual areas, i.e., they satisfy

$$\text{area } A + \text{area } B = \text{area } (A \cup B), \quad \text{area } (A \cap B) = 0. \quad (12.12)
$$

All standard geometrical subsets — rectangles, triangles, polygons, disks, ellipses, etc. — and unions thereof are mutually area compatible. Only pathological subsets like that in Example 12.3, certain fractals, etc. are incompatible. Area compatibility will play an essential role in our approach to integration. One special case is contained in the following useful result.

**Lemma 12.4.** Let $a \in \mathbb{R}$. Suppose $A \subset \{ (x, y) \mid x \leq a \}$ and $B \subset \{ (x, y) \mid x \geq a \}$. Then $A, B$ are area compatible. Area compatibility also holds when $A \subset \{ (x, y) \mid y \leq a \}$ and $B \subset \{ (x, y) \mid y \geq a \}$.
Proof: Let $R_1, \ldots, R_n$ be an essentially disjoint collection of closed rectangles, with each $R_i \subset A \cup B$. We split the collection into two subcollections $R_A, R_B$ that are, respectively, contained in $A$ and $B$. If $R_i \subset A$, then it is in $R_A$; if $R_i \subset B$, then it is in $R_B$. The remaining $R_i$ can be divided into two subrectangles $R_i^- = R_i \cap \{x \leq a\} \subset A$ and $R_i^+ = R_i \cap \{x \geq a\} \subset B$, and the two halves are added to their respective subcollections. Note that

$$\text{area } R_i^+ + \text{area } R_i^- = \text{area } R_i.$$ 

Thus,

$$\sum_{i=1}^{m} \text{area } R_i = \sum_{R \subset R_A \cup R_B} \text{area } R \leq \text{area } A + \text{area } B,$$

and hence

$$\text{area } (A \cup B) = \sup \left\{ \sum_{i=1}^{m} \text{area } R_i \right\} \leq \text{area } A + \text{area } B,$$

which, when compared with (12.11), proves their compatibility (12.12). The second statement is proved in the same manner. \(Q.E.D.\)

Before proceeding further, we note that areas are unchanged under several elementary transformations of the plane. In each case, we need only prove the result for closed rectangles; the general case will follow from our Definition 12.2 of area.

**Lemma 12.5.** If $A \subset \mathbb{R}^2$ and $(a, b) \in \mathbb{R}^2$, define the translated region

$$\tilde{A} = (a, b) + A = \{ (x + a, y + b) \mid (x, y) \in A \}.$$ 

Then $\text{area } \tilde{A} = \text{area } A$. 

**Figure 12.** Translated Region.
See Figure 12. Indeed, translation preserves the area of rectangles:

\[
R = R_{e,c}^{f,d}, \quad \text{then} \quad \tilde{R} = (a,b) + R = R_{e+a,c+b}^{f+a,d+b},
\]

and hence

\[
\text{area } \tilde{R} = [(f + a) - (e + a)] [(d + b) - (c + b)] = (e - f) (d - c) = \text{area } R.
\]

As noted above, this suffices to prove the result in general.

**Lemma 12.6.** If \( A \subset \mathbb{R}^2 \), then the reflected regions

\[
\tilde{A}_x = \{ (x, -y) \mid (x, y) \in A \}, \quad \tilde{A}_y = \{ (-x, y) \mid (x, y) \in A \},
\]

both have the same area as \( A \).

Here \( \tilde{A}_x \) is the reflection of \( A \) with the \( x \) axis serving as the mirror, while \( \tilde{A}_y \) is the reflection of \( A \) through the \( y \) axis; see Figure 13. Again, one checks that neither reflection affects the areas of rectangles.

Other examples of area-preserving transformations are the rotations

\[
(x, y) \mapsto (x \cos \theta - y \sin \theta, \ x \sin \theta + y \cos \theta), \quad \text{where} \quad 0 \leq \theta < 2\pi,
\]

and the shearing transformations

\[
(x, y) \mapsto \begin{cases} 
(x + \beta y, \ y), \\
(x, \gamma x + y),
\end{cases} \quad \text{where} \quad \beta, \gamma \in \mathbb{R}.
\]

The former rotate rectangles into “rectangular diamonds” of the same area; the latter shear rectangles into parallelograms again of the same area; see Figure 14. Proof that they both preserve areas of rectangles is a bit more challenging, and will be left to the reader. One can, of course, compose such transformations to produce other area-preserving transformations. In general, an affine transformation

\[
(x, y) \mapsto (\alpha x + \beta y + a, \ \gamma x + \delta y + b)
\]

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{reflected_regions.png}
\caption{Reflected Regions.}
\end{figure}
where $\alpha, \beta, \gamma, \delta, a, b \in \mathbb{R}$, preserves area if and only if the absolute value of its determinant is one:

$$|\alpha \delta - \beta \gamma| = 1.$$  \hfill (12.17)

All such affine transformations map parallelograms, including rectangles, into parallelograms of the same area. There also exist a wide variety of nonlinear transformations that preserve area, [18].

On the other hand, stretching the plane in one or both directions multiplies the area by the corresponding factor(s).

**Lemma 12.7.** Let $\lambda, \mu \in \mathbb{R}$. If $A \subset \mathbb{R}^2$, then the stretched regions

$$\hat{A}_\lambda = \{(x, \lambda y) \mid (x, y) \in A\}, \quad \hat{A}_\mu = \{(\mu x, y) \mid (x, y) \in A\},$$  \hfill (12.18)

satisfy

$$\text{area } \hat{A}_\lambda = |\lambda| \text{ area } A, \quad \text{area } \hat{A}_\mu = |\mu| \text{ area } A.$$

Again the result is proved by showing that it has the given effect when $A = R$ is a rectangle. If $\lambda = 0$ and/or $\mu = 0$ in (12.18) the region is squashed down to a subset of the $x$ or $y$ axis, respectively, with zero area. If $\lambda < 0$ or $\mu < 0$ the region is reflected and stretched, with the cases $\lambda = \mu = -1$ being the pure reflections in Lemma 12.6. We can combine individual scalings; for example if we scale both $x$ and $y$ by $\lambda = \mu$ (the effect of zooming on a camera), then the area of the scaled region is multiplied by $\lambda^2$, which is the reason area is measured in units of length$^2$. If one subjects a circle to any combination of stretches, the result is, in general, an ellipse, and hence one can use Lemma 12.7 to deduce the formula for the area of an ellipse from that of a circle. More generally, a general affine transformation (12.16) scales area by the overall factor $\delta = |\alpha \delta - \beta \gamma|$. A general calculus formula for the behavior of area under a nonlinear transformation of the plane can be found in [1, 5]
13. Integration.

In calculus, we are particularly interested in computing the areas of regions that lie under the graphs of functions, as illustrated in Figure 15. Indeed, the areas of the geometric examples mentioned above can all be determined once we know how to compute the areas of this particular type of region since we can glue such regions together to make more complicated regions.

To begin with, suppose $f: [a, b] \to [0, \infty)$ is a positive function, so $f(x) \geq 0$ for all $a \leq x \leq b$. We define its lower region

$$L = \{ (x, y) | \ a \leq x \leq b, \ 0 \leq y \leq f(x) \},$$

meaning that part of the plane lying on or above the $x$ axis and on or below its graph

$$G = \{ (x, f(x)) | \ a \leq x \leq b \}.$$  \hfill (13.1)

As per Definition 12.2, the area of $L$ equals the supremum of the total areas of essentially disjoint collections of closed subrectangles $R_1, \ldots, R_n \subset L$.

Let us further assume that $f$ is bounded from above on the interval $[a, b]$, so there exists $\beta \geq 0$ such that $0 \leq f(x) \leq \beta$ for all $a \leq x \leq b$; equivalently, $f[a, b] \subset [0, \beta]$. Let

$$U_\beta = \{ (x, y) | \ a \leq x \leq b, \ f(x) \leq y \leq \beta \}$$

be the complementary upper region, consisting of points lying on or above the graph of $f$ and on or below the line $x = \beta$. Observe, as sketched in Figure 16, that the union of the lower region $L$ and the upper region $U_\beta$ is the closed rectangle of height $\beta$ sitting on the

**Figure 15.** The Lower Region under the Graph of a Function.
Figure 16. Upper and Lower Regions and Graph.

interval \([a, b]\):

\[
R_{\beta} = L \cup U_{\beta} = R_{a,b}^{b,\beta} = \{(x, y) \mid a \leq x \leq b, \ 0 \leq y \leq \beta\}.
\]

On the other hand, the intersection of the upper and lower regions is the graph: \(L \cap U_{\beta} = G \), which has area 0 because it contains no open rectangles. The basic premise of integration is that the upper and lower regions be area compatible, in the sense of (12.12), which serves to inspire the following key definition.

**Definition 13.1.** A function \(f: [a, b] \rightarrow [0, \beta]\) is integrable if its upper and lower regions (13.1, 3) are area compatible, meaning

\[
\text{area } L + \text{area } U_{\beta} = \text{area } R_{\beta} = \beta (b - a).
\]

In this case, we define the integral of \(f\) on the interval \([a, b]\) to be the area of its lower region:

\[
\int_{a}^{b} f(x) \, dx = \text{area } L = \text{area } \{(x, y) \mid a \leq x \leq b, \ 0 \leq y \leq f(x)\}.
\]

The left hand side of (13.5) is sometimes referred to as a definite integral, indicating that the endpoints \(a, b\) of the interval of integration are to be viewed as fixed.

**Remark:** The integral symbol \(\int\) is an elongated “S” which stands for “sum”, indicating that we are calculating areas by summing up (and then taking a supremum) of the areas of rectangles. The notation dates back to the German mathematician and philosopher Gottfried Leibniz, who was Newton’s contemporaneous rival inventor of the calculus. While Newton can legitimately claim priority, Leibniz’s notation proved to be much better and
is one of the reasons much of the initial post-Newton development of calculus took place on the European continent rather than in England. The integration variable $x$, which is indicated by the symbol $dx$, is a “dummy variable” and any other letter, except $a$ or $b$, can be used instead:

$$\int_a^b f(x) \, dx = \int_a^b f(y) \, dy = \int_a^b f(t) \, dt,$$

and so on. These are just different ways of denoting the same area. In Leibniz’s thinking, $dx$ represents the width of an “infinitesimally” narrow rectangle lying under the graph of the function, and so the product $f(x) \, dx$ is to viewed as its “infinitesimal” area. The total area or integral is then obtained by “summing” these infinitely many infinitesimal quantities. Mathematicians have endeavored to make this idea rigorous, [20], but we will not pursue this line of reasoning here.

For example, any constant function $f(x) \equiv c$, where $c \geq 0$, is integrable on any closed bounded interval $[a, b]$. (Later we will see that this is also true when $c < 0$.) Indeed, the associated lower region $L = \{(x, y) | a \leq x \leq b, \ 0 \leq y \leq c\}$ is just a closed rectangle, namely $R_{a,0}^{b,c}$, and hence its area is $c(b - a)$. On the other hand, given $\beta \geq c$, the upper region is also a closed rectangle, $U_\beta = \{(x, y) | a \leq x \leq b, \ c \leq y \leq \beta\} = R_{a,c}^{b,\beta}$, with area $(\beta - c)(b - a)$. The sum of their areas is thus the area of the entire rectangle of height $\beta$:

$$\text{area } L + \text{area } U_\beta = c(b - a) + (\beta - c)(b - a) = \beta(b - a) = \text{area } R_\beta,$$

verifying the integrability criterion (13.4). Thus, the integral of the constant function is the area of the lower region:

$$\int_a^b c \, dx = c(b - a). \quad (13.6)$$

A second relatively easy case is the function $f(x) = x$ on the interval $[a, b]$ where $0 \leq a \leq b$ to ensure positivity. The lower region $L = \{(x, y) | a \leq x \leq b, \ 0 \leq y \leq x\}$ is a trapezoid with vertical side lengths $a, b$ — see Figure 17 — and hence its area is

![Figure 17. Integral of $x$.](image-url)
known to be \( \frac{1}{2}(a+b)(b-a) = \frac{1}{2}b^2 - \frac{1}{2}a^2 \). Further, given \( \beta > b \), the upper region 
\( U_\beta = \{ (x, y) \mid a \leq x \leq b, \ x \leq y \leq \beta \} \) is also a trapezoid, now with vertical side lengths 
\( \beta-a \) and \( \beta-b \), and hence its area is 
\[ \frac{1}{2} \left[ (\beta-a) + (\beta-b) \right] (b-a) = \beta (b-a) - \frac{1}{2} b^2 + \frac{1}{2} a^2. \]
The sum of their areas is 
\[ \text{area } L + \text{area } U_\beta = \left[ \frac{1}{2} b^2 - \frac{1}{2} a^2 \right] + \left[ \beta (b-a) - \frac{1}{2} b^2 + \frac{1}{2} a^2 \right] = \beta (b-a) = \text{area } R_\beta, \]
thus establishing the integrability criterion (13.4). We conclude that 
\[ \int_a^b x \, dx = \frac{1}{2} b^2 - \frac{1}{2} a^2. \quad (13.7) \]
We will defer trying to compute any more complicated examples until we have established
the remarkable connection between integration and differentiation known as the Fundamental
Theorem of Calculus. With this powerful tool in hand, both of the preceding
integrals, and many more, will reduce to easy one-line calculations. It is important to
note at the outset that all “reasonable” functions, including, as we will shortly learn, those
that are not everywhere positive, are integrable. Later we will exhibit an example of a
pathological non-integrable function.

