Vector Calculus in Two Dimensions

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1. Introduction.

The purpose of these notes is to review the basics of vector calculus in the two dimensions. We will assume you are familiar with the basics of partial derivatives, including the equality of mixed partials (assuming they are continuous), the chain rule, implicit differentiation. In addition, some familiarity with multiple integrals is assumed, although we will review the highlights. Proofs and full details can be found in most vector calculus texts, including [**1**, **4**].

We begin with a discussion of plane curves and domains. Many physical quantities, including force and velocity, are determined by vector fields, and we review the basic concepts. The key differential operators in planar vector calculus are the gradient and divergence operations, along with the Jacobian matrix for maps from $\mathbb{R}^2$ to itself. There are three basic types of line integrals: integrals with respect to arc length, for computing lengths of curves, masses of wires, center of mass, etc., ordinary line integrals of vector fields for computing work and fluid circulation, and flux line integrals for computing flux of fluids and forces. Next, we review the basics of double integrals of scalar functions over plane domains. Line and double integrals are connected by the justly famous Green's theorem, which

2. Plane Curves.

We begin our review by collecting together the basic facts concerning geometry of plane curves. A *curve* $C \subset \mathbb{R}^2$ is parametrized by a pair of continuous functions

$$\mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \in \mathbb{R}^2, \tag{2.1}$$

where the scalar *parameter* t varies over an (open or closed) interval $I \subset \mathbb{R}$. When it exists, the *tangent vector* to the curve at the point $\mathbf{x}$ is described by the derivative,

$$\frac{d\mathbf{x}}{dt} = \dot{\mathbf{x}} = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}. \tag{2.2}$$

We shall often use Newton's dot notation to abbreviate derivatives with respect to the parameter t.

Physically, we can think of a curve as the trajectory described by a particle moving in the plane. The parameter t is identified with the time, and so $\mathbf{x}(t)$ gives the position of the particle at time t. The tangent vector $\dot{\mathbf{x}}(t)$ measures the velocity of the particle at time t;

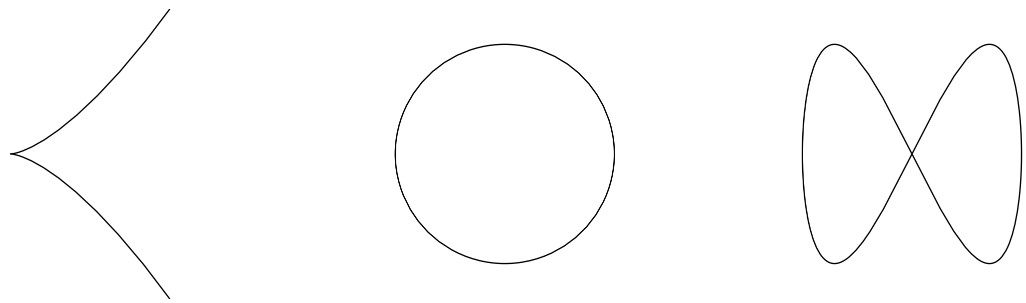


Figure 1. Planar Curves.

its magnitude† $\| \dot{\mathbf{x}} \| = \sqrt{\dot{x}^2 + \dot{y}^2}$ is the speed, while its orientation (assuming the velocity is nonzero) indicates the instantaneous direction of motion of the particle as it moves along the curve. Thus, by the *orientation* of a curve, we mean the direction of motion or parametrization, as indicated by the tangent vector. Reversing the orientation amounts to moving backwards along the curve, with the individual tangent vectors pointing in the opposite direction.

The curve parametrized by $\mathbf{x}(t)$ is called *smooth* provided its tangent vector is continuous and everywhere *nonzero*: $\dot{\mathbf{x}} \neq \mathbf{0}$. This is because curves with vanishing derivative may have corners or cusps; a simple example is the first curve plotted in Figure 1, which has parametrization

$$\mathbf{x}(t) = \begin{pmatrix} t^2 \\ t^3 \end{pmatrix}, \qquad \dot{\mathbf{x}}(t) = \begin{pmatrix} 2\,t \\ 3\,t^2 \end{pmatrix},$$

and has a cusp at the origin when $t = 0$ and $\dot{\mathbf{x}}(0) = 0$. Physically, a particle trajectory remains smooth as long as the speed of the particle is never zero, which effectively prevents the particle from instantaneously changing its direction of motion. A closed curve is *smooth* if, in addition to satisfying $\dot{\mathbf{x}}(t) \neq \mathbf{0}$ at all points $a \leq t \leq b$, the tangents at the endpoints match up: $\dot{\mathbf{x}}(a) = \dot{\mathbf{x}}(b)$. A curve is called *piecewise smooth* if its derivative is piecewise continuous and nonzero everywhere. The corners in a piecewise smooth curve have well-defined right and left tangents. For example, polygons, such as triangles and rectangles, are piecewise smooth curves. In this book, all curves are assumed to be at least piecewise smooth.

A curve is *simple* if it has no self-intersections: $\mathbf{x}(t) \neq \mathbf{x}(s)$ whenever $t \neq s$. Physically, this means that the particle is never in the same position twice. A curve is *closed* if $\mathbf{x}(t)$ is defined for $a \leq t \leq b$ and its endpoints coincide: $\mathbf{x}(a) = \mathbf{x}(b)$, so that the particle ends up where it began. For example, the unit circle

$$\mathbf{x}(t) = (\cos t, \sin t)^T \qquad \text{for} \qquad 0 \leq t \leq 2\pi,$$

† Throughout, we always use the standard Euclidean inner product and norm. With some care, all of the concepts can be adapted to other choices of inner product. In differential geometry and relativity, one even allows the inner product and norm to vary from point to point, [**2**].

is closed and simple[†], while the curve

$$\mathbf{x}(t) = (\cos t, \sin 2t)^T \qquad \text{for} \qquad 0 \le t \le 2\pi,$$

is not simple since it describes a figure eight that intersects itself at the origin. Both curves are illustrated in Figure 1.

Assuming the tangent vector $\dot{\mathbf{x}}(t) \neq \mathbf{0}$, then the *normal* vector to the curve at the point $\mathbf{x}(t)$ is the orthogonal or perpendicular vector

$$\dot{\mathbf{x}}^{\perp} = \begin{pmatrix} \dot{y} \\ -\dot{x} \end{pmatrix} \tag{2.3}$$

of the same length $\| \dot{\mathbf{x}}^{\perp} \| = \| \dot{\mathbf{x}} \|$. Actually, there are two such normal vectors, the other being the negative $-\dot{\mathbf{x}}^{\perp}$. We will always make the "right-handed" choice (2.3) of normal, meaning that as we traverse the curve, the normal always points to our right. If a simple closed curve C is oriented so that it is traversed in a counterclockwise direction — the standard mathematical orientation — then (2.3) describes the outwards-pointing normal. If we reverse the orientation of the curve, then both the tangent vector and normal vector change directions; thus (2.3) would give the inwards-pointing normal for a simple closed curve traversed in the clockwise direction.

The same curve C can be parametrized in many different ways. In physical terms, a particle can move along a prescribed trajectory at a variety of different speeds, and these correspond to different ways of parametrizing the curve. Conversion from one parametrization $\mathbf{x}(t)$ to another $\widetilde{\mathbf{x}}(\tau)$ is effected by a *change of parameter*, which is a smooth, invertible function $t = g(\tau)$; the reparametrized curve is then $\widetilde{\mathbf{x}}(\tau) = \mathbf{x}(g(\tau))$. We require that $dt/d\tau = g'(\tau) > 0$ everywhere. This ensures that each t corresponds to a unique value of τ, and, moreover, the curve remains smooth and is traversed in the same overall direction under the reparametrization. On the other hand, if $g'(\tau) < 0$ everywhere, then the orientation of the curve is reversed under the reparametrization. We shall use the notation $-C$ to indicate the curve having the same shape as C, but with the reversed orientation.

Example 2.1. The function $\mathbf{x}(t) = (\cos t, \sin t)^T$ for $0 < t < \pi$ parametrizes a semi-circle of radius 1 centered at the origin. If we set[†] $\tau = -\cot t$ then we obtain the less evident parametrization

$$\widetilde{\mathbf{x}}(\tau) = \left(\frac{1}{\sqrt{1+\tau^2}}\,, \; -\frac{\tau}{\sqrt{1+\tau^2}} \right)^T \qquad \text{for} \qquad -\infty < \tau < \infty$$

of the *same* semi-circle, in the *same* direction. In the familiar parametrization, the velocity vector has unit length, $\| \dot{\mathbf{x}} \| \equiv 1$, and so the particle moves around the semicircle in the counterclockwise direction with unit speed. In the second parametrization, the particle

† For a closed curve to be simple, we require $\mathbf{x}(t) \neq \mathbf{x}(s)$ whenever $t \neq s$ *except* at the ends, where $\mathbf{x}(a) = \mathbf{x}(b)$ is required for the ends to close up.

† The minus sign is to ensure that $d\tau/dt > 0$.

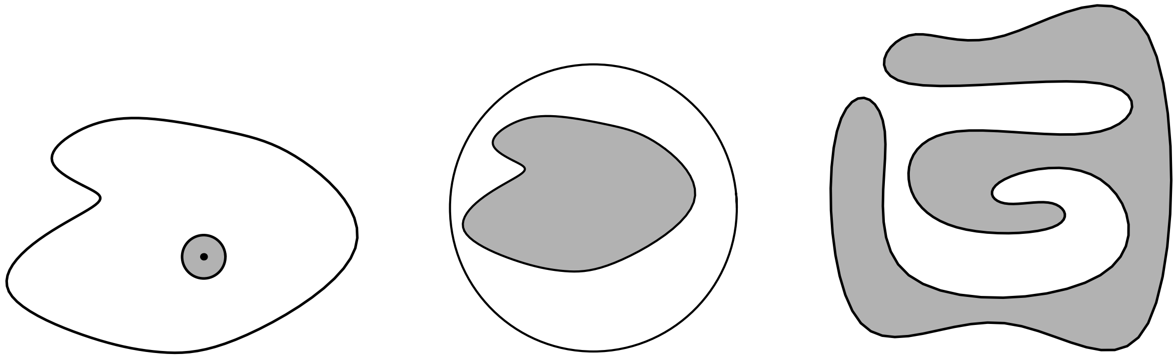


Figure 2. Topology of Planar Domains.

slows down near the endpoints, and, in fact, takes an infinite amount of time to traverse the semicircle from right to left.

3. Planar Domains.

A plate or other two-dimensional body occupies a region in the plane, known as a *domain*. The simplest example is an open circular disk

$$D_r(\mathbf{a}) = \{\, \mathbf{x} \in \mathbb{R}^2 \;\big|\; \|\, \mathbf{x} - \mathbf{a} \,\| < r \,\} \tag{3.1}$$

of radius r centered at a point $\mathbf{a} \in \mathbb{R}^2$. In order to properly formulate the mathematical tools needed to understand boundary value problems and dynamical equations for such bodies, we first need to review basic terminology from point set topology of planar sets. Many of the concepts carry over as stated to subsets of any higher dimensional Euclidean space $\mathbb{R}^n$.

Let $\Omega \subset \mathbb{R}^2$ be any subset. A point $\mathbf{a} \in \Omega$ is called an *interior point* if some small disk centered at $\mathbf{a}$ is entirely contained within the set: $D_\varepsilon(\mathbf{a}) \subset \Omega$ for some $\varepsilon > 0$; see Figure 2. The set Ω is *open* if every point is an interior point. A set K is *closed* if and only if its complement $\Omega = \mathbb{R}^2 \setminus K = \{\mathbf{x} \notin K\}$ is open.

Example 3.1. If $f(x,y)$ is any continuous real-valued function, then the subset $\{\, f(x,y) > 0 \,\}$ where f is strictly positive is open, while the subset $\{\, f(x,y) \geq 0 \,\}$ where f is non-negative is closed. One can, of course, replace 0 by any other constant, and also reverse the direction of the inequalities, without affecting the conclusions.

In particular, the set

$$D_r = \{\, x^2 + y^2 < r^2 \,\} \tag{3.2}$$

consisting of all points of (Euclidean) norm strictly less than r, defines an *open disk* of radius r centered at the origin. On the other hand,

$$K_r = \{\, x^2 + y^2 \leq r^2 \,\} \tag{3.3}$$

is the *closed disk* of radius r, which includes the bounding circle

$$C_r = \{\, x^2 + y^2 = r^2 \,\}. \tag{3.4}$$

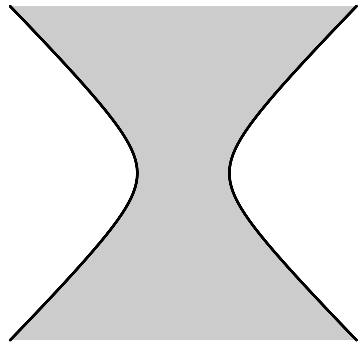


Figure 3. Open Sets Defined by a Hyperbola.

A point $\mathbf{x}^\star$ is a *limit point* of a set Ω if there exists a sequence of points $\mathbf{x}^{(n)} \in \Omega$ converging to it, so that $\mathbf{x}^{(n)} \to \mathbf{x}^\star$ as $n \to \infty$. Every point $\mathbf{x} \in \Omega$ is a limit point (just take all $\mathbf{x}^{(n)} = \mathbf{x}$) but the converse is not necessarily valid. For example, the points on the circle (3.4) are all limit points for the open disk (3.2). The *closure* of a set Ω, written $\overline{\Omega}$, is defined as the set of all limit points of Ω. In particular, a set K is closed if and only if it contains all its limit points, and so $K = \overline{K}$. The *boundary* $\partial\Omega$ of a subset Ω consists of all limit points which are not interior points. If Ω is open, then its closure is the disjoint union of the set and its boundary $\overline{\Omega} = \Omega \cup \partial\Omega$. Thus, the closure of the open disk D_r is the closed disk $\overline{D}_r = D_r \cup C_r$; the circle $C_r = \partial D_r = \partial \overline{D}_r$ forms their common boundary.

An open subset that can be written as the union, $\Omega = \Omega_1 \cup \Omega_2$, of two disjoint, nonempty, open subsets, so $\Omega_1 \cap \Omega_2 = \varnothing$, is called *disconnected*. For example, the open set

$$\Omega = \{ x^2 - y^2 > 1 \} \tag{3.5}$$

is disconnected, consisting of two disjoint "sectors" bounded by the two branches of the hyperbola $x^2 - y^2 = 1$; see Figure 3. On the other hand, the complementary open set

$$\widehat{\Omega} = \{ x^2 - y^2 < 1 \} \tag{3.6}$$

is *connected*, and consists of all points between the two hyperbolas.

A subset is called *bounded* if it is contained inside a (possibly large) disk, i.e., $\Omega \subset D_r$ for some $r > 0$, as in the second picture in Figure 2. Thus, both the closed and the open disks (3.2), (3.3) are bounded, whereas the two hyperbolic sectors (3.5), (3.6) are both unbounded.