The integrability requirement (13.4) is, reassuringly, independent of the upper bound \( \beta \).
Indeed, if \( \gamma > \beta \), then the corresponding upper region
\( U_\gamma = \{ (x, y) \mid a \leq x \leq b, \ f(x) \leq y \leq \gamma \} \)
\[ = \{ (x, y) \mid a \leq x \leq b, \ f(x) \leq y \leq \beta \} \cup \{ (x, y) \mid a \leq x \leq b, \ \beta \leq y \leq \gamma \} = U_\beta \cup \overline{R}^{b,\gamma}_{a,\beta} \]
is obtained by gluing a closed rectangle of height \( \gamma - \beta \) lying directly above \( U_\beta \), meeting
at the horizontal line \( y = \beta \). In view of Lemma 12.4,
\[ \text{area } U_\gamma = \text{area } U_\beta + \text{area } \overline{R}^{b,\gamma}_{a,\beta} = \text{area } U_\beta + (\gamma - \beta) (b-a). \]
Since we are assuming (13.4) holds, this implies
\[ \text{area } L + \text{area } U_\gamma = \text{area } L + \text{area } U_\beta + \text{area } \overline{R}^{b,\gamma}_{a,\beta} = \beta (b-a) + (\gamma - \beta) (b-a) = \gamma (b-a) = \text{area } R_\gamma, \]
and thus (13.4) also holds with \( \gamma \) replacing \( \beta \).

Some basic properties of integration follow. First, if \( a = b \), every function is integrable
on the singleton \( [a, a] = \{ a \} \), with
\[ \int_a^a f(x) \, dx = 0. \quad (13.8) \]
Indeed, both the lower and upper regions have zero area, so the integrability requirement
(13.4) is trivially satisfied.

**Proposition 13.2.** If \( c \geq 0 \) is constant and \( f \) is integrable on \( [a, b] \), then the function
\( cf(x) \) is also integrable, and
\[ \int_a^b \left[ cf(x) \right] \, dx = c \int_a^b f(x) \, dx. \quad (13.9) \]
Proof: The lower region of the function $c f(x)$ is obtained by stretching the lower region of $f(x)$ by the factor $c$:

$$\{(x, y) \mid a \leq x \leq b, \ 0 \leq y \leq c f(x)\} = \{(x, cy) \mid a \leq x \leq b, \ 0 \leq y \leq f(x)\}.$$ 

Since $c \geq 0$, Lemma 12.7 implies that the area of the stretched region is $c$ times the area of the original, which implies (13.9), subject to verification that $c f(x)$ is integrable. For this, using $c \beta$ for the upper bound for $c f(x)$, the upper region is stretched by a similar amount, as is the corresponding rectangle: $R_{\beta} \mapsto R_{c \beta}$, and one thus verifies a scaled version of the integrability condition (13.4). \[Q.E.D.\]

It will be useful to extend formula (13.9) to all real $c \in \mathbb{R}$, which will allow us to integrate negative functions. If $0 \leq f(x) \leq \beta$ on $[a, b]$, then its negative satisfies $-\beta \leq -f(x) \leq 0$, and has integral

$$\int_{a}^{b} \left[-f(x)\right] \, dx = -\int_{a}^{b} f(x) \, dx,$$

(13.10)

which is the case $c = -1$ of (13.9). The right hand side is the negative of area of the lower region $L$ of $f$, namely the region (13.1) lying below the graph of $f(x)$ and above the $x$ axis. The corresponding region for $-f(x)$ is

$$\tilde{L} = \{(x, y) \mid a \leq x \leq b, \ -f(x) \leq y \leq 0\} = \{(x, -y) \mid a \leq x \leq b, \ 0 \leq y \leq f(x)\},$$

which is the reflection of the lower region $L$ through the $x$ axis, which, according to Lemma 12.6, has the same area $\tilde{L} = \text{area } L$. Equivalently, $\tilde{L}$ is the region lying above the graph of $-f(x)$ and below the $x$ axis; see Figure 18. The right hand side of formula (13.10) is the negative of the area of this region, and hence the integral of a negative function is minus the area of the region $\tilde{L}$ lying between its graph and the $x$ axis. In other words, from the point of view of integration, regions below the $x$ axis make negative contributions.
to the integral. For example, if \( f(x) = -x \) for \( 0 \leq a \leq x \leq b \), then

\[
\int_a^b (-x) \, dx = - \int_a^b x \, dx = \frac{1}{2} a^2 - \frac{1}{2} b^2
\]
is minus the area of the trapezoid lying between its graph and the \( x \) axis. More generally, if the graph of \( f(x) \) lies both above and below the \( x \) axis, its integral is the sum of the areas of the subregions lying above the axis minus the sum of the areas of the regions lying below; in Figure 19, the blue regions make positive contributions to the integral while the purple regions make negative contributions, and the integral is the total of all such contributions. A justification will appear shortly.

One can always convert a bounded but non-positive function into an everywhere positive function by adding a large positive constant to it, effectively translating its graph upwards. The basic formula is

\[
\int_a^b [f(x) + c] \, dx = \int_a^b f(x) \, dx + \int_a^b c \, dx = \int_a^b f(x) \, dx + c (b - a),
\]
which is a special case of a later formula, namely (13.23), but is easy to prove directly when \( f \) and \( c \) are both positive; we then extend its validity to all integrable functions and all \( c \in \mathbb{R} \). In the former case, the lower region for \( f(x) + c \) is

\[
\{(x, y) \mid a \leq x \leq b, \ 0 \leq y \leq f(x) + c\}
\]

\[
= \{(x, y) \mid a \leq x \leq b, \ 0 \leq y \leq c\} \cup \{(x, y) \mid a \leq x \leq b, \ c \leq y \leq f(x) + c\},
\]
which is the union of the rectangle \( R_{a,0}^{b,c} \) and a vertically translated version of the lower region of \( f \):

\[
\{(x, y) \mid a \leq x \leq b, \ c \leq y \leq f(x) + c\} = \{(x, y + c) \mid a \leq x \leq b, \ 0 \leq y \leq f(x)\};
\]
see Figure 20. Lemma 12.5 says that the translated region has the same area as the
original, hence (13.11) reduces to the result that the area of this disjoint union is just the sum of the individual areas, which follows from Lemma 12.4. One can prove integrability of \( f(x) + c \) by replacing the upper bound \( \beta \) for \( f(x) \) by \( \beta + c \), and noting that its upper region is the same translate of the upper region of \( f(x) \):

\[
\{(x, y) \mid a \leq x \leq b, \ f(x) + c \leq y \leq \beta + c \} = \{(x, y + c) \mid a \leq x \leq b, \ f(x) \leq y \leq \beta \},
\]

and hence also has the same area.

For example, if \( f(x) = x - 1 \) for \( 0 \leq x \leq 1 \), then we can chose \( c = -1 \) so that \( \tilde{f}(x) = f(x) + 1 = x \) and hence, according to (13.7),

\[
\int_{0}^{1} (x - 1) \, dx = \int_{0}^{1} x \, dx + (-1)(1 - 0) = -\frac{1}{2}.
\]

The function \( x - 1 \) lies entirely below the \( x \) axis when \( 0 \leq x \leq 1 \), and we are effectively computing the negative of the area of the triangle that lies between its graph and the \( x \) axis.

In general we only need to establish an integration property for positive functions, and then extend it as stated to general functions by the same technique of adding in a suitably large constant.

**Example 13.3.** Given \( a < b \), a function that is constant on a finite number of disjoint open subintervals is known as a *step function*. In other words, given \( I_j = (a_j, b_j) \) for \( j = 1, \ldots, m \), with

\[
a \leq a_1 < b_1 \leq a_2 < b_2 \leq \cdots \leq a_m < b_m \leq b
\]

and nonzero constants \( c_1, \ldots, c_m \in \mathbb{R} \setminus \{0\} \), the corresponding step function is

\[
\sigma(x) = \begin{cases} 
  c_j, & a_j < x < b_j, \\
  0, & \text{otherwise}. 
\end{cases}
\]
For simplicity, we assign the value 0 outside the given rectangles; indeed, the values at the endpoints \(a_j, b_j\) of the subintervals play no role in its integration. An example is the sign function (4.1), restricted to a bounded interval \([-a, a]\).

If the step function is non-negative, so all \(c_j > 0\) then the corresponding lower region consists of the closed rectangles \(R_{a_j,0}^{b_j,c_j}\) that lie below its graph over each interval. The integral of the step function (13.13) is the sum of their areas:

\[
\int_a^b \sigma(x) \, dx = \sum_{i=1}^m c_j (b_j - a_j).
\]

We leave the verification of the integrability criterion (13.4) for a suitable upper bound \(\beta \geq \max\{c_1, \ldots, c_m\}\) to the reader. The same integration formula (13.14) holds in general, with the right hand side (13.14) being the sum of the areas of the rectangles lying above the \(x\) axis minus the sum of the areas of the rectangles lying below.

For example, consider the simple positive step function

\[
\sigma(x) = \begin{cases} 
5, & -1 < x < 0, \\
3, & 0 < x < 2, \\
0, & \text{otherwise}.
\end{cases}
\]

On the interval \([-1, 2]\), its lower region consists of two closed rectangles, \(R_{-1,0}^{1,5}\) and \(R_{0,0}^{2,3}\), of respective areas 5 and 6. Thus, its integral equals the total area:

\[
\int_{-1}^{2} \sigma(x) \, dx = 5 \cdot 1 + 3 \cdot 2 = 11.
\]

On the other hand, the step function

\[
\sigma(x) = \begin{cases} 
-2, & -3 < x < 0, \\
4, & 0 < x < 1 \\
-5, & 1 < x < 3, \\
0, & \text{otherwise},
\end{cases}
\]

is not everywhere non-negative. Its integral

\[
\int_{-3}^{3} \sigma(x) \, dx = -2 \cdot 3 + 4 \cdot 1 - 5 \cdot 2 = -12
\]

combines the area of the rectangle \(R_{-3,0}^{1,4}\) above the \(x\) axis with the negatives of the areas of the rectangles \(R_{-3,-2}^{0,0}\) and \(R_{1,-5}^{3,0}\) that are below.

As noted above, most “reasonably behaved” functions are integrable. In particular, this includes all bounded monotone functions.

**Theorem 13.4.** \(f: [a, b] \to \mathbb{R}\) is monotone and bounded, then \(f\) is integrable.

Another fundamental result states that all continuous functions are integrable. Proofs of both of these results will appear below.
Theorem 13.5. If \( f: [a, b] \rightarrow \mathbb{R} \) is continuous, then \( f \) is integrable.

Remark: There are many bounded discontinuous functions that are nevertheless integrable, including all piecewise constant step functions, as in Example 13.3. More generally any piecewise integrable function, meaning that \( f \) is integrable when restricted to each of a finite union of subintervals, is integrable on the entire interval. Thus piecewise monotone and piecewise continuous functions are all integrable. This result follows by repeated application of Proposition 13.7 below.

On the other hand, the pathological function

\[
f(x) = \begin{cases} 
1, & x \in \mathbb{Q}, \\
0, & x \in \mathbb{R} \setminus \mathbb{Q}, 
\end{cases}
\]

is not integrable on any closed interval \( a \leq x \leq b \). Indeed, using the upper bound \( \beta = 1 \), the lower and upper regions are

\[
L = \{(x, y) \mid a \leq x \leq b, \ 0 \leq y \leq f(x)\}
= \{(x, y) \mid a \leq x \leq b, \ 0 \leq y \leq 1 \text{ if } x \in \mathbb{Q}, \ x = 0 \text{ if } y \not\in \mathbb{Q}\},
\]

\[
U_\beta = \{(x, y) \mid a \leq x \leq b, \ f(x) \leq y \leq 1\}
= \{(x, y) \mid a \leq x \leq b, \ y = 1 \text{ if } x \in \mathbb{Q}, \ 0 \leq y \leq 1 \text{ if } x \not\in \mathbb{Q}\}.
\]

By the same reasoning as used in Example 12.3, we find

\[
\text{area } L + \text{area } U_\beta = 0 < \text{area } R_\beta = b - a,
\]

and hence \( L \) and \( U \) are not area compatible and the integrability condition (13.4) is not satisfied.