The class of subsets for which the boundary value problems for the partial differential equations of equilibrium mechanics are properly prescribed can now be defined.

Definition 3.2. A *planar domain* is a connected, open subset $\Omega \subset \mathbb{R}^2$ whose boundary $\partial\Omega$ consists of one or more piecewise smooth, simple curves, such that Ω lies entirely on one side of each of its boundary curve(s).

The last condition is to avoid dealing with pathologies. For example, the subset $\Omega \setminus C$ obtained by cutting out a curve C from the interior of an open set Ω would not be an allowable domain.

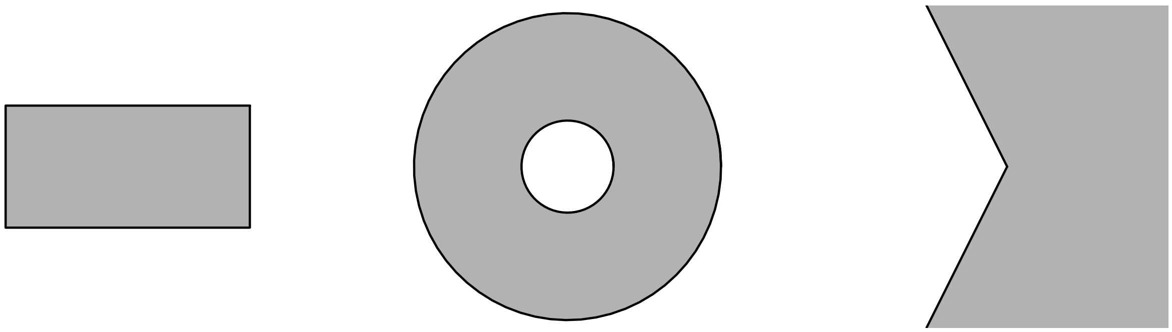


Figure 4. Planar Domains.

Example 3.3. The open rectangle $R = \{ a < x < b, c < y < d \}$ is an open, connected and bounded domain. Its boundary is a piecewise smooth curve, since there are corners where the tangent does not change continuously.

The *annulus*

$$r^2 < x^2 + y^2 < R^2, \qquad \text{for fixed} \qquad 0 < r < R, \tag{3.7}$$

is an open, connected, bounded domain whose boundary consists of two disjoint concentric circles. The degenerate case of a *punctured disk*, when $r = 0$, is *not* a domain since its boundary consists of a circle and a single point — the origin.

Another well-studied example is the wedge-shaped domain $W = \{ \alpha < \theta < \beta \}$ consisting of all points whose angular coordinate $\theta = \tan^{-1} y/x$ lies between two prescribed values. If $0 < \beta - \alpha < 2\pi$, then the wedge is a domain whose boundary consists of two connected rays. However, if $\beta = \alpha + 2\pi$, then the wedge is obtained by cutting the plane along a single ray at angle α. The latter case does not comply with our definition of a domain since the wedge now lies on both sides of its boundary ray.

Any connected domain is automatically *pathwise connected* meaning that any two points can be connected by (i.e., are the endpoints of) a curve lying entirely within the domain. If the domain is bounded, which is the most important case for boundary value problems, then its boundary consists of one or more piecewise smooth, simple, closed curves. A bounded domain Ω is called *simply connected* if it has just one such boundary curve; this means that Ω is connected and has no holes, and so its boundary $\partial\Omega = C$ is a simple closed curve that contains Ω in its interior. For instance, an open disk and a rectangle are both simply connected, whereas an annulus is not.

The Jordan Curve Theorem states the intuitively obvious, but actually quite deep, result that any simple closed curve divides the plane $\mathbb{R}^2$ into two disjoint, connected, open domains — its *interior*, which is bounded and simply connected, and its *exterior*, which is unbounded and not simply connected. This result is illustrated in the final figure in Figure 2; the interior of the indicated simple closed curve is shaded in gray while the exterior is in white. Note that the each subdomain lies entirely on one side of the curve, which forms their common boundary.

The following result is often used to characterize the simple connectivity of more general planar subsets, including unbounded domains.

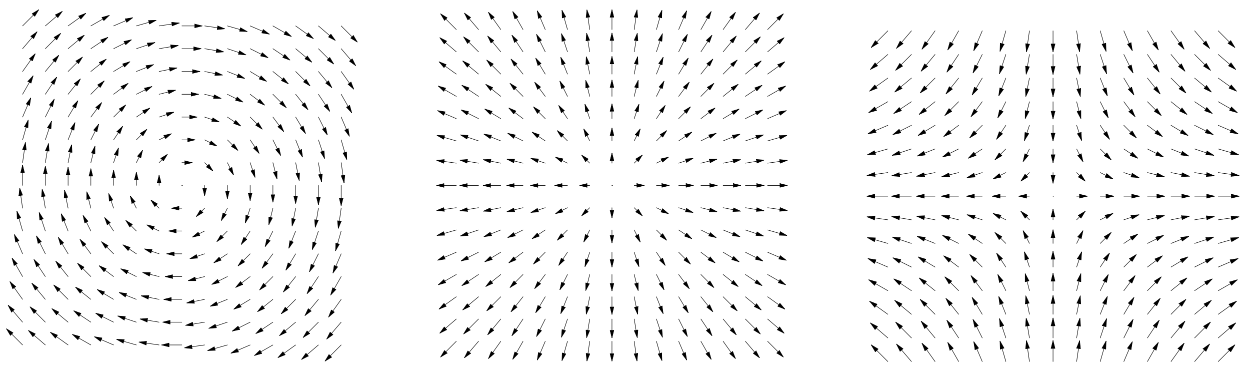


Figure 5. Vector Fields.

Lemma 3.4. *A planar domain* $\Omega \subset \mathbb{R}^2$ *is* simply connected *if it is connected and, moreover, if the interior of any simple closed curve* $C \subset \Omega$ *is also contained in* Ω*.*

For example, an annulus (3.7) is not simply connected because the interior of a circle going around the hole is not entirely contained within the annulus. On the other hand, the unbounded domain (3.6) lying between two branches of a hyperbola is simply connected, even though its boundary consists of two disjoint, unbounded curves.

4. Vector Fields.

A vector-valued function $\mathbf{v}(x,y) = \begin{pmatrix} v_1(x,y) \\ v_2(x,y) \end{pmatrix}$ is known as a (planar) *vector field*. A vector field assigns a vector $\mathbf{v}(x,y)$ to each point $(x,y)^T$ in its domain of definition, and hence defines a (in general nonlinear) function $\mathbf{v}\colon \Omega \to \mathbb{R}^2$. The vector field can be conveniently depicted by drawing an arrow representing the vector $\mathbf{v} = \mathbf{v}(x,y)$ starting at its point of definition $(x,y)^T$. See Figure 5 for some representative sketches.

Example 4.1. Vector fields arise very naturally in physics and engineering applications from physical forces: gravitational, electrostatic, centrifugal, etc. A *force field* $\mathbf{f}(x,y) = (f_1(x,y), f_2(x,y))^T$ describes the direction and magnitude of the force experienced by a particle at position (x,y). In a planar universe, the gravitational force field exerted by a point mass concentrated at the origin has, according to Newtonian gravitational theory, magnitude proportional to† $1/r$, where $r = \|\mathbf{x}\|$ is the distance to the origin, and is directed towards the origin. Thus, the vector field describing gravitational force has the form

$$\mathbf{f} = -\gamma \frac{\mathbf{x}}{\|\mathbf{x}\|} = \left(\frac{-\gamma x}{\sqrt{x^2+y^2}}, \frac{-\gamma y}{\sqrt{x^2+y^2}} \right)^T, \tag{4.1}$$

where $\gamma > 0$ denotes the constant of proportionality, namely the product of the two masses times the universal gravitational constant. The same force law applies to the attraction, $\gamma > 0$, and repulsion, $\gamma < 0$, of electrically charged particles.

† In three-dimensional Newtonian gravity, $1/r$ is replaced by $1/r^2$.

Newton's Laws of planetary motion produce the second order system of differential equations

$$m \, \frac{d^2 \mathbf{x}}{dt^2} = \mathbf{f}.$$

The solutions $\mathbf{x}(t)$ describe the trajectories of planets subject to a central gravitational force, e.g., the sun. They also govern the motion of electrically charged particles under a central electric charge, e.g., classical (i.e., not quantum) electrons revolving around a central nucleus. In three-dimensional Newtonian mechanics, planets move along conic sections — ellipses in the case of planets, and parabolas and hyperbolas in the case of non-recurrent objects like some comets. Interestingly (and not as well-known), the corresponding two-dimensional theory is not as neatly described — the typical orbit of a planet around a planar sun does not form a simple closed curve, [**3**]!

Example 4.2. Another important example is the velocity vector field $\mathbf{v}$ of a steady-state fluid flow. The vector $\mathbf{v}(x,y)$ measures the instantaneous velocity of the fluid particles (molecules or atoms) as they pass through the point (x,y). "Steady-state" means that the velocity at a point (x,y) does not vary in time — even though the individual fluid particles are in motion. If a fluid particle moves along the curve $\mathbf{x}(t) = (x(t), y(t))^T$, then its velocity at time t is the derivative $\mathbf{v} = \dot{\mathbf{x}}$ of its position with respect to t. Thus, for a time-independent velocity vector field $\mathbf{v}(x,y) = (\, v_1(x,y), v_2(x,y)\,)^T$, the fluid particles will move in accordance with an autonomous, first order system of ordinary differential equations

$$\frac{dx}{dt} = v_1(x,y), \qquad\qquad \frac{dy}{dt} = v_2(x,y). \tag{4.2}$$

According to the basic theory of systems of ordinary differential equations , an individual particle's motion $\mathbf{x}(t)$ will be uniquely determined solely by its initial position $\mathbf{x}(0) = \mathbf{x}_0$. In fluid mechanics, the trajectories of particles are known as the *streamlines* of the flow. The velocity vector $\mathbf{v}$ is everywhere tangent to the streamlines. When the flow is steady, the streamlines do not change in time. Individual fluid particles experience the same motion as they successively pass through a given point in the domain occupied by the fluid.

As a specific example, consider the vector field

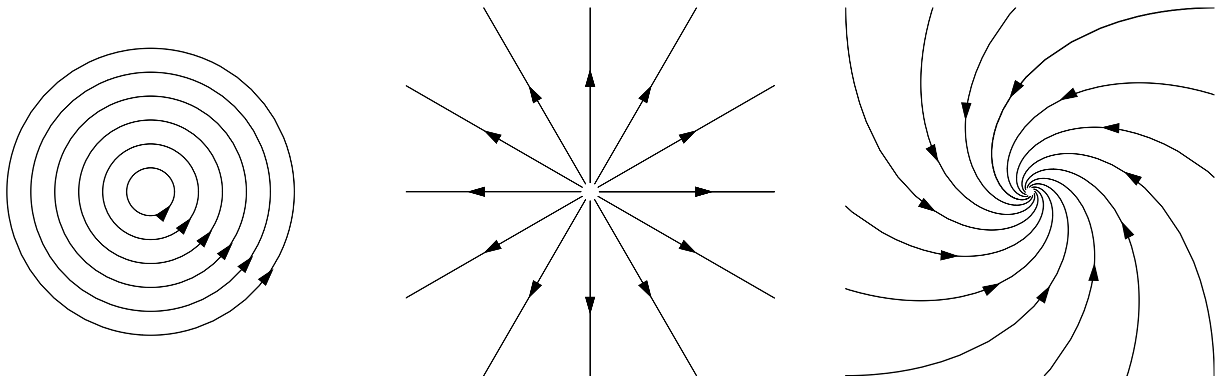
$$\mathbf{v}(x,y) = \begin{pmatrix} -\,\omega\, y \\ \omega\, x \end{pmatrix}, \tag{4.3}$$

for fixed $\omega > 0$, which is plotted in the first figure in Figure 5. The corresponding fluid trajectories are found by solving the associated first order system of ordinary differential equations

$$\dot{x} = -\,\omega\, y, \qquad\qquad \dot{y} = \omega\, x,$$

with initial conditions $x(0) = x_0$, $y(0) = y_0$. This is a linear system, and can be solved by the eigenvalue and eigenvector techniques presented in [**6**; Chapter 9]. The resulting flow

$$x(t) = x_0 \cos \omega\, t - y_0 \sin \omega\, t, \qquad\qquad y(t) = x_0 \sin \omega\, t + y_0 \cos \omega\, t,$$



Rotational Flow Source Spiral Sink

Figure 6. Steady State Fluid Flows.

corresponds to a fluid that is uniformly rotating around the origin. The streamlines are concentric circles, and the fluid particles rotate around the circles in a counterclockwise direction with angular velocity ω, as illustrated in Figure 6. Note that the fluid velocity $\mathbf{v}$ is everywhere tangent to the circles. The origin is a stagnation point, since the velocity field $\mathbf{v} = \mathbf{0}$ vanishes there, and the particle at the origin does not move.

As another example, the radial vector field

$$\mathbf{v}(x,y) = \alpha\,\mathbf{x} = \begin{pmatrix} \alpha\, x \\ \alpha\, y \end{pmatrix} \tag{4.4}$$

corresponds to a fluid source, $\alpha > 0$, or sink, $\alpha < 0$, at the origin, and is plotted in the second figure in Figure 5. The solution to the first order system of ordinary differential equations $\dot{\mathbf{x}} = \alpha\,\mathbf{x}$ with initial conditions $\mathbf{x}(0) = \mathbf{x}_0$ gives the radial flow $\mathbf{x}(t) = e^{\alpha\, t}\,\mathbf{x}_0$. The streamlines are the rays emanating from the origin, and the motion is outwards (source) or inwards (sink) depending on the sign of α. As in the rotational flow, the origin is a stagnation point.

Combining the radial and circular flow vector fields,

$$\mathbf{v}(x,y) = \begin{pmatrix} \alpha\, x - \omega\, y \\ \omega\, x + \alpha\, y \end{pmatrix} \tag{4.5}$$

leads to a swirling source or sink — think of the water draining out of your bathtub. Again, the flow is found by integrating a linear system of ordinary differential equations

$$\dot{x} = \alpha\, x - \omega\, y, \qquad \dot{y} = \omega\, x + \alpha\, y.$$

Solving as in [**6**; Chapter 9], we find that the fluid particles follow the spiral streamlines

$$x(t) = e^{\alpha\, t}\big(\, x_0 \cos \omega\, t - y_0 \sin \omega\, t\,\big), \qquad y(t) = e^{\alpha\, t}\big(\, x_0 \sin \omega\, t + y_0 \cos \omega\, t\,\big),$$

again illustrated in Figure 6.

Remark: Of course, physical fluid motion occurs in three-dimensional space. However, any planar flow can also be viewed as a particular type of three-dimensional fluid motion that does not depend upon the vertical coordinate. The motion on every horizontal plane is the same, and so the planar flow represents a cross-section of the full three-dimensional motion. For example, slicing a steady flow past a vertical cylinder by a transverse horizontal plane results in a planar flow around a circle.