Let us summarize the basic properties of the integral. Some proofs will be deferred until later in this section. We begin with a basic inequality.

Proposition 13.6. If \( f(x) \leq g(x) \) for all \( x \in [a, b] \) and both are integrable, then

\[
\int_a^b f(x) \, dx \leq \int_a^b g(x) \, dx. \tag{13.16}
\]

Remark: The result also holds for the strict inequalities, but is harder to prove; see Theorem 13.13 below.

Proof: Assuming \( 0 \leq f(x) \leq g(x) \), the lower regions are

\[
\int_a^b f(x) \, dx = \text{area } L_f, \quad \text{where} \quad L_f = \{(x, y) \mid a \leq x \leq b, \ 0 \leq y \leq f(x)\},
\]

\[
\int_a^b g(x) \, dx = \text{area } L_g, \quad \text{where} \quad L_g = \{(x, y) \mid a \leq x \leq b, \ 0 \leq y \leq g(x)\}.
\]

The inequality implies \( L_f \subset L_g \), and hence, according to (12.3), area \( L_f \leq \text{area } L_g \), which is the same as (13.16). More generally, we add the same suitably large constant to \( f \) and \( g \) to make them both positive while preserving the inequality, using (13.11) to complete the proof.

Q.E.D.
In particular, if \( \alpha \leq f(x) \leq \beta \) for all \( x \in [a, b] \), then in view of (13.6),
\[
\alpha (b - a) \leq \int_a^b f(x) \, dx \leq \beta (b - a). \tag{13.17}
\]

A second consequence is the absolute value integral inequality
\[
\left| \int_a^b f(x) \, dx \right| \leq \int_a^b |f(x)| \, dx. \tag{13.18}
\]

This follows from (13.16) using the inequalities \(-|f(x)| \leq f(x) \leq |f(x)|\).

The next result says that we can evaluate an integral by splitting up the domain into subintervals.

**Proposition 13.7.** Suppose \( a < b < c \). Then \( f \) is integrable on the intervals \( [a, b] \) and \( [b, c] \) if and only if it is integrable on the combined interval \( [a, c] \), in which case
\[
\int_a^c f(x) \, dx = \int_a^b f(x) \, dx + \int_b^c f(x) \, dx. \tag{13.19}
\]

**Proof:** As before, we prove this assuming \( 0 \leq f(x) \leq \beta \) for \( a \leq x \leq c \), and then extend the formula to general functions by adding in suitable constant. Under this assumption, the full lower and upper regions
\[
L = \{ (x, y) \mid a \leq x \leq c, \ 0 \leq y \leq f(x) \}, \quad U_\beta = \{ (x, y) \mid a \leq x \leq c, \ f(x) \leq y \leq \beta \},
\]
are unions of their counterparts
\[
\tilde{L} = \{ (x, y) \mid a \leq x \leq b, \ 0 \leq y \leq f(x) \}, \quad \tilde{U}_\beta = \{ (x, y) \mid a \leq x \leq b, \ f(x) \leq y \leq \beta \},
\]
\[
\hat{L} = \{ (x, y) \mid b \leq x \leq c, \ 0 \leq y \leq f(x) \}, \quad \hat{U}_\beta = \{ (x, y) \mid b \leq x \leq c, \ f(x) \leq y \leq \beta \},
\]
Since
\[
\tilde{L}, \ \tilde{U}_\beta \subset \{ (x, y) \mid x \leq b \}, \quad \hat{L}, \ \hat{U}_\beta \subset \{ (x, y) \mid x \geq b \},
\]
Lemma 12.4 implies
\[
\text{area } L = \text{area } \tilde{L} + \text{area } \hat{L}, \quad \text{area } U_\beta = \text{area } \tilde{U}_\beta + \text{area } \hat{U}_\beta. \tag{13.20}
\]

On the other hand, we always have
\[
\text{area } L + \text{area } U_\beta \leq \text{area } \tilde{R}^{c,\beta}_{a,0}, \quad \text{area } \tilde{L} + \text{area } \tilde{U}_\beta \leq \text{area } \tilde{R}^{b,\beta}_{a,0}, \quad \text{area } \hat{L} + \text{area } \hat{U}_\beta \leq \text{area } \hat{R}^{c,\beta}_{b,0}. \tag{13.21}
\]

If \( f \) is integrable on \( [a, c] \), the first inequality is an equality. Combined with (13.20) and the fact that area \( \tilde{R}^{c,\beta}_{a,0} = \text{area } \tilde{R}^{b,\beta}_{a,0} + \text{area } \hat{R}^{c,\beta}_{b,0} \) proves that the second and third are also equalities, thus proving integrability of \( f \) on \( [a, b] \) and \( [b, c] \). The converse is proved in a similar manner. With integrability of \( f \) on all three intervals established, the first equation in (13.20) yields (13.19).

\(Q.E.D.\)
Let us drop the assumption that \( c > b \) in (13.19), but still require that the equation as written holds. In particular, if \( c = a \), then, by (13.8),

\[
0 = \int_a^a f(x) \, dx = \int_a^b f(x) \, dx + \int_b^a f(x) \, dx.
\]

Thus,

\[
\int_b^a f(x) \, dx = - \int_a^b f(x) \, dx. \tag{13.22}
\]

In other words, assuming \( f(x) \geq 0 \), integrating in the “wrong” direction, from right to left, produces the negative of the area of the lower region lying between the graph of \( f \) and the \( x \) axis, whereas integrating in the correct direction, from left to right, gives the area. On the other hand, if \( f(x) \leq 0 \), then integration in the wrong direction produces the area of the region lying above the graph of \( f \) and below the \( x \) axis, while integrating in the correct direction yields its negative.

The next result says that the integral of the sum of two functions is the sum of their integrals. The proof will appear later in the section.

**Theorem 13.8.** If \( f: [a, b] \to [\alpha, \beta] \) and \( g: [a, b] \to [\gamma, \delta] \) are both bounded and integrable, then their sum \( f: [a, b] \to [\alpha + \gamma, \beta + \delta] \) is also bounded and integrable, with

\[
\int_a^b \left[ f(x) + g(x) \right] \, dx = \int_a^b f(x) \, dx + \int_a^b g(x) \, dx. \tag{13.23}
\]

**Warning:** The integral of the product of two functions is *not* the product of their integrals!

In particular, if we combine (13.23) with (13.10), and assume that \( f(x) \geq g(x) \) for all \( a \leq x \leq b \), we find the integral of their difference can be used to compute the area of the region between their graphs, as illustrated in Figure 21:

\[
\int_a^b \left[ f(x) - g(x) \right] \, dx = \int_a^b f(x) \, dx - \int_a^b g(x) \, dx = \text{area} \left\{ (x, y) \ \bigg| \ a \leq x \leq b, \ g(x) \leq y \leq f(x) \right\}. \tag{13.24}
\]

Indeed, the region between their graphs is the set theoretic difference between the lower region for \( f \) and the lower region for \( g \), whose areas are represented by the two individual integrals. If we drop the assumption that \( f \) is everywhere greater than \( g \), then the integral of their difference represents the “signed area” between their graphs, where sections with \( f(x) \geq g(x) \) are counted with positive area while those where \( f(x) \leq g(x) \) have negative area, thereby generalizing our earlier observations concerning positive and negative functions, which corresponds to the case \( g(x) \equiv 0 \), whose graph is the \( x \) axis.

With the basic properties in hand, let us return to general considerations. Suppose we wish to prove that a given bounded non-negative function \( f: [a, b] \to [0, \beta] \) is integrable, i.e., satisfies (13.4) where \( L, U_\beta \) are its lower and upper regions, whose union is the entire
rectangle of height $\beta$:

$$L \cup U_\beta = R_\beta = \overline{R_{a,0}^{b,\beta}}.$$  

**Definition 13.9.** An essentially disjoint collection of closed rectangles

$$V = \{ S_1, \ldots, S_m, T_1, \ldots, T_n \}$$

is called $f$-adapted if $S_1, \ldots, S_n \subset L$ and $T_1, \ldots, T_m \subset U_\beta$.

Given an $f$-adapted collection, we let

$$S^* = \bigcup_{i=1}^{n} S_i, \quad T^* = \bigcup_{i=1}^{m} T_i, \quad V^* = S^* \cup T^* \subset R_\beta. \quad (13.25)$$

Further, let

$$Z^* = \overline{R_\beta \setminus V^*}, \quad (13.26)$$

denote the closure of the complementary subset, which, as in (12.8), is also the union of a set of rectangles $Z = \{ Z_1, \ldots, Z_i \}$. Since we are using the rectangles in $V$ to approximate the areas of the lower and upper regions, they will fill up an increasingly large fraction of the overall rectangle $R_\beta$. Thus, we expect that $Z^*$ will be the union of a collection of smaller and smaller rectangles. Indeed, according to (12.8),

$$\text{area } S^* + \text{area } T^* + \text{area } Z^* = \text{area } V^* + \text{area } Z^* = \text{area } R_\beta = \beta(b-a). \quad (13.27)$$

Moreover, by definition,

$$\text{area } L = \sup \{ \text{area } S^* \}, \quad \text{area } U_\beta = \sup \{ \text{area } T^* \}, \quad (13.28)$$

where the suprema are taken over all possible $f$-adapted collections $V$. Thus,

$$\text{area } L + \text{area } U_\beta = \sup \{ \text{area } S^* \} + \sup \{ \text{area } T^* \} = \sup \{ \text{area } S^* + \text{area } T^* \}
= \text{sup } \{ \text{area } R_\beta - \text{area } Z^* \} = \text{area } R_\beta - \text{inf } \{ \text{area } Z^* \}.$$
We thus arrive at the following criterion for integrability.

**Theorem 13.10.** The nonnegative bounded function \( f: [a, b] \to [0, \beta] \) is integrable if and only if

\[
\inf \{ \text{area } Z^* \} = 0,
\]

where the infimum is over all possible \( f \)-adapted collections of rectangles \( V \) with complement \( Z^* \) constructed as above in (13.25, 26).

**Lemma 13.11.** If \( f(x) \geq 0 \) is integrable on \([a, b]\) and \( a \leq c < d \leq b \), then \( f(x) \) is integrable on \([c, d]\) and

\[
\int_c^d f(x) \, dx \leq \int_a^b f(x) \, dx.
\]

**Proof:** Assuming integrability, the inequality follows immediately because the two lower regions satisfy

\[
\{(x, y) \mid a \leq x \leq b, \ 0 \leq y \leq f(x)\} \subset \{(x, y) \mid c \leq x \leq d, \ 0 \leq y \leq f(x)\},
\]

and hence the area of the first is \( \leq \) the area of the second. To prove integrability, given \( Z^* \) as above, its intersection \( \tilde{Z}^* = Z^* \cap X_{cd} \) with the vertical strip \( X_{cd} = \{(x, y) \mid c \leq x \leq d\} \) is a complementary region for the integral on \([c, d]\), associated with the rectangles \( S_i \cap X_{cd} \) and \( T_j \cap X_{cd} \). Since \( \tilde{Z}^* \subset Z^* \), we have area \( \tilde{Z}^* \leq \text{area } Z^* \). Thus, (13.29) implies \( \inf \{ \text{area } \tilde{Z}^* \} = 0 \), thereby proving integrability of \( f(x) \) on the subinterval \([c, d]\). \( Q.E.D. \)

It is often impractical to consider all such collections of rectangles. Rather, we typically concentrate on those of a certain type. For this purpose, we introduce a refinement of the integrability condition of Theorem 13.10.

**Theorem 13.12.** Let \( \Lambda \) be a set. Each \( \lambda \in \Lambda \) indexes a collection of \( f \)-adapted rectangles \( V_\lambda = \{ S_{\lambda,1}, \ldots, S_{\lambda,n_\lambda}, T_{\lambda,1}, \ldots, T_{\lambda,m_\lambda} \} \) for some \( n_\lambda, m_\lambda \in \mathbb{N} \), with

\[
S_{\lambda,i}^* = \bigcup_{i=1}^{n_\lambda} S_{\lambda,i} \subset L, \quad T_{\lambda,j}^* = \bigcup_{j=1}^{m_\lambda} T_{\lambda,j} \subset U_\beta, \quad V_\lambda^* = S_\lambda^* \cup T_\lambda^* \subset R_\beta.
\]

Let \( Z_\lambda^* = \overline{R_\beta \setminus V_\lambda^*} \) be the corresponding complementary set. If

\[
\inf \{ \text{area } Z_\lambda^* \mid \lambda \in \Lambda \} = 0,
\]

then the function \( f(x) \) is integrable and, moreover,

\[
\int_a^b f(x) \, dx = \text{area } L = \sup \{ \text{area } S_\lambda^* \mid \lambda \in \Lambda \}.
\]

Intuitively, Theorem 13.12 says that if we are able to cover the graph of \( f \) by collections of closed rectangles — namely those forming the sets \( Z^* \) — that have arbitrarily small total area, then integrability of the function follows. Except in very special cases, such as constant functions and step functions, the set \( \Lambda \) indexing the collections in Theorem 13.12 is infinite, and typically contains one or more real parameters.
Figure 22. Lower, Upper, and Intermediate Rectangles.