5. Gradient and Curl.

In the same vein, a scalar-valued function $u(x, y)$ is often referred to as a *scalar field*, since it assigns a scalar to each point $(x, y)^T$ in its domain of definition. Typical physical examples of scalar fields include temperature, deflection of a membrane, height of a topographic map, density of a plate, and so on.

The *gradient* operator ∇ maps a scalar field $u(x, y)$ to the vector field

$$\nabla u = \operatorname{grad} u = \begin{pmatrix} \partial u/\partial x \\ \partial u/\partial y \end{pmatrix} \tag{5.1}$$

consisting of its two first order partial derivatives. The scalar field u is often referred to as a *potential function* for its gradient vector field ∇u. For example, the gradient of the potential function $u(x, y) = x^2 + y^2$ is the radial vector field $\nabla u = (2x, 2y)^T$. Similarly, the gradient of the logarithmic potential function

$$u(x, y) = -\gamma \log r = -\tfrac{1}{2}\gamma \log(x^2 + y^2)$$

is the gravitational force (4.1) exerted by a point mass concentrated at the origin. Additional physical examples include the velocity potential of certain fluid velocity vector fields and the electromagnetic potential whose gradient describes the electromagnetic force field.

Not every vector field admits a potential because not every vector field lies in the range of the gradient operator ∇. Indeed, if $u(x, y)$ has continuous second order partial derivatives, and

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \mathbf{v} = \nabla u = \begin{pmatrix} u_x \\ u_y \end{pmatrix},$$

then, by the equality of mixed partials,

$$\frac{\partial v_1}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial v_2}{\partial x}\,.$$

The resulting equation

$$\frac{\partial v_1}{\partial y} = \frac{\partial v_2}{\partial x} \tag{5.2}$$

constitutes one of the necessary conditions that a vector field must satisfy in order to be a gradient. Thus, for example, the rotational vector field (4.3) does not satisfy (5.2), and hence is *not* a gradient. There is *no* potential function for such circulating flows.

The difference between the two terms in (5.2) is known as the *curl* of the planar vector field $\mathbf{v} = (v_1, v_2)$, and denoted by[†]

$$\nabla \wedge \mathbf{v} = \operatorname{curl} \mathbf{v} = \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} \,. \tag{5.3}$$

Notice that the curl of a planar vector field is a scalar field. (In contrast, in three dimensions, the curl of a vector field is a vector field, [**1**, **4**].) Thus, a necessary condition for a vector field to be a gradient is that its curl vanish identically: $\nabla \wedge \mathbf{v} \equiv 0$.

Even if the vector field has zero curl, it still may not be a gradient. Interestingly, the general criterion depends only upon the topology of the domain of definition, as clarified in the following theorem.

Theorem 5.1. *Let* $\mathbf{v}$ *be a smooth vector field defined on a domain* $\Omega \subset \mathbb{R}^2$. *If* $\mathbf{v} = \nabla u$ *for some scalar function* u, *then* $\nabla \wedge \mathbf{v} \equiv 0$. *If* Ω *is simply connected, then the converse holds: if* $\nabla \wedge \mathbf{v} \equiv 0$ *then* $\mathbf{v} = \nabla u$ *for some potential function* u *defined on* Ω.

As we shall see, this result is a direct consequence of Green's Theorem 8.1.

Example 5.2. The vector field

$$\mathbf{v} = \left(\frac{-y}{x^2 + y^2} \,,\; \frac{x}{x^2 + y^2} \right)^T \tag{5.4}$$

satisfies $\nabla \wedge \mathbf{v} \equiv 0$. However, there is no potential function defined for all $(x, y) \neq (0, 0)$ such that $\nabla u = \mathbf{v}$. As the reader can check, the angular coordinate

$$u = \theta = \tan^{-1} \frac{y}{x} \tag{5.5}$$

satisfies $\nabla \theta = \mathbf{v}$, but θ is not well-defined on the entire domain since it experiences a jump discontinuity of magnitude 2π as we go around the origin. Indeed, $\Omega = \{ \mathbf{x} \neq \mathbf{0} \}$ is *not* simply connected, and so Theorem 5.1 does not apply. On the other hand, if we restrict $\mathbf{v}$ to any simply connected subdomain $\widehat{\Omega} \subset \Omega$ that does not encircle the origin, then the angular coordinate (5.5) can be unambiguously and smoothly defined on $\widehat{\Omega}$, and does serve as a single-valued potential function for $\mathbf{v}$.

In fluid mechanics, the curl of a vector field measures the local circulation in the associated steady state fluid flow. If we place a small paddle wheel in the fluid, then its rate of spinning will be in proportion to $\nabla \wedge \mathbf{v}$. (An explanation of this fact will appear below.) The fluid flow is called *irrotational* if its velocity vector field has zero curl, and hence, assuming Ω is simply connected, is a gradient: $\mathbf{v} = \nabla u$. In this case, the paddle wheel will not spin. The scalar function $u(x, y)$ is known as the *velocity potential* for the fluid motion. Similarly, a force field that is given by a gradient $\mathbf{f} = \nabla \varphi$ is called a *conservative force field*, and the function φ defines the force potential.

[†] In this text, we adopt the more modern wedge notation $\wedge$ for what is often denoted by a cross $\times$.

Suppose $u(\mathbf{x}) = u(x,y)$ is a scalar field. Given a parametrized curve $\mathbf{x}(t) = (\, x(t), y(t)\,)^T$, the composition $f(t) = u(\mathbf{x}(t)) = u(x(t), y(t))$ indicates the behavior as we move along the curve. For example, if $u(x,y)$ represents the elevation of a mountain range at position (x,y), and $\mathbf{x}(t)$ represents our position at time t, then $f(t) = u(\mathbf{x}(t))$ is our altitude at time t. Similarly, if $u(x,y)$ represents the temperature at (x,y), then $f(t) = u(\mathbf{x}(t))$ measures our temperature at time t.

The rate of change of the composite function is found through the chain rule

$$\frac{df}{dt} = \frac{d}{dt}\, u(x(t), y(t)) = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} = \nabla u \cdot \dot{\mathbf{x}}, \tag{5.6}$$

and hence equals the dot product between the gradient $\nabla u(\mathbf{x}(t))$ and the tangent vector $\dot{\mathbf{x}}(t)$ to the curve at the point $\mathbf{x}(t)$. For instance, our rate of ascent or descent as we travel through the mountains is given by the dot product of our velocity vector with the gradient of the elevation function. The dot product between the gradient and a fixed vector $\mathbf{a} = (\, a, b\,)^T$ is known as the *directional derivative* of the scalar field $u(x,y)$ in the direction $\mathbf{a}$, and denoted by

$$\frac{\partial u}{\partial \mathbf{a}} = \mathbf{a} \cdot \nabla u = a\, u_x + b\, u_y. \tag{5.7}$$

Thus, the rate of change of u along a curve $\mathbf{x}(t)$ is given by its directional derivative $\partial u/\partial \dot{\mathbf{x}} = \nabla u \cdot \dot{\mathbf{x}}$, as in (5.6), in the tangent direction. This leads us to one important interpretation of the gradient vector.