Proof: According to (12.11), (13.27),
\[
\text{area } R_\beta \geq \text{area } L + \text{area } U_\beta \geq \text{area } S^*_\lambda + \text{area } T^*_\lambda = \text{area } R_\beta - \text{area } Z^*_\lambda.
\]
By our assumption, the infimum of the final term over all \( \lambda \in \Lambda \) is zero, and hence
\[
\text{area } R_\beta = \text{area } L + \text{area } U_\beta = \sup \{ \text{area } S^*_\lambda \} + \sup \{ \text{area } T^*_\lambda \}.
\]
The first equality proves integrability; the second, combined with the facts that
\[
\sup \{ \text{area } S^*_\lambda \} \leq \text{area } L,
\]
\[
\sup \{ \text{area } T^*_\lambda \} \leq \text{area } U_\beta,
\]
establishes the formula (13.32) for the integral. Q.E.D.

Perhaps the most interesting collections of open subrectangles are those constructed as follows. We assume \( 0 \leq f(x) \leq \beta \) for \( a \leq x \leq b \). Given \( n \in \mathbb{N} \), let us choose a partition
\[
a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b \tag{13.33}
\]
of the interval \([a, b]\) into subintervals \([x_{i-1}, x_i]\) whose lengths, \( x_i - x_{i-1} \), are all deemed to be “small”, although we do not impose any a priori restriction on their size. Choose
\[
0 \leq \alpha_i \leq \beta_i \leq \beta
\]
that are lower and upper bounds for \( f(x) \) on the \( i \)th subinterval, so
\[
\alpha_i \leq f(x) \leq \beta_i \quad \text{for all } x_{i-1} \leq x \leq x_i.
\]
Define the upper and lower rectangles associated with the partition and bounds by, respectively,
\[
S_i = R_{x_{i-1},x_i,\alpha_i}^{x_i,\alpha_i} = \{(x,y) \mid x_{i-1} \leq x \leq x_i, \ 0 \leq y \leq \alpha_i \} \subset L,
\]
\[
T_i = R_{x_{i-1},x_i,\beta_i}^{x_i,\gamma} = \{(x,y) \mid x_{i-1} \leq x \leq x_i, \ \beta_i \leq y \leq \beta \} \subset U_\beta.
\tag{13.34}
\]
These clearly satisfy the properties for forming an \( f \)-adapted system of rectangles. In particular,
\[
S^* = \bigcup_{i=1}^n S_i \subset L, \quad \text{and hence } \text{area } S^* = \sum_{i=1}^n \alpha_i (x_i - x_{i-1}) \leq \text{area } L. \tag{13.35}
\]
We will call area $S^*$ a lower sum for the integral of $f$. The union of the intermediate rectangles

$$Z_i = R_{x_{i-1}, \alpha_i}^{x_i, \beta_i} = \{ (x,y) \mid x_{i-1} \leq x \leq x_i, \ \alpha_i \leq y \leq \beta_i \}$$  \hspace{1cm} (13.36)

forms the complementary set $Z^* = \bigcup_{i=1}^{n} Z_i$. Since $\alpha_i \leq f(x) \leq \beta_i$ for $x_{i-1} \leq x \leq x_i$, the graph of $f$ is contained in the complementary set:

$$G = \{ (x,f(x)) \mid a \leq x \leq b \} \subset Z^*,$$

which thus forms a system of small rectangles covering its graph; see Figure 22.

On the other hand, if we set

$$W_i = S_i \cup Z_i = R_{x_{i-1}, 0}^{x_i, \beta_i},$$

then, since $0 \leq f(x) \leq \beta_i$ when $x_{i-1} \leq x \leq x_i$,

$$W^* = \bigcup_{i=1}^{n} W_i \supset L, \quad \text{and hence} \quad \text{area } W^* = \sum_{i=1}^{n} \beta_i (x_i - x_{i-1}) \geq \text{area } L. \quad (13.37)$$

The latter expression is referred to as an upper sum; see Figure 23. The area of the lower region $L$ is thus bounded between the lower and upper sums:

$$\sum_{i=1}^{n} \alpha_i (x_i - x_{i-1}) \leq \text{area } L \leq \sum_{i=1}^{n} \beta_i (x_i - x_{i-1}). \quad (13.38)$$

\* If $\alpha_i = \beta_i$, which implies that $f(x) = \alpha_i = \beta_i$ is constant on the interval $[x_{i-1}, x_i]$, then the degenerate rectangle $Z_i$ is not included in the complementary region $Z^*$ as defined earlier, (13.26). However, since it has zero area, it can be included here without affecting the overall calculation of areas.

\[\text{Figure 23. Lower and Upper Sums.}\]
According to Theorem 13.12, the function \( f(x) \) is integrable on \([a, b]\) provided the areas of the complementary sets \( Z^* \) have zero infimum over all partitions of \([a, b]\) and all choices of upper and lower bounds on the subintervals:

\[
\inf \{ \text{area } Z^* \} = 0 \quad \text{where} \quad \text{area } Z^* = \sum_{i=1}^{n} \text{area } Z_i = \sum_{i=1}^{n} (\beta_i - \alpha_i) (x_i - x_{i-1}). \tag{13.39}
\]

Once integrability is established, the value of the integral is the supremum of the lower sums (13.35) and the infimum of the upper sums (13.37):

\[
\int_a^b f(x) \, dx = \sup \left\{ \sum_{i=1}^{n} \alpha_i (x_i - x_{i-1}) \right\} = \inf \left\{ \sum_{i=1}^{n} \beta_i (x_i - x_{i-1}) \right\}. \tag{13.40}
\]

In fact, this condition is both necessary and sufficient for integrability. This follows from the fact that one can subdivide any \( f \) adapted system of rectangles to obtain one with the same lower and upper sums as the above. One can even restrict the allowed partitions in some suitable manner. For example, a common choice is to require that they be regular meaning that the length of the subintervals are the same, and so

\[
x_i = a + i h, \quad i = 0, \ldots, n, \quad \text{where} \quad h = \frac{b-a}{n} \quad \text{for} \quad n \in \mathbb{N}. \tag{13.41}
\]

We derived Theorem 13.12 and the integrability criterion (13.40) under the assumption that \( f \) is nonnegative, but by the usual device of adding in a constant, they also apply as stated to general functions. Let us use this construction to prove the fundamental results that all monotone and all continuous functions are integrable.

**Proof of Theorem 13.4:** We prove integrability of monotone increasing functions, leaving the decreasing case as a simple adaptation. Thus we assume \( f(x) \leq f(y) \) whenever \( a \leq x \leq y \leq b \). We use the regular partitions (13.41) to subdivide the interval into \( n \in \mathbb{N} \) equally spaced subintervals; we thus view \( \Lambda = \mathbb{N} \) as our index set for the integrability criterion (13.31). Because \( f \) is increasing on any subinterval, its values at the endpoints provide lower and upper bounds:

\[
\alpha_i = f(x_{i-1}) \leq f(x_i) = \beta_i, \quad \text{when} \quad x_{i-1} \leq x \leq x_i.
\]

Thus, assuming regularity of the partition, so \( x_i - x_{i-1} = (b-a)/n \) for each \( i = 1, \ldots, n \), the area of the intermediate region (13.39) is

\[
\text{area } Z^*_n = \sum_{i=1}^{n} (\beta_i - \alpha_i) (x_i - x_{i-1}) = \sum_{i=1}^{n} \left[ f(x_i) - f(x_{i-1}) \right] \frac{b-a}{n}
\]

\[
= \left\{ \left[ f(x_1) - f(x_0) \right] + \left[ f(x_2) - f(x_1) \right] + \cdots + \left[ f(x_n) - f(x_{n-1}) \right] \right\} \frac{b-a}{n}
\]

\[
= \left[ f(x_n) - f(x_0) \right] \frac{b-a}{n} = \frac{f(b) - f(a)}{n} (b-a),
\]

where we observe that all the terms in the summation cancel except the first and the last. Since this holds for all \( n \in \mathbb{N} \), the infimum of these intermediate areas is zero, and hence
Proof of Theorem 13.5: Since \( f \) is continuous, according to Theorem 5.13, it is bounded: \( \alpha \leq f(x) \leq \beta \) for all \( a \leq x \leq b \). Let \( s > 0 \). The Uniform Continuity Theorem 10.13 implies that there exists \( r > 0 \) such that \( |f(x) - f(y)| < s \) whenever \( x, y \in [a, b] \) and \( |x - y| < r \). Let us then choose a partition

\[
a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b,
\]
such that each subinterval has length \( 0 < x_i - x_{i-1} < r \); for example, we could choose a regular partition (13.41) with \( n > (b-a)/r \). Let

\[
\alpha_i = \inf \{ f(x) \mid x_{i-1} \leq x \leq x_i \}, \quad \beta_i = \sup \{ f(x) \mid x_{i-1} \leq x \leq x_i \}.
\]

According to Theorem 5.14, there exist \( y_i, z_i \in [x_{i-1}, x_i] \) such that \( f(y_i) = \alpha_i, \ f(z_i) = \beta_i \). Since

\[
|y_i - z_i| \leq x_i - x_{i-1} < r,
\]
we conclude that \( 0 \leq \beta_i - \alpha_i = f(z_i) - f(y_i) < s \).

Thus, in view of (13.36), we can bound the areas of the intermediate rectangles:

\[
\text{area } Z_i = (\beta_i - \alpha_i) (x_i - x_{i-1}) < s \ (x_i - x_{i-1}),
\]
and hence their total area is bounded by

\[
\text{area } Z^* = \sum_{i=1}^{n} \text{area } Z_i < s \ (b-a).
\]

Since this holds for all \( s > 0 \), the infimum over all such \( Z^* \) is zero, and hence the integrability criterion of Theorem 13.12 is satisfied. \( \square \)

Proof of Theorem 13.8: We will assume \( 0 \leq f(x) \leq \beta \) and \( 0 \leq g(x) \leq \delta \), whereby \( 0 \leq f(x) + g(x) \leq \beta + \delta \), for all \( a \leq x \leq b \), the general case following by our usual device of adding in a constant to both functions. Given a partition

\[
a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b,
\]
choose \( \alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n, \gamma_1, \ldots, \gamma_n, \delta_1, \ldots, \delta_n \) such that

\[
0 \leq \alpha_i \leq f(x) \leq \beta_i \leq \beta, \quad 0 \leq \gamma_i \leq g(x) \leq \delta_i \leq \delta, \quad \text{ for all } x_{i-1} \leq x \leq x_i.
\]
This implies

\[
0 \leq \alpha_i + \gamma_i \leq f(x) + g(x) \leq \beta_i + \delta_i \leq \beta + \delta, \quad \text{ for all } x_{i-1} \leq x \leq x_i.
\]
The corresponding intermediate rectangles are

\[
Z_i = R_{x_{i-1}, \alpha_i}^{x_i, \beta_i}, \quad \hat{Z}_i = R_{x_{i-1}, \gamma_i}^{x_i, \delta_i}.
\]
Since we are assuming integrability of \( f \) and \( g \), their combined areas

\[
\text{area } Z^* = \text{area } \bigcup_{i=1}^{n} Z_i, \quad \text{area } \hat{Z}^* = \text{area } \bigcup_{i=1}^{n} \hat{Z}_i,
\]
both have zero infimum as we vary over all partitions and upper and lower bounds. Thus, in view of (13.42), the “summed” intermediate rectangles
\[
\tilde{Z}_i = R^{x_{i-1}, \alpha_i + \gamma_i}_{x_i, \beta_i + \delta_i}
\]
play the same role for \( f(x) + g(x) \), and, moreover, area \( \tilde{Z}_i = \text{area } Z_i + \text{area } \hat{Z}_i \). We conclude that their combined areas
\[
\text{area } \tilde{Z}^* = \text{area } \bigcup_{i=1}^{n} \tilde{Z}_i = \text{area } Z^* + \text{area } \hat{Z}^*,
\]
also have zero infimum, thus proving the integrability of \( f(x) + g(x) \).

Finally, to prove the integration formula (13.23), we note that the corresponding lower sums for, respectively, \( f + g, f, g \), also add up:
\[
\sum_{i=1}^{n} (\alpha_i + \gamma_i)(x_i - x_{i-1}) = \sum_{i=1}^{n} \alpha_i (x_i - x_{i-1}) + \sum_{i=1}^{n} \gamma_i (x_i - x_{i-1}),
\]
which implies that the same holds for their integrals, thus proving (13.23). \( Q.E.D. \)

Finally, with some additional work, one can replace the inequality in (13.16) by a strict inequality.