Proposition 5.3. *The gradient ∇u of a scalar field points in the direction of steepest increase of u. The negative gradient, $-\nabla u$, which points in the opposite direction, indicates the direction of steepest decrease of u.*

For example, if $u(x,y)$ represents the elevation of a mountain range at position (x,y) on a map, then ∇u tells us the direction that is steepest uphill, while $-\nabla u$ points directly downhill — the direction water will flow. Similarly, if $u(x,y)$ represents the temperature of a two-dimensional body, then ∇u tells us the direction in which it gets hottest the fastest. Heat energy (like water) will flow in the opposite direction, namely in the direction of the vector $-\nabla u$. This basic fact underlies the derivation of the multi-dimensional heat and diffusion equations.

You need to be careful in how you interpret Proposition 5.3. Clearly, the faster you move along a curve, the faster the function $u(x,y)$ will vary, and one needs to take this into account when comparing the rates of change along different curves. The easiest way to normalize is to assume that the tangent vector $\mathbf{a} = \dot{\mathbf{x}}$ has norm 1, so $\|\, \mathbf{a}\,\| = 1$ and we are going through $\mathbf{x}$ with unit speed. Once this is done, Proposition 5.3 is an immediate consequence of the Cauchy–Schwarz inequality. Indeed,

$$\left| \frac{\partial u}{\partial \mathbf{a}} \right| = |\, \mathbf{a} \cdot \nabla u\,| \leq \|\, \mathbf{a}\,\|\; \|\, \nabla u\,\| = \|\, \nabla u\,\|, \qquad \text{when} \qquad \|\, \mathbf{a}\,\| = 1,$$

with equality if and only if $\mathbf{a} = c\, \nabla u$ points in the same direction as the gradient. Therefore, the maximum rate of change is when $\mathbf{a} = \nabla u / \|\, \nabla u\,\|$ is the unit vector in the direction of the gradient, while the minimum is achieved when $\mathbf{a} = -\nabla u/\|\, \nabla u\,\|$ points in the opposite direction. As a result, Proposition 5.3 tells us how to move if we wish to minimize a scalar function as rapidly as possible.

Theorem 5.4. *A curve* $\mathbf{x}(t)$ *will realize the steepest decrease in the scalar field* $u(\mathbf{x})$ *if and only if it satisfies the gradient flow equation*

$$\dot{\mathbf{x}} = -\nabla u, \qquad \text{or} \qquad \begin{aligned} \frac{dx}{dt} &= -\frac{\partial u}{\partial x}(x,y), \\ \frac{dy}{dt} &= -\frac{\partial u}{\partial y}(x,y). \end{aligned} \tag{5.8}$$

The only points at which the gradient does not tell us about the directions of increase/decrease are the *critical points*, which are, by definition, points where the gradient vanishes: $\nabla u = \mathbf{0}$. These include local maxima or minima of the function, i.e., mountain peaks or bottoms of valleys, as well as other types of critical points like saddle points that represent mountain passes. In such cases, we must look at the second or higher order derivatives to tell the directions of increase/decrease.

Remark: Theorem 5.4 forms the basis of gradient descent methods for numerically approximating the maxima and minima of functions. One begins with a guess (x_0, y_0) for the minimum and then follows the gradient flow in to the minimum by numerically integrating the system of ordinary differential equations (5.8).

Example 5.5. Consider the function $u(x,y) = x^2 + 2y^2$. Its gradient vector field is $\nabla u = (\,2x, 4y\,)^T$, and hence the gradient flow equations (5.8) take the form

$$\dot{x} = -2x, \qquad \dot{y} = -4y.$$

The solution that starts out at initial position $(\,x_0, y_0\,)^T$ is

$$\mathbf{x}(t) = x_0\, e^{-2t}, \qquad y(t) = y_0\, e^{-4t}. \tag{5.9}$$

Note that the origin is a stable fixed point for this linear dynamical system, and the solutions $\mathbf{x}(t) \to \mathbf{0}$ converge exponentially fast to the minimum of the function $u(x,y)$. If we start out not on either of the coordinate axes, so $x_0 \neq 0$ and $y_0 \neq 0$, then the trajectory (5.9) is a semi-parabola of the form $y = c x^2$, where $c = y_0/x_0^2$. These curves, along with the four coordinate semi-axes, are the paths to follow to get to the minimum $\mathbf{0}$ the fastest.

Level Sets

Let $u(x,y)$ be a scalar field. The curves defined by the implicit equation

$$u(x,y) = c \tag{5.10}$$

holding the function $u(x,y)$ constant are known as its *level sets*. For instance, if $u(x,y)$ represents the elevation of a mountain range, then its level sets are the usual contour lines on a topographic map. The Implicit Function Theorem tells us that, away from critical points, the level sets of a planar function are simple, though not necessarily closed, curves.

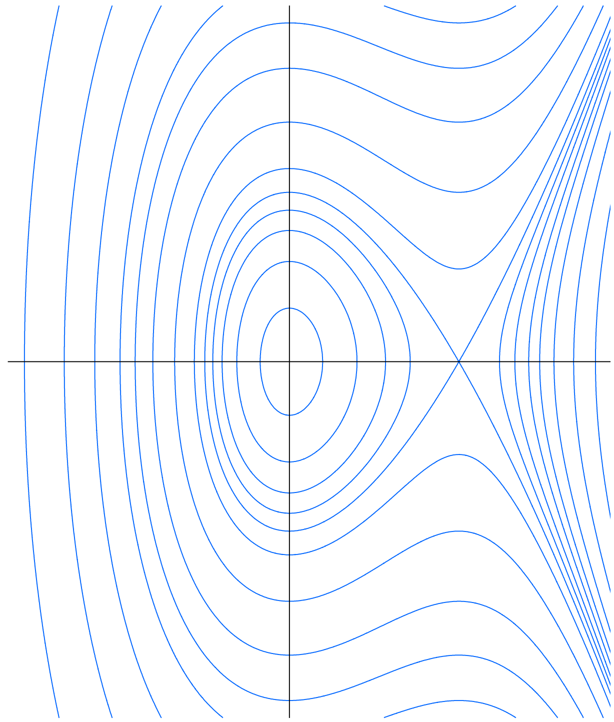


Figure 7. Level Sets of $u(x, y) = 3x^2 - 2x^3 + y^2$.

Theorem 5.6. *If the function* $u(x, y)$ *has continuous partial derivatives, and, at a point,* $\nabla u(x_0, y_0) \neq \mathbf{0}$*, then the level set passing through* $(x_0, y_0)^T$ *is, at least nearby, a smooth curve.*

Thus, if $\nabla u \neq \mathbf{0}$ at all points on the level set $S = \{u(x, y) = c\}$ for $c \in \mathbb{R}$, then each connected component of S is a smooth curve[†]. Critical points, where $\nabla u = \mathbf{0}$, are either isolated points, or points of intersection of level sets. For example the level sets of the function $u(x, y) = 3x^2 - 2x^3 + y^2$ are plotted in Figure 7. The function has critical points at $(0, 0)^T$ and $(1, 0)^T$. The former is a local minimum, and forms an isolated level point, while the latter is a saddle point, and is the point of intersection of the level curves $\{u = 1\}$.

If we parametrize an individual level set by $\mathbf{x}(t) = (x(t), y(t))^T$, then (5.10) tells us that the composite function $u(x(t), y(t)) = c$ is constant along the curve and hence its derivative

$$\frac{d}{dt}\, u(x(t), y(t)) = \nabla u \cdot \dot{\mathbf{x}} = 0$$

vanishes. We conclude that the tangent vector $\dot{\mathbf{x}}$ to the level set is *orthogonal* to the gradient direction ∇u at each point. In this manner, we have established the following additional important interpretation of the gradient, which is illustrated in Figure 8.

† This, of course, assumes that $S \neq \varnothing$.

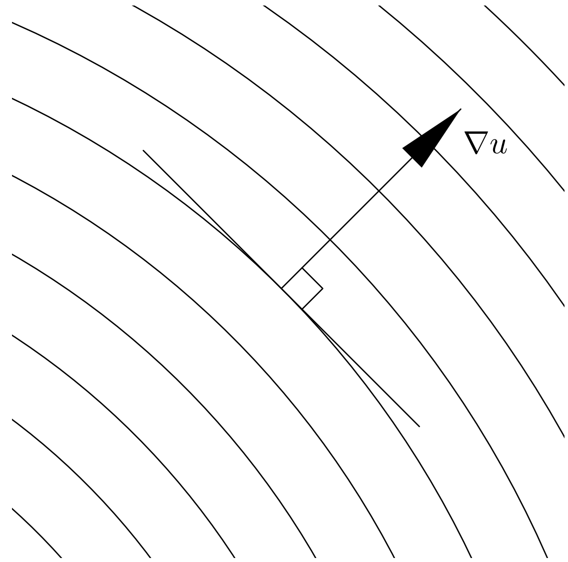


Figure 8. Level Sets and Gradient.

Theorem 5.7. *The gradient* ∇u *of a scalar field* u *is everywhere orthogonal to its level sets* $\{ u = c \}$.

Comparing Theorems 5.4 and 5.7, we conclude that the curves of steepest descent are always orthogonal (perpendicular) to the level sets of the function. Thus, if we want to hike uphill the fastest, we should keep our direction of travel always perpendicular to the contour lines. Similarly, if $u(x,y)$ represents temperature in a planar body at position (x,y) then the level sets are the curves of constant temperature, known as the *isotherms*. Heat energy will flow in the negative gradient direction, and hence orthogonally to the isotherms.

Example 5.8. Consider again the function $u(x,y) = x^2 + 2y^2$ from Example 5.5. Its level sets $u(x,y) = x^2 + 2y^2 = c$ form a system of concentric ellipses centered at the origin. Theorem 5.7 implies that the parabolic trajectories (5.9) followed by the solutions to the gradient flow equations form an orthogonal system of curves to the ellipses. This is evident in Figure 9, showing that the ellipses and parabolas intersect everywhere at right angles.

6. Integrals on Curves.

As you know, integrals of scalar functions, $\int_a^b f(t)\,dt$, are taken along real intervals $[a,b] \subset \mathbb{R}$. In higher dimensional calculus, there are a variety of possible types of integrals. The closest in spirit to one-dimensional integration are "line[†] integrals", in which one integrates along a curve. In planar calculus, line integrals come in three flavors. The most

[†] A more accurate term would be "curve integral", but the terminology is standard and will not be altered in this text.

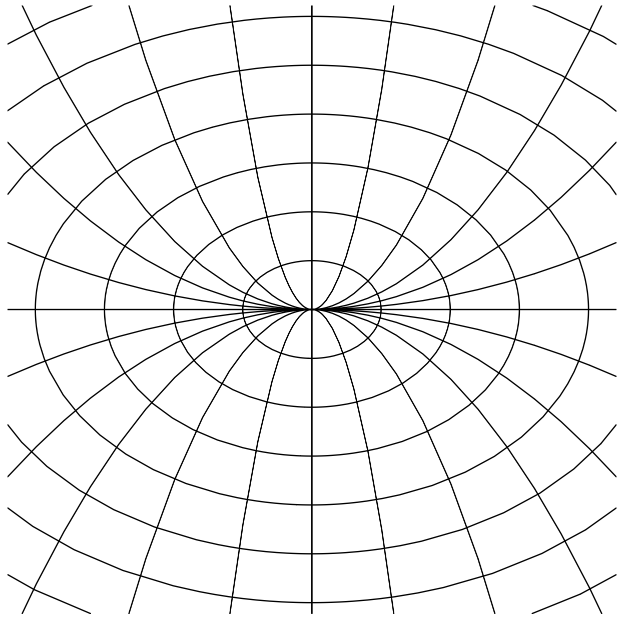


Figure 9. Orthogonal System of Ellipses and Parabolas.

basic are the integrals of scalar functions with respect to arc length. Such integrals are used to compute lengths of curves, and masses of one-dimensional objects like strings and wires. The second and third varieties are used to integrate a vector field along a curve. Integrating the tangential component of the vector field is used, for instance, to compute work and measure circulation along the curve. The last type integrates the normal component of the vector field along the curve, and represents flux (fluid, heat, electromagnetic, etc.) along the curve.

Arc Length

The *length* of the plane curve $\mathbf{x}(t)$ over the parameter range $a \le t \le b$ is computed by integrating the (Euclidean) norm[†] of its tangent vector:

$$\mathcal{L}(C) = \int_a^b \left\| \frac{d\mathbf{x}}{dt} \right\| dt = \int_a^b \sqrt{\dot{x}^2 + \dot{y}^2}\; dt. \tag{6.1}$$

The formula is justified by taking the limit of sums of lengths of small approximating line segments, [**1**, **4**]. For example, if the curve is given as the graph of a function $y = f(x)$ for $a \le x \le b$, then its length is computed by the familiar calculus formula

$$\mathcal{L}(C) = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\; dx. \tag{6.2}$$

It is important to verify that the length of a curve does not depend upon any particular parametrization (or even direction of traversal) of the curve.

[†] Alternative norms lead to alternative notions of curve length, of importance in the study of curved spaces in differential geometry. In Einstein's theory of relativity, one allows the norm to vary from point to point, and hence length will vary over space.

Example 6.1. The length of a circle $\mathbf{x}(t) = \begin{pmatrix} r\cos t \\ r\sin t \end{pmatrix}$, $0 \le t \le 2\pi$, of radius r is given by

$$\mathcal{L}(C) = \int_0^{2\pi} \left\| \frac{d\mathbf{x}}{dt} \right\| dt = \int_0^{2\pi} r\, dt = 2\pi r,$$

verifying the well-known formula for its circumference. On the other hand, the curve

$$\mathbf{x}(t) = \begin{pmatrix} a\cos t \\ b\sin t \end{pmatrix}, \qquad 0 \le t \le 2\pi, \tag{6.3}$$

parametrizes an ellipse with semi-axes a, b. Its arc length is given by the integral

$$s = \int_0^{2\pi} \sqrt{a^2\sin^2 t + b^2\cos^2 t}\; dt.$$

Unfortunately, this integral cannot be expressed in terms of elementary functions. It is, in fact, an *elliptic integral*, [**5**], so named for this very reason!

A curve is said to be parametrized by *arc length*, written $\mathbf{x}(s) = (\, x(s), y(s)\,)^T$, if one traverses it with constant, unit speed, which means that

$$\left\| \frac{d\mathbf{x}}{ds} \right\| = 1 \tag{6.4}$$

at all points. In other words, the length of that part of the curve between arc length parameter values $s = s_0$ and $s = s_1$ is exactly equal to $s_1 - s_0$. To convert from a more general parameter t to arc length $s = \sigma(t)$, we must compute

$$s = \sigma(t) = \int_a^t \left\| \frac{d\mathbf{x}}{dt} \right\| dt \qquad \text{and so} \qquad ds = \|\,\dot{\mathbf{x}}\,\|\, dt = \sqrt{\dot{x}^2 + \dot{y}^2}\; dt. \tag{6.5}$$

The *unit tangent* to the curve at each point is obtained by differentiating with respect to the arc length parameter:

$$\mathbf{t} = \frac{d\mathbf{x}}{ds} = \frac{\dot{\mathbf{x}}}{\|\,\dot{\mathbf{x}}\,\|} = \left(\frac{\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}}, \frac{\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}} \right), \qquad \text{so that} \qquad \|\,\mathbf{t}\,\| = 1. \tag{6.6}$$