**Theorem 13.13.** Let \( a < b \). If \( f, g \) are integrable and \( f(x) < g(x) \) for all \( x \in [a, b] \), then
\[
\int_{a}^{b} f(x) \, dx < \int_{a}^{b} g(x) \, dx.
\]

**Proof:** We can use (13.24) to rewrite (13.43) as
\[
\int_{a}^{b} \left[ g(x) - f(x) \right] \, dx > 0 \quad \text{provided } g(x) - f(x) > 0 \quad \text{for all } a \leq x \leq b.
\]

Thus it suffices to prove that if \( f \) is integrable and \( f(x) > 0 \) for all \( a \leq x \leq b \), then
\[
\int_{a}^{b} f(x) \, dx > 0.
\]

The proof of that the integral is strictly positive is easy if \( f \) is continuous. Indeed, set \( \alpha = \inf f[a, b] \). Theorem 5.14 tells us that there exists \( x_* \in [a, b] \) such that \( \alpha = f(x_*) > 0 \). Thus, \( \int_{a}^{b} f(x) \, dx \geq \alpha (b - a) > 0 \), using (13.17).

If \( f \) is merely integrable, we argue by contradiction. Thus, suppose
\[
f(x) > 0 \quad \text{for all } a \leq x \leq b, \quad \text{but } \int_{a}^{b} f(x) \, dx = 0.
\]

Condition (13.40) implies that the upper sums have zero infimum:
\[
\inf \left\{ \sum_{i=1}^{n} \beta_i (x_i - x_{i-1}) \right\} = 0.
\]

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Thus, for every $r > 0$ we can find a partition and a set of upper bounds $\beta_i$ such that the corresponding upper sum < $r$. We claim that this implies that there exists a sequence of closed intervals $I_k = [c_k, d_k]$ for $k \in \mathbb{N}$ such that $I_{k+1} \subset I_k$ and $f(x) \leq 2^{-k}$ for all $x \in I_k$. In this case, $c_1 \leq c_2 \leq c_3 \leq \cdots \leq d_3 \leq d_2 \leq d_1$. Thus, $x_* = \sup\{c_k\}$ satisfies $c_k \leq x_* \leq d_k$, and hence $x_* \in I_k$ for all $k \in \mathbb{N}$. But this implies $f(x_*) \leq 2^{-k}$ for all $k$, and hence $f(x_*) \leq 0$, in contradiction to our assumption that $f$ is everywhere positive.

To construct the intervals $I_k$ we proceed by induction. We begin by invoking (13.44) to choose a partition of $[a, b]$ and corresponding upper bounds $\beta_i \geq 0$ such that

$$\sum_{i=1}^{n} \beta_i (x_i - x_{i-1}) \leq \frac{1}{2} (b - a).$$

Let $\beta_* = \min \{\beta_1, \ldots, \beta_n\}$ be the minimum of these upper bounds. Then

$$\frac{1}{2} (b - a) \geq \sum_{i=1}^{n} \beta_i (x_i - x_{i-1}) \geq \beta_* \sum_{i=1}^{n} (x_i - x_{i-1}) = \beta_* (b - a),$$

and hence $\beta_* \leq \frac{1}{2}$.

We thus can set $I_1 = [x_{i-1}, x_i]$ to be the interval where $\beta_i = \beta_*$. (If there are several with this property, choose any one.) Then by the above inequality, $f(x) \leq \beta_i = \beta_* < \frac{1}{2} = 2^{-1}$ whenever $x \in I_1$.

Now, given $I_k = [c_k, d_k]$ as above, in order to construct an appropriate $I_{k+1} \subset I_k$, we note that $\int_{c_k}^{d_k} f(x) \, dx = 0$ also, because if it were positive, Proposition 13.7 would imply that the integral over the entire interval $[a, b]$ must also be positive. By the same reasoning, we can find a partition of $I_k$, say $c_k = x_0 < x_1 < \cdots < x_{n-1} < x_n = d_k$, and a set of upper bounds $\tilde{\beta}_i \geq f(x)$ for $x_{i-1} \leq x \leq x_i$, with minimum $\tilde{\beta}_* = \min \{\tilde{\beta}_1, \ldots, \tilde{\beta}_n\}$, such that

$$\tilde{\beta}_* (d_k - c_k) = \tilde{\beta}_* \sum_{i=1}^{n} (x_i - x_{i-1}) = \sum_{i=1}^{n} \tilde{\beta}_i (x_i - x_{i-1}) \leq 2^{-k-1} (d_k - c_k).$$

We then set $I_{k+1} = [x_{i-1}, x_i]$ to be one of these intervals where $\tilde{\beta}_i = \tilde{\beta}_*$, so that if $x \in I_{k+1}$, then $f(x) \leq \tilde{\beta}_* \leq 2^{-k-1}$. This completes the construction of the intervals $I_k$, and hence the proof of the theorem. \(\text{Q.E.D.}\)

The next step is to look at what happens as we vary the endpoints of the interval of integration. Let us replace the upper limit of integration by a variable point $x$. The resulting function

$$I(x) = \int_{a}^{x} f(t) \, dt$$

is known as the indefinite integral of $f(x)$ since its upper endpoint $x$ is no longer viewed as being fixed. As noted earlier, we should use a different “dummy” variable $t$ inside the integral sign so as not to confuse the integration variable with the limits of integration.
The key result is the continuity of the indefinite integral, irrespective of whether or not the function being integrated is continuous.

**Theorem 13.14.** Let \( a \leq c \leq b \). If \( f(x) \) is bounded and integrable on \([a, b]\), then its indefinite integral

\[
I(x) = \int_c^x f(t) \, dt
\]

is a continuous function of \( x \in [a, b] \).

**Proof:** If \( a \leq x, y \leq b \), then, using (13.19, 22),

\[
I(x) - I(y) = \int_c^x f(t) \, dt - \int_c^y f(t) \, dt = \int_y^x f(t) \, dt.
\]

Thus, given a bound \(|f(x)| \leq \beta\), by (13.17, 18),

\[
|I(x) - I(y)| = \left| \int_y^x f(t) \, dt \right| \leq \int_y^x |f(t)| \, dt \leq \int_y^x \beta \, dt = \beta |y - x|.
\]

Thus, \( I(x) \) satisfies the Lipschitz continuity condition (4.2) with \( \lambda = \beta \), and hence Theorem 4.4 implies that \( I \) is continuous. \(Q.E.D.\)

**Proposition 13.15.** If \( f(x) \) is integrable and \( > 0 \) then its indefinite integral (13.45) is a strictly increasing monotone function of \( x \).

**Proof:** If \( x < y \), then, using (13.43),

\[
I(y) - I(x) = \int_a^y f(t) \, dt - \int_a^x f(t) \, dt = \int_x^y f(t) \, dt > 0.
\]

\(Q.E.D.\)

**Remark:** The conclusion of Proposition 13.15 remains valid even when \( f(x) \) has isolated zeros. For example,

\[
\int_0^x t^2 \, dt = \frac{1}{3} x^3
\]

is strictly monotone. On the other hand, if \( f(x) = 0 \) on some open subinterval, then its integral is no longer strictly monotone.

Finally, let us prove the Fundamental Theorem of Calculus, which states the remarkable fact that differentiation is the opposite of integration. It was this amazing discovery of Newton and Leibniz that sparked the calculus revolution in mathematics.

**Theorem 13.16.** Suppose \( f:\mathbb{R} \to \mathbb{R} \) is continuous. Then its indefinite integral

\[
I(x) = \int_a^x f(t) \, dt.
\]

is differentiable and its derivative is the original function: \( I'(x) = f(x) \).
Proof: As in Definition 7.3, given \( b \in \mathbb{R} \), we form the difference quotient

\[
Q(x) = \begin{cases} 
\frac{I(x) - I(b)}{x - b}, & x \neq b, \\
I(b), & x = b.
\end{cases}
\]

According to (7.3), continuity of \( Q(x) \) implies differentiability of \( I(x) \) at \( x = b \), with \( I'(b) = f(b) \). If we can show that this holds for all \( b \), the Fundamental Theorem follows.

To prove continuity, note first that, when \( x \neq b \),

\[
Q(x) = \frac{I(x) - I(b)}{x - b} = \frac{1}{x - b} \left( \int_{a}^{x} f(t) \, dt - \int_{a}^{b} f(t) \, dt \right) = \frac{1}{x - b} \int_{b}^{x} f(t) \, dt,
\]

where we applied (13.19, 22). We further note that, again using (13.22),

\[
Q(x) = \frac{1}{x - b} \int_{b}^{x} f(t) \, dt = -\frac{1}{x - b} \int_{b}^{x} f(t) \, dt = \frac{1}{b - x} \int_{x}^{b} f(t) \, dt.
\]  \( (13.46) \)

Now, suppose \( \alpha < f(b) < \beta \). Since \( f \) is continuous and \( b \in f^{-1}(\alpha, \beta) \), there exists an open interval \( (c, d) \) containing \( b \) such that \( (c, d) \subset f^{-1}(\alpha, \beta) \). Thus, if \( c < x < d \), then \( \alpha < f(x) < \beta \). Thus, as a consequence of Theorem 13.13, if \( b < x < d \),

\[
\alpha (x - b) < \int_{b}^{x} f(t) \, dt < \beta (x - b),
\]

whereas if \( c < x < b \), then

\[
\alpha (b - x) < \int_{x}^{b} f(t) \, dt < \beta (b - x).
\]

In view of (13.46), this implies \( \alpha < Q(x) < \beta \) whenever \( c < x < d \), including \( x = b \). We deduce that \( (c, d) \subset Q^{-1}(\alpha, \beta) \), which implies that the latter set is open. Since this holds for all such intervals, we deduce that \( Q \) is continuous, which establishes the Fundamental Theorem.

Q.E.D.

Thus, to integrate a function \( f(x) \), if we know an antiderivative, meaning a function \( F(x) \) such that \( F'(x) = f(x) \), then

\[
\int_{a}^{b} f(x) \, dx = F(b) - F(a).
\]  \( (13.47) \)

Indeed, \( F \) and \( I \) have the same derivative,

\[
F'(x) - I'(x) = f(x) - f(x) \equiv 0,
\]

and hence Corollary 7.15 implies that

\[
F(x) - I(x) = c
\]

is constant. Since \( I(a) = 0 \), we have \( F(a) = c \) and hence

\[
\int_{a}^{x} f(t) \, dt = I(x) = F(x) - c = F(x) - F(a).
\]

Setting \( x = b \) proves (13.47).
**Example 13.17.** We already know that if
\[ F(x) = x^{n+1} \]
then
\[ F'(x) = (n + 1) x^n. \quad (13.48) \]
Thus, using (13.47), we can evaluate
\[ \int_a^b x^n \, dx = \frac{b^{n+1}}{n+1} - \frac{a^{n+1}}{n+1}, \quad (13.49) \]
where we divide both sides by \( n + 1 \). The case \( n = 0 \) reproduces our formula (13.6) for the integral of the constant function \( x^0 = 1 \), while the case \( n = 1 \) reproduces our formula (13.7) for the integral of \( x^1 = x \), namely the area of the trapezoidal region under its graph. Formula (13.48) is also valid when \( n < 0 \), and hence so is the integration formula (13.49) provided \( n \neq -1 \).

The one case not covered by the preceding Example 13.17 is when \( n = -1 \). Indeed, there is no algebraic function \( f(x) \) whose derivative is \( 1/x \). Its indefinite integral will define the *natural logarithm*
\[ \log x = \int_1^x \frac{dx}{x} \quad \text{for} \quad x > 0. \quad (13.50) \]
We will usually drop the qualifier “natural” since it is essentially the only logarithm we discuss in these notes. Theorems 13.14 and 13.16 imply that, when \( x > 0 \), the logarithm is a continuous function whose derivative \( \log'(x) = 1/x \). Corollary 7.15 says that the only other functions with this property are \( \log x + c \) for a constant \( c \in \mathbb{R} \). Proposition 13.15 implies that \( \log x \) is a strictly increasing function on \( \mathbb{R}^+ \). Since \( \log 1 = 0 \), we have \( \log x > 0 \) when \( x > 1 \), while \( \log x < 0 \) when \( x < 0 \).

It can be shown that \( \log x \) is unbounded, from both above and below, on the entire positive real line \( \mathbb{R}^+ \). Indeed, consider the region
\[ L_\infty = \{(x, y) \mid x > 1, \; 0 < y < 1/x \}. \]
It contains the essentially disjoint rectangles
\[ R_n = R_{n-1,0}^{1/n} \subset L_\infty \quad \text{for} \quad n \in \mathbb{N}. \]
The combined area of these rectangles is
\[ \sum_{n=1}^{\infty} \text{area } R_n = \sum_{n=1}^{\infty} \frac{1}{n} = \infty, \]
being the well known divergent harmonic series, [24]. Since
\[ \int_1^{\infty} \frac{dx}{x} = \text{area } L_\infty \geq \sum_{n=1}^{\infty} \text{area } R_n, \]
this proves that \( \log x \) is unbounded from above as \( x \to \infty \). Similarly, the region
\[ \hat{L}_\infty = \{(x, y) \mid 0 < x < 1, \; 0 < y < 1/x \} \]
contains the essentially disjoint rectangles
\[ \hat{R}_n = R_{1/(n-1), n-1} \subset \hat{L}_\infty \quad \text{for} \quad n \in \mathbb{N} \]
that have combined area
\[ \sum_{n=1}^{\infty} \text{area} \hat{R}_n = \sum_{n=1}^{\infty} (n-1) \left( \frac{1}{n-1} - \frac{1}{n} \right) = \sum_{n=1}^{\infty} \frac{1}{n} = \infty. \]
Thus,
\[ \int_0^1 \frac{dx}{x} = -\sum_{n=1}^{\infty} \text{area} \hat{R}_n, \]
which shows that \( \log x \) is not bounded from below as \( x \to 0^+ \).

The inverse function of the natural logarithm is the exponential function \( e^x \), i.e., \( \exp(x) = e^x \) satisfies \( \exp(x) = \log^{-1}(x) \). Thus,
\[ e^{\log x} = x \quad \text{or, equivalently,} \quad \log(e^x) = x. \]
In other words,
\[ y = e^x \quad \text{if and only if} \quad x = \log y. \]
In particular, the base \( e \) of the exponential is defined by\( ^{†} \)
\[ y = e = e^1 \approx 2.71287 \ldots, \quad \text{which satisfies} \quad 1 = \log e = \int_1^e \frac{dx}{x}. \]
In other words, \( e > 1 \) is the real number with the property that the area of the region lying below the graph of \( 1/x \) between \( 1 \leq x \leq e \) is exactly \( 1 \). Since \( \log: \mathbb{R}^+ \to \mathbb{R}^+ \) is continuous and strictly increasing, the same is true for \( \exp: \mathbb{R}^+ \to \mathbb{R}^+ \).

Let us compute the derivative of the exponential. Using (7.13), since \( \log'(y) = 1/y \), we have
\[ \exp'(x) = \frac{1}{\log'(\exp(x))} = \exp(x). \quad (13.51) \]
Thus, the exponential function is its own derivative. In fact, we can completely characterize such functions.

**Proposition 13.18.** If \( f'(x) = f(x) \) then \( f(x) = c e^x \) for some constant \( c \in \mathbb{R} \).

**Proof:** Suppose first that \( f(x) > 0 \). Define \( g(x) = \log f(x) \). Then, applying the Chain Rule (7.10),
\[ g'(x) = f'(x) \log f(x) = \frac{1}{f(x)} f(x) = 1. \]
Thus, \( g(x) = x + a \) for some constant \( a \in \mathbb{R} \), and hence
\[ f(x) = e^{g(x)} = e^{x+a} = c e^x, \quad \text{where} \quad c = e^a > 0. \]

\( ^{†} \) The decimal expansion of \( e \) is found by numerical integration.
If \( f(x) < 0 \), then the same proof using \( g(x) = \log[-f(x)] \) implies \( f(x) = c e^x \) for some \( c < 0 \). Finally, if \( f(a) = 0 \) for some \( a \in \mathbb{R} \), then we claim that \( f(x) \equiv 0 \). Indeed, if \( f(b) \neq 0 \), then, by the preceding argument, \( f(x) = c e^x \) for some \( c \neq 0 \). But then \( f(x) \neq 0 \) for all \( x \in \mathbb{R} \), contradicting our original assumption. Q.E.D.

We close this section with the Mean Value Theorem for integrals.

**Proposition 13.19.** Let \( a < b \) and suppose \( f:[a,b] \to \mathbb{R} \) is continuous. Then there exists \( a < c < b \) such that

\[
\int_a^b f(x) \, dx = f(c) (b-a).
\] (13.52)

The proof is an immediate application of the Mean Value Theorem 7.13 to the indefinite integral \( I(x) \), using the Fundamental Theorem 13.16 to evaluate its derivative. Equation (13.52) can be interpreted as saying that the area under the graph of \( f \) equals the area of a rectangle of height \( f(c) \) for some \( a < c < b \).


We now turn to the integration of a function \( F(x,y) \) depending on two variables, leading to the notion of double integral, which, when non-negative, will measure the volume of the region lying under its graph. Thus the first step is to define what is meant by volume of a region \( A \subset \mathbb{R}^3 \).

We begin with the analog of a bounded closed rectangle, which is a closed rectangular solid or, for short, a box

\[
\mathcal{B}_{a,c,e}^{b,d,f} = \{ (x,y,z) \in \mathbb{R}^3 \mid a \leq x \leq b, c \leq y \leq d, e \leq z \leq f \}.
\] (14.1)

The volume of the box (14.1) is the product of its length, width and height:

\[
\text{vol } \mathcal{B}_{a,c,e}^{b,d,f} = (b-a) (d-c) (f-e).
\] (14.2)

We also note that its volume is the product of its height, which is \( f-e \), times the area of its projection onto the \( xy \) plane, which is the closed rectangle

\[
\overline{R}_{a,c}^{b,d} = \{ (x,y,0) \in \mathbb{R}^3 \mid a \leq x \leq b, c \leq y \leq d \}, \quad \text{with area } \overline{R}_{a,c}^{b,d} = (b-a) (d-c).
\] (14.3)

In particular, the volume of \( \mathcal{B}_{a,c,e}^{b,d,f} \) is 0 if and only if \( a < b, c < d, e < f \), in which case its interior is the nonempty open box

\[
B_{a,c,e}^{b,d,f} = \{ (x,y,z) \in \mathbb{R}^3 \mid a < x < b, c < y < d, e < z < f \}.
\] (14.4)

On the other hand, if \( a = b \) and/or \( c = d \) and/or \( e = f \), the box collapses to either a closed rectangle, or a closed line segment, or a point, each of which has zero volume and empty interior.

The boundary of the box (14.1) consists of 6 closed rectangles, obtained by setting \( x = a \) or \( x = b \) or \( y = c \) or \( y = d \) or \( z = e \) of \( z = f \), with zero volume. A half open box consists of the union of an open box and a part of its boundary, typically one or more of its rectangular
Two boxes are said to be \textit{essentially disjoint} if their intersection is either empty or is contained in one of their boundary rectangles and hence of zero volume. The volume of an essentially disjoint union of boxes is the sum of their individual volumes. With this in hand, the general definition of the volume of a subset of $\mathbb{R}^3$ is in direct analogy with the definition of the area of a planar set.

\textbf{Definition 14.1.} Let $A \subset \mathbb{R}^3$. Let
\[
V_A = \left\{ \sum_{i=1}^{n} \text{vol } B_i \bigg| \begin{array}{c}
B_1, \ldots, B_n \text{ are essentially disjoint boxes } \\
\text{and } B_i \subset A \text{ for } i = 1, \ldots, n.
\end{array} \right\} \subset \mathbb{R}^+.
\]
Then we define the \textit{volume} of $A$ as
\[
\text{vol } A = \begin{cases}
0, & A = \emptyset, \\
\sup V_A, & V_A \subset \mathbb{R}^+ \text{ is bounded,} \\
\infty, & V_A \subset \mathbb{R}^+ \text{ is unbounded.}
\end{cases} \quad (14.5)
\]

Thus, $A$ has zero volume if and only if it contains no open boxes. If $A \subset B$ then $\text{vol } A \leq \text{vol } B$. Thus, if $A$ is bounded, meaning that $A \subset B$ is contained in some bounded box $B$, then its volume is finite, $0 \leq \text{vol } A \leq \text{vol } B < \infty$. There are unbounded sets with finite volume. Further,
\[
\text{vol } A + \text{vol } B \leq \text{vol } (A \cup B) \quad \text{when} \quad \text{vol } (A \cap B) = 0. \quad (14.6)
\]
The proof mimics that of the corresponding statement (12.11) for areas. We call $A$ and $B$ \textit{volume compatible} if the above inequality is an equality. In particular, as in Lemma 12.4, if $A \subset \{ x \leq a \}$ and $B \subset \{ x \geq a \}$, then they are volume compatible. One can replace $x$ by $y$ or $z$ and the result remains valid.

The proof that the preceding definition of volume is consistent, i.e., the volume of a box equals the least upper bound of the volumes of any essentially disjoint collection of sub-boxes, proceeds via subdivision in analogy with the areas of rectangles; details are left to the reader. In particular, closed, open, and half open boxes having the same boundary have the same volume. Another useful property is that the volume of a (generalized) \textit{cylinder}\footnote{If $D \subset \mathbb{R}^2$ is a circular disk, then $C$ is a circular cylinder.}
\[
C = \{ (x, y, z) \mid (x, y) \in D, \ 0 \leq z \leq h \} \quad (14.7)
\]
of height $h$ lying over a planar region $D \subset \mathbb{R}^2$ is equal to the area of its base times its height:
\[
\text{vol } C = h \text{ area } D. \quad (14.8)
\]

Before proving this, we first note that we can identify the cylinder as a Cartesian product of $D$ with a closed interval of length $h$, so $C = D \times [0, h]$. In general, mimicking the planar
case (9.3), given subsets $A \subset \mathbb{R}^2$ and $B \subset \mathbb{R}$, define their Cartesian product to be the subset

$$A \times B = \{ (x, y, z) \mid (x, y) \in A, \ z \in B \} \subset \mathbb{R}^3. \quad (14.9)$$

The first set might itself be a Cartesian product, $A = S \times T$ for $S, T \subset \mathbb{R}$, in which case $A \times B = S \times T \times B$ is a triple Cartesian product. For example, a box is the Cartesian product of a closed rectangle and a closed interval, which is turn is the Cartesian product of three closed intervals:

$$\mathcal{B}_{a,c,e} = \mathcal{R}_{a,b,d} \times [e, f] = [a, b] \times [c, d] \times [e, f]. \quad (14.10)$$

To prove the formula (14.8) for the cylinder’s volume, given any collection $R_1, \ldots, R_n$ of essentially disjoint closed rectangles $R_i \subset D$, we can form a corresponding essentially disjoint collection of boxes $B_i = R_i \times [0, h] \subset C$ of height $h$. Using (14.3), the volume of each $B_i$ is equal to $h$ times the area of the corresponding $R_i$. Since the supremum of the total areas over all possible choices of rectangles equals the area of $D$, the same result holds for the corresponding boxes:

$$\sup \left\{ \sum_{i=1}^n \text{vol } B_i \right\} = h \sup \left\{ \sum_{i=1}^n \text{area } R_i \right\} = h \text{ area } D.$$  

Technically, we should check that no other disjoint collection of boxes contained in $C$ can have a larger volume, but one can always use subdivision to produce a collection of the above form with the same or larger volume, and hence the supremum is as advertised.

As with areas, volumes are unchanged under certain basic transformations of $\mathbb{R}^3$, including translations, reflections, rotations, and shears. Scaling in one of the coordinate directions multiplies the volume by the absolute value of the scaling factor. One proves these results by simply establishing them for boxes.

15. Double Integrals.

The basic theory of integration for functions of several variables proceeds in direct analogy with the univariate case developed in Section 13, replacing area by volume throughout. Integration of a positive function $F: D \to \mathbb{R}^+$ over a closed, bounded region $D \subset \mathbb{R}^2$ will measure the volume of the solid region lying between the graph of $F$ and the $xy$ plane $\{ z = 0 \} = \{ (x, y, 0) \mid x, y \in \mathbb{R} \} \subset \mathbb{R}^3$.

Let $D \subset \mathbb{R}^2$ be a bounded closed subset. Suppose $F: D \to [0, \beta]$ is a bounded non-negative function, so $0 \leq F(x, y) \leq \beta$ for all $(x, y) \in D$. Define the lower and upper regions

$$L = \{ (x, y, z) \mid (x, y) \in D, \ 0 \leq z \leq F(x, y) \},$$
$$U_\beta = \{ (x, y, z) \mid (x, y) \in D, \ F(x, y) \leq z \leq \beta \}. \quad (15.1)$$

Their union

$$L \cup U_\beta = C_\beta = \{ (x, y, z) \mid (x, y) \in D, \ 0 \leq z \leq \beta \} \quad (15.2)$$
is the cylinder of height $\beta$ with base $D$. On the other hand,

$$ L \cap U_\beta = G = \{ (x, y, F(x, y)) \mid (x, y) \in D \} $$

(15.3)

is a surface representing the graph of $F$ over $D \subset \mathbb{R}^2$, which has zero volume because it contains no open boxes.

**Definition 15.1.** Let $D \subset \mathbb{R}^2$ be a bounded closed subset. A bounded nonnegative function $F: D \to [0, \beta]$ is *integrable* if its lower and upper regions are volume compatible, meaning

$$ \text{vol } L + \text{vol } U_\beta = \text{vol } C_\beta = \beta \text{ area } D. $$

(15.4)

In this case, we define the *double integral* of $F$ on $D$ to be the volume of its lower region:

$$ \iint_D F(x, y) \, dx \, dy = \text{vol } L = \text{vol } \{ (x, y, z) \mid (x, y) \in D, \ 0 \leq z \leq F(x, y) \}. $$

(15.5)

In particular, if $F(x, y) \equiv c \geq 0$ is constant, then its lower region is the cylinder $D \times [0, c]$, and hence

$$ \iint_D c \, dx \, dy = c \text{ area } D. $$

The confirmation that constant functions satisfy the integrability criterion (15.4) for any choice of $\beta \geq c$ is left to the reader. In particular, one can compute the area of a domain by integrating the constant function $1$ over it:

$$ \text{area } D = \iint_D dx \, dy. $$

(15.6)

As in the single variable case, one extends double integration to functions that are not everywhere positive by adding in a constant. Specifically, if $F(x, y) \geq c$ for some $c \in \mathbb{R}$ and all $(x, y) \in \overline{D}$, then $\tilde{F}(x, y) = F(x, y) - c \geq 0$, and hence, assuming $\tilde{F}$ is integrable,

$$ \iint_D F(x, y) \, dx \, dy = \iint_D [ \tilde{F}(x, y) + c ] \, dx \, dy = \iint_D \tilde{F}(x, y) \, dx \, dy + c \text{ area } D. $$

(15.7)

It is not difficult to show that this gives a consistent value for the double integral of $F$, independent of the choice of the lower bound $\alpha$. In general, the double integral counts volumes of subregions lying above the $xy$ plane with a positive sign, and those lying below it with a negative sign.

The basic properties of double integrals mimic those of single integrals. The proofs are analogous, and thus left to the motivated reader.

- If $\text{area } D = 0$, then $\iint_D F(x, y) \, dx \, dy = 0$,

(15.8)

for any function $F(x, y)$. This is because there are no open rectangles contained in $D$ and hence no open boxes contained in either the lower of upper regions, which thus both
have zero volume. More generally,

$$\text{if } F(x, y) = 0 \text{ for } (x, y) \in A \setminus D \text{ and area } D = 0, \text{ then } \iint_A F(x, y) \, dx \, dy = 0. \quad (15.9)$$

- If $F(x, y) \leq G(x, y)$ or $F(x, y) < G(x, y)$ for all $(x, y) \in D$, then the corresponding double integral inequality holds:

$$\iint_D F(x, y) \, dx \, dy \leq \iint_D G(x, y) \, dx \, dy, \quad \iint_D F(x, y) \, dx \, dy < \iint_D G(x, y) \, dx \, dy. \quad (15.10)$$

As with single integrals, the first result is immediate, while the second requires work.

- The double integral of the sum of two functions is the sum of the individual double integrals:

$$\iint_D [F(x, y) + G(x, y)] \, dx \, dy = \iint_D F(x, y) \, dx \, dy + \iint_D G(x, y) \, dx \, dy. \quad (15.11)$$

- Multiplying the function by a constant $c \in \mathbb{R}$ multiplies the double integral by the same constant:

$$\iint_D [c \, F(x, y)] \, dx \, dy = c \iint_D F(x, y) \, dx \, dy. \quad (15.12)$$

- If $D \subset E$, and $F(x, y) \geq 0$, then

$$\iint_D F(x, y) \, dx \, dy \leq \iint_E F(x, y) \, dx \, dy. \quad (15.13)$$

- The integral over the union $D \cup E$ of two regions whose intersection has zero volume, $\text{vol}(D \cap E) = 0$, is the sum of the individual double integrals:

$$\iint_{D \cup E} F(x, y) \, dx \, dy = \iint_D F(x, y) \, dx \, dy + \iint_E F(x, y) \, dx \, dy. \quad (15.14)$$

For example, if we split the unit disk

$$D = \{ x^2 + y^2 \leq 1 \} \quad \text{into two half disks} \quad H^+ = \{ x^2 + y^2 \leq 1, \ y \geq 0 \}, \quad H^- = \{ x^2 + y^2 \leq 1, \ y \geq 0 \},$$

then

$$\iint_D F(x, y) \, dx \, dy = \iint_{H^+} F(x, y) \, dx \, dy + \iint_{H^-} F(x, y) \, dx \, dy.$$ 

In this case, their intersection $D = H^+ \cap H^- = \{ (x, 0) \mid -1 < x < 1 \}$ is an open line segment with zero area.

Let us establish an integrability criterion for a bounded non-negative function defined on a closed bounded domain $D \subset \mathbb{R}^2$, so that $F: D \to [0, \beta]$. More general functions are treated by the usual device of adding in a suitably large positive constant, using (15.7). There are analogs of Theorems 13.10 and 13.12, that the reader is invited to formulate.
Instead, we skip directly to the analog of the integrability criterion (13.40) based on lower and upper sums.

Let \( R_1, \ldots, R_n \subset D \) be an essentially disjoint collection of closed rectangles. We choose upper and lower bounds on each rectangle, so

\[
0 \leq \alpha_m \leq F(x, y) \leq \beta_m \leq \beta \quad \text{for} \quad (x, y) \in R_m.
\]

The corresponding lower, upper, and intermediate boxes are thus given by

\[
S_m = R_m \times [0, \alpha_m] \subset L, \quad T_m = R_m \times [\beta_m, \beta] \subset U, \quad Z_m = R_m \times [\alpha_m, \beta_m]. \quad (15.15)
\]

As in Theorem 13.10, the function \( F \) is integrable over \( D \) if and only if the total volumes of the intermediate boxes has zero infimum:

\[
0 = \inf \left\{ \text{vol} \left( \bigcup_{m=1}^n Z_m \right) \right\} = \inf \left\{ \sum_{m=1}^n \text{vol} Z_m \right\} = \inf \left\{ \sum_{m=1}^n (\beta_m - \alpha_m) \text{ area } R_m \right\}.
\]

In this case, the double integral equals the supremum of the corresponding lower sums, which are the total volumes of the boxes \( L_m \), or, equivalently, the infimum of the upper sums, which are the total volumes of the boxes \( L_m \cup Z_m \):

\[
\iint_D F(x, y) \, dx \, dy = \sup \left\{ \sum_{m=1}^n \alpha_m \text{ area } R_m \right\} = \inf \left\{ \sum_{m=1}^n \beta_m \text{ area } R_m \right\}. \quad (15.17)
\]

Assuming \( D \subset \overline{R}^{b,d}_{a,c} \), the simplest system comes from choosing partitions

\[
a = x_0 < x_1 < x_2 < \cdots < x_{k-1} < x_k = b, \quad c = y_0 < y_1 < y_2 < \cdots < y_{l-1} < y_l = d,
\]

and letting \( R_1, \ldots, R_n \) be all the associated rectangles \( R_{i,j} \) that satisfy

\[
R_{i,j} = R_{x_{i-1},y_{j-1}}^{x_i,y_j} \subset D \quad \text{for} \quad 1 \leq i \leq k, \quad 1 \leq j \leq l. \quad (15.19)
\]
See Figure 24; the integration domain is shaded in the left hand plot, and its partition into subrectangles is on the right.

The proof that any continuous function is integrable follows that of Theorem 13.5 using the Uniform Continuity Theorem 10.14.

**Theorem 15.2.** If $D \subset \mathbb{R}^2$ is closed and bounded, and $F : D \to \mathbb{R}$ is continuous, then $F$ is integrable.

As for the integrability of monotone functions, we define $F(x, y)$ to be **monotone non-decreasing** if

$$F(x, y) \leq F(z, w) \quad \text{whenever} \quad x \leq z, \; y \leq w. \quad (15.20)$$

Reversing one or both of the latter inequalities produces three other forms of monotonicity for functions of two variables.

**Theorem 15.3.** If $R \subset \mathbb{R}^2$ is a bounded, closed rectangle, and $F : \overline{R} \to \mathbb{R}$ is monotone and bounded, then $F$ is integrable on $R$.

**Remark:** One can extend this result to more general bounded closed domains $D \subset \mathbb{R}^2$ provided its boundary $\partial D$ has finite length.

**Proof:** Suppose $\alpha \leq F(x, y) \leq \beta$ for $(x, y) \in R$; the other three cases are similar. We choose uniform partitions

$$x_i = i \frac{b - a}{n}, \quad y_j = j \frac{d - c}{n}, \quad i, j = 0, \ldots, n,$$

for $n \in \mathbb{N}$, and subdivide $R = \bigcup_{i, j} R_{i, j}$ as in (15.18–19). Assuming monotone nondecreasing as in (15.20), we can choose lower and upper bounds to be the values of $F$ at the bottom left and top right corners of each rectangle, respectively:

$$\alpha_{i, j} = F(x_{i-1}, y_{j-1}) \leq F(x, y) \leq F(x_i, y_j) = \beta_{i, j}, \quad x_{i-1} \leq x \leq x_i, \; y_{j-1} \leq y \leq y_j.$$

Thus the intermediate volumes are given by

$$\text{vol } Z^* = \sum_{i=1}^{n} \sum_{j=1}^{n} (\beta_{i, j} - \alpha_{i, j}) (x_i - x_{i-1}) (y_j - y_{j-1})$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} \left[ F(x_i, y_j) - F(x_{i-1}, y_{j-1}) \right] \frac{(b-a)(d-c)}{n^2}$$

$$= \left[ F(b, d) - F(a, c) + \sum_{i=1}^{n-1} \left[ F(b, y_i) - F(a, y_i) + F(x_i, d) - F(x_i, c) \right] \right] \frac{(b-a)(d-c)}{n^2}$$

$$\leq 2n (\beta - \alpha) \frac{(b-a)(d-c)}{n^2} = \frac{2(\beta - \alpha)(b-a)(d-c)}{n},$$

noting that most of the terms in the intermediate summation cancel each other. (Compare the proof of the one variable version Theorem 13.4.) The latter quantity has zero infimum for $n \in \mathbb{N}$, thus proving integrability. \(Q.E.D.\)
Unfortunately, there is no direct analog of the Fundamental Theorem of Calculus for double integrals. The best analog is Green’s Theorem, [1, 15], but this involves line integrals that we will not cover here. In the absence of such a result, an effective way to evaluate a double integral is to reduce it to a pair of ordinary single integrals. This result is known as Fubini’s Theorem, named after the early twentieth century Italian mathematician Guido Fubini, although simpler versions, e.g., for bounded continuous functions, were known much earlier.

**Theorem 15.4.** Let \( R = \mathbb{R}_{a,c}^{d} \subset \mathbb{R}^{2} \) be a closed rectangle. Let \( F: R \to \mathbb{R} \) be integrable. Assume further that, for each fixed \( c \leq y \leq d \), the functions \( f_{y}(x) = F(x, y) \) are integrable on the interval \( a \leq x \leq b \). Then the function

\[
h(y) = \int_{a}^{b} f_{y}(x) \, dx = \int_{a}^{b} F(x, y) \, dx
\]

is integrable on the interval \( c \leq y \leq d \), and

\[
\iint_{R} F(x, y) \, dx \, dy = \int_{c}^{d} h(y) \, dy = \int_{c}^{d} \left( \int_{a}^{b} F(x, y) \, dx \right) \, dy.
\] (15.21)

Similarly, if, for each fixed \( a \leq x \leq b \), the function \( g_{x}(y) = F(x, y) \) is integrable on the interval \( c \leq y \leq d \), then

\[
k(x) = \int_{c}^{d} g_{x}(y) \, dy = \int_{c}^{d} F(x, y) \, dy
\]

is integrable on the interval \( a \leq x \leq b \), and

\[
\iint_{R} F(x, y) \, dx \, dy = \int_{a}^{b} k(x) \, dx = \int_{a}^{b} \left( \int_{c}^{d} F(x, y) \, dy \right) \, dx.
\] (15.22)

In the inner expressions in the Fubini formulae, the variable not being integrated is to be treated as a constant parameter. For example, let \( R = \mathbb{R}^{2}_{0,0} = \{ 0 \leq x \leq 2, \ 0 \leq y \leq 1 \} \). Then

\[
\iint_{R} (xy + y^{2}) \, dx \, dy = \int_{0}^{2} \left[ \int_{0}^{1} (xy + y^{2}) \, dy \right] \, dx = \int_{0}^{2} \left( \frac{1}{2} x + \frac{1}{3} \right) \, dx = \frac{5}{3},
\]

or, alternatively,

\[
\iint_{R} (xy + y^{2}) \, dx \, dy = \int_{0}^{1} \left[ \int_{0}^{2} (xy + y^{2}) \, dx \right] \, dy = \int_{0}^{1} \left( 2y + 2y^{2} \right) \, dy = \frac{5}{3},
\]

which give the same answer, as Theorem 15.4 says they must.

**Remark:** According to Proposition 10.3, continuity of \( F(x, y) \) implies continuity, and hence integrability, of \( f_{y}(x) \) and \( g_{x}(y) \). However, integrability of \( F(x, y) \) does not necessarily imply their integrability. For example, the function

\[
F(x, y) = \begin{cases} 
1, & \text{if } x \in \mathbb{Q} \text{ and } y = \frac{1}{2}, \\
0, & \text{otherwise},
\end{cases}
\]

for \( 0 \leq x, y \leq 1 \).
is integrable over the unit square $\mathbb{R}_{0,0}^{1,1}$ with double integral zero. This is because $F(x,y) \neq 0$ only on the line segment $\{(x, \frac{1}{2}) | 0 \leq x \leq 1\}$, which has zero area; see (15.9). On the other hand, $f_{1/2}(x) = F(x, \frac{1}{2})$ is the non-integrable function (13.15). Intuitively, the fact that $f_y(x)$ is not integrable for only one value of $x$ does not affect the overall integrability of $F(x,y)$. On the other hand, if we fix $0 \leq x \leq 1$, every one of the functions $g_x(y) = F(x,y)$ is zero at every point except possibly $y = \frac{1}{2}$, and hence is integrable with zero integral. Thus, while in this case we cannot justify the first Fubini formula (15.21), we can apply the second Fubini formula (15.22) to compute the double integral.

**Proof of Theorem 15.4:** We will prove the first Fubini formula (15.21) in which one first integrates with respect to $x$ and then with respect to $y$; proof of the second simply amounts to interchanging the roles of $x$ and $y$. We will employ the scheme based on partitions (15.18) of the intervals $a \leq x \leq b$ and $c \leq y \leq d$, their associated rectangles (15.19) and lower and upper bounds $\alpha_{i,j} \leq F(x,y) \leq \beta_{i,j}$ for $(x,y) \in R_{i,j}$. In this case, since the domain $D = \mathbb{R}_{a,c}^{b,d}$ is a closed rectangle, all of the corresponding rectangles $R_{i,j}$ are subsets thereof, and their total area equals the area of $D$.

Let us fix a value $y_{j-1} \leq y \leq y_j$, and consider the function $f_y(x) = F(x,y)$ for $a \leq x \leq b$. The lower and upper bounds for $F(x,y)$ on $R_{i,j}$ also serve as lower and upper bounds for $f_y(x)$:

$$\alpha_{i,j} \leq f_y(x) = F(x,y) \leq \beta_{i,j} \quad \text{for} \quad x_{i-1} \leq x \leq x_i.$$ 

Because we are assuming integrability of $f_y(x)$, in view of (13.35, 37), the corresponding lower and upper sums, which we denote by $\lambda_j, \mu_j$, respectively, bound its integral:

$$\lambda_j = \sum_{i=1}^{k} \alpha_{i,j} (x_i - x_{i-1}) \leq h(y) = \int_{a}^{b} f_y(x) \, dx \leq \sum_{i=1}^{k} \beta_{i,j} (x_i - x_{i-1}) = \mu_j.$$  \hspace{1cm} (15.23)

**Warning:** It is not necessarily true that the supremum of these lower sums or the infimum of the upper sums equals the integral of $f_y(x)$, because the lower and upper bounds $\alpha_{i,j}, \beta_{i,j}$ are required to hold on the entire rectangle $R_{i,j}$, not just on the interval $[x_{i-1}, x_i]$.

Now let us form the corresponding lower and upper sums for the function $h(y)$, which similarly bound its integral:

$$\sum_{j=1}^{l} \lambda_j (y_j - y_{j-1}) \leq \int_{c}^{d} h(y) \, dy \leq \sum_{j=1}^{l} \mu_j (y_j - y_{j-1}).$$

Using (15.23), these lower and upper sums are

$$\sum_{j=1}^{l} \lambda_j (y_j - y_{j-1}) = \sum_{j=1}^{l} \sum_{i=1}^{k} \alpha_{i,j} (x_i - x_{i-1}) (y_j - y_{j-1}),$$

$$\sum_{j=1}^{l} \mu_j (y_j - y_{j-1}) = \sum_{j=1}^{l} \sum_{i=1}^{k} \beta_{i,j} (x_i - x_{i-1}) (y_j - y_{j-1}).$$  \hspace{1cm} (15.24)
and we recognize the right hand sides as the lower and upper sums (15.17) for the double integral of $F(x, y)$. Their difference is

$$
\sum_{j=1}^{l} (\mu_j - \lambda_j) (y_j - y_{j-1}) = \sum_{i=1}^{k} \sum_{j=1}^{l} (\beta_{i,j} - \alpha_{i,j}) (x_i - x_{i-1}) (y_j - y_{j-1}).
$$

(15.25)

The left hand side of (15.25) is the total area of the intermediate rectangles associated with the given lower and upper bounds for $h(y)$. The right hand side of (15.25) is the total volume of the intermediate boxes for the double integral. Since $F(x, y)$ is integrable, the infimum of the latter volumes is zero, and hence its double integral is the supremum of the lower sums or, equivalently, the infimum of the upper sums on the right hand side of (15.24). We conclude that the intermediate area sums on the left hand side of (15.25) also have zero infimum, which, using the criterion in Theorem 13.12, implies that $h(y)$ is integrable, and its integral equals the supremum of the lower sums appearing on the left hand side of (15.24), which also equals the infimum of the upper sums. Since these quantities agree, we conclude that

$$
\int \int_{\mathbb{R}} F(x, y) \, dx \, dy = \int_{c}^{d} h(y) \, dy,
$$

which is equivalent to the first Fubini formula (15.21).

Q.E.D.

Example 15.5. Here is a cautionary example. Suppose we try to integrate the function $F(x, y) = (x^2 - y^2)/(x^2 + y^2)^2$ over the unit square $Q = \{0 \leq x, y \leq 1\}$. If we use the first Fubini formula (15.21) we obtain

$$
\int \int_{Q} \frac{(x^2 - y^2)}{(x^2 + y^2)^2} \, dx \, dy = \int_{0}^{1} \left( \int_{0}^{1} \frac{(x^2 - y^2)}{(x^2 + y^2)^2} \, dx \right) \, dy = -\int_{0}^{1} \frac{dy}{1 + y^2} = -\frac{\pi}{4}.
$$

The second equality is a consequence of the integrand being the partial derivative with respect to $x$ of the function $-x/(x^2 + y^2)$; the final integral involves the inverse tangent function and can be found in any calculus text. On the other hand, if we use the second Fubini formula (15.22) we obtain

$$
\int \int_{Q} \frac{(x^2 - y^2)}{(x^2 + y^2)^2} \, dx \, dy = \int_{0}^{1} \left( \int_{0}^{1} \frac{(x^2 - y^2)}{(x^2 + y^2)^2} \, dy \right) \, dx = \int_{0}^{1} \frac{dx}{1 + x^2} = +\frac{\pi}{4}.
$$

Thus, the two iterated integrals give different values! The problem is that $F(x, y)$ is not bounded on $Q$ — indeed, $F(x, 0) = 1/x^2$, while $F(0, y) = -1/y^2$ — and hence is not integrable, so Theorem 15.4 does not apply. On the other hand, one can use the Fubini formulas to integrate $F$ on any rectangle that does not contain the origin.

One can generalize Fubini’s formula (15.22) to a region lying between the graphs of two scalar functions $f, g: [a, b] \to \mathbb{R}$ such that $f(x) < g(x)$ for all $x \in [a, b].$ Let

$$
D = \{ (x, y) \mid a < x < b, \ f(x) < y < g(x) \}.
$$
Then, under the appropriate integrability assumptions,
\[
\int \int_D F(x, y) \, dx \, dy = \int_a^b \left( \int_{f(x)}^{g(x)} F(x, y) \, dy \right) \, dx.
\] (15.26)

In this case one cannot immediately reverse the order of integration, unless the region can be alternatively written as
\[
D = \{ (x, y) \mid c < y < d, \ h(y) < x < k(y) \},
\]
in which case
\[
\int \int_D F(x, y) \, dx \, dy = \int_c^d \left( \int_{h(y)}^{k(y)} F(x, y) \, dx \right) \, dy.
\] (15.27)

In particular, one can compute the area of \( D \) using (15.6):
\[
\text{area } D = \int \int_D \, dx \, dy = \int_a^b [g(x) - f(x)] \, dx,
\]
thus reproducing a standard univariate calculus formula (13.24).

With the foundations in place, further development of vector calculus, including extensions to three-dimensional space, change of variables, line and surface integrals, Green’s Theorem, Stokes Theorem, Gauss’ Theorem, proceeds in essentially the same manner as in traditional texts, e.g., [1, 5, 13]. Also see my lecture notes [15, 16] on Vector Calculus in Two and Three Dimensions, available on my website. We thus view this as an appropriate point to close off our development of “Continuous Calculus”.


For the reader who is already familiar with calculus in its usual rigorous incarnation, a few remarks are in order.

First of all, after the initial idea crossed my mind, I was pleasantly surprised how easily all the parts ended up fitting together and, in many cases, simplified the proofs of many of the important theorems, especially those involving differentiation. Of course, one needs to spend a little time in advance discussing the topology of the real line and, later, the plane, but most of the results are not especially difficult, and are, of course, well worth learning in their own right. Initially I was only going to cover basic differential calculus, but as I gradually realized that all of the fundamental results, including Cauchy sequences and uniform continuity, could be handled by this framework, the more these notes expanded in scope and sophistication. Indeed, I could not resist including the simple proof of the Fundamental Theorem 13.16, and hence ended up including a somewhat innovative approach to Riemann integration, in which the topological approach to continuity also plays a role, albeit less prominent. I am also surprised that, as noted in the introduction, there appears to be only one precedent for taking this approach, [2], although perhaps there are others that I am simply unaware of.
Pedagogically, it would probably be better to initially skip over a number of the results collected together in the second and fifth sections, and get to the heart of the matter sooner, referring back to needed topological results as they arise. On the other hand, the distractions and complications of epsilons and deltas completely disappear without any sacrifice in rigor! Moreover, while the language of limits proves to still be useful, they no longer play a central role and, in fact, could be done away with entirely.

There are several possible objections to this approach. First, since one is imposing continuity from the beginning, what about discontinuous functions with limits and derivatives? As noted, piecewise continuous functions are easily handled in this framework. The derivative of a piecewise continuous function, e.g., the sign function at \( x = 0 \), does not exist in the usual sense. And, in conformity with all introductory calculus texts, I am not attempting to introduce generalized functions (distributions) like the Dirac delta function here, \([17]\). Other examples of discontinuous functions with limits and derivatives include the function

\[
f(x) = \begin{cases} 
  x^2, & x \in \mathbb{Q}, \\
  0, & x \in \mathbb{R} \setminus \mathbb{Q},
\end{cases}
\]

which has the limit 0 at \( x = 0 \) and also \( f'(0) = 0 \), but is not included in this approach, which does not apply to such isolated points of continuity. On the other hand, such pathological functions do not play a role in any of the usual applications of calculus, especially those encountered by students from other disciplines, and, in my opinion, are only of interest at this level to the mathematical diehard. So I am not especially concerned by omitting them, and do not think much is lost thereby. In my opinion, the simplicity of the framework overcomes the need to allow such pathologies into the picture. Keep in mind that I am not reverting to pre-nineteenth century mathematics, where the only functions considered were analytic functions, perhaps with isolated singularities. Indeed, the notes do not get to Taylor series and the definition of analyticity. Non-analytic but (piecewise) continuous functions of a non-pathological nature are eminently worthy of consideration. Furthermore, such pathologies can and should be deferred until an advanced course in real analysis that also replaces Riemann’s integration theory by Lebesgue’s more powerful approach.

The more accurate title of these notes, “Topological Calculus”, confirms that this is an essentially topological theory, and independent of any choice of metric\(^\dagger\). Thus, like the proverbial topologist who cannot distinguish between a doughnut and a coffee cup, topological calculus does not distinguish between calculus on doughnuts and calculus on a coffee cups! As such, I believe it is the right approach to calculus on manifolds, \([22]\), before one introduces any sort of metric — Riemannian, Finsler, etc. In this approach,

\(^\dagger\) Actually, this is not 100% correct, since we implicitly use the standard \( |x - y| \) distance on \( \mathbb{R} \) and the corresponding \( L^\infty \) norm on \( \mathbb{R}^2 \) at certain points in the development. Nevertheless, the underlying theme of this approach is topological.
differential forms and the deRham theory, [26], can now be formulated in a topological (continuous) and, I would argue, more natural framework.

In any event, I am very interested in your reactions to these notes. Please send me email with comments, concerns, criticisms, corrections, etc.

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References