(As always, we require $\dot{\mathbf{x}} \ne \mathbf{0}$.) The *unit normal* to the curve is orthogonal to the unit tangent,

$$\mathbf{n} = \mathbf{t}^\perp = \left(\frac{dy}{ds}, -\frac{dx}{ds} \right) = \left(\frac{\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}}, \frac{-\,\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}} \right), \qquad \text{so that} \qquad \begin{aligned} \|\,\mathbf{n}\,\| &= 1, \\ \mathbf{n}\cdot\mathbf{t} &= 0. \end{aligned} \tag{6.7}$$

At each point on the curve, the vectors $\mathbf{t}, \mathbf{n}$ form an orthonormal basis of $\mathbb{R}^2$ known as the *moving frame* along the curve. For example, for the ellipse (6.3) with semi-axes a, b, the unit tangent and normal are given by

$$\mathbf{t} = \frac{1}{a^2+b^2} \begin{pmatrix} -\,a\sin t \\ b\cos t \end{pmatrix}, \qquad \mathbf{n} = \frac{1}{a^2+b^2} \begin{pmatrix} b\cos t \\ a\sin t \end{pmatrix},$$

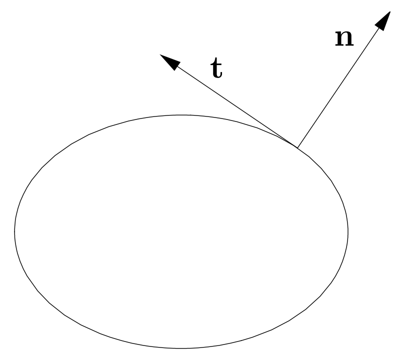


Figure 10. The Moving Frame for an Ellipse.

and graphed in Figure 10. Actually, a curve has two unit normals at each point — one points to our right side and the other to our left side as we move along the curve. The normal $\mathbf{n}$ in (6.7) is the right-handed normal, and is the traditional one to choose; the opposite, left-handed normal is its negative $-\mathbf{n}$. If we traverse a simple closed curve in a counterclockwise direction, then the right-handed normal $\mathbf{n}$ is the unit *outward* normal, pointing to the curve's exterior.

Arc Length Integrals

We now explain how to integrate scalar functions along curves. Suppose first that C is a (piecewise) smooth curve that is parametrized by arc length, $\mathbf{x}(s) = (x(s), y(s))$ for $0 \le s \le \ell$, where ℓ is the total length of C. If $u(\mathbf{x}) = u(x, y)$ is any scalar field, we define its *arc length integral* along the curve C to be

$$\int_C u\, ds = \int_0^\ell u(x(s), y(s))\, ds. \tag{6.8}$$

For example, if $\rho(x, y)$ represents the density at position (x, y) of a wire bent in the shape of a curve C, then the arc length integral $\displaystyle\int_C \rho(x, y)\, ds$ computes the total mass of the wire. In particular, the length of the curve is (tautologously) given by

$$\mathcal{L}(C) = \int_C ds = \int_0^\ell ds = \ell. \tag{6.9}$$

If we use an alternative parametrization $\mathbf{x}(t)$, with $a \le t \le b$, then the arc length integral is computed using the change of parameter formula (6.5), and so

$$\int_C u\, ds = \int_a^b u(\mathbf{x}(t)) \left\| \frac{d\mathbf{x}}{dt} \right\| dt = \int_a^b u(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt. \tag{6.10}$$

Changing the orientation of the curve does *not* alter the value of this type of line integral. Moreover, if we break up the curve into two nonoverlapping pieces, then the arc length integral decomposes into a sum:

$$\int_C u\, ds = \int_{-C} u\, ds, \qquad \int_C u\, ds = \int_{C_1} u\, ds \, + \int_{C_2} u\, ds, \qquad C = C_1 \cup C_2. \tag{6.11}$$

Example 6.2. A circular wire radius 1 has density proportional to the distance of the point from the x axis. The mass of the wire is computed by the arc length integral

$$\oint_C |\, y\, |\, ds = \int_0^{2\pi} |\sin t\,|\, dt = 4.$$

The arc length integral was evaluated using the parametrization $\mathbf{x}(t) = (\, a\cos t, a\sin t\,)^T$ for $0 \le t \le 2\pi$, whereby $ds = \|\,\dot{\mathbf{x}}\,\|\, dt = dt$.

Line Integrals of Vector Fields

There are two intrinsic ways of integrating a vector field along a curve. In the first version, we integrate its tangential component $\mathbf{v} \cdot \mathbf{t}$, where $\mathbf{t} = d\mathbf{x}/ds$ is the unit tangent vector, with respect to arc length.

Definition 6.3. The *line integral* of a vector field $\mathbf{v}$ along a parametrized curve $\mathbf{x}(t)$ is given by

$$\int_C \mathbf{v} \cdot d\mathbf{x} = \int_C v_1(x,y)\, dx + v_2(x,y)\, dy = \int_C \mathbf{v} \cdot \mathbf{t}\; ds. \tag{6.12}$$

To evaluate the line integral, we parametrize the curve by $\mathbf{x}(t)$ for $a \le t \le b$, and then

$$\int_C \mathbf{v} \cdot d\mathbf{x} = \int_a^b \mathbf{v}(\mathbf{x}(t)) \cdot \frac{d\mathbf{x}}{dt}\, dt = \int_a^b \left[v_1(x(t),y(t))\, \frac{dx}{dt} + v_2(x(t),y(t))\, \frac{dy}{dt} \right] dt. \tag{6.13}$$

This result follows from the formulae (6.5, 6) for the arc length and unit tangent vector. In general, line integrals are independent of how the curve is parametrized — as long as it is traversed in the same direction. Reversing the direction of parameterization, i.e., changing the orientation of the curve, changes the sign of the line integral — because it reverses the direction of the unit tangent. As before, line integrals can be decomposed into sums over components:

$$\int_{-C} \mathbf{v} \cdot d\mathbf{x} = -\int_C \mathbf{v} \cdot d\mathbf{x}, \qquad \int_C \mathbf{v} \cdot d\mathbf{x} = \int_{C_1} \mathbf{v} \cdot d\mathbf{x} \;+\; \int_{C_2} \mathbf{v} \cdot d\mathbf{x}, \qquad C = C_1 \cup C_2. \tag{6.14}$$

In the second formula, one must take care to orient the two parts C_1, C_2 in the same direction as C.

Example 6.4. Let C denote the circle of radius r centered at the origin. We will compute the line integral of the rotational vector field (5.4), namely

$$\oint_C \mathbf{v} \cdot d\mathbf{x} = \oint_C \frac{y\, dx - x\, dy}{x^2 + y^2}\,.$$

The circle on the integral sign serves to remind us that we are integrating around a closed curve. We parameterize the circle by

$$x(t) = r\cos t, \qquad y(t) = r\sin t, \qquad 0 \le t \le 2\pi.$$

Applying the basic line integral formula (6.13), we find

$$\oint_C \frac{y\,dx - x\,dy}{x^2 + y^2} = \int_0^{2\pi} \frac{-\,r^2 \sin^2 t - r^2 \cos^2 t}{r^2}\,dt = -\,2\pi,$$

independent of the circle's radius. Note that the parametrization goes around the circle once in the counterclockwise direction. If we go around once in the clockwise direction, e.g., by using the parametrization $\mathbf{x}(t) = (r \sin t, r \cos t)$, then the resulting line integral equals $+\,2\pi$.

If $\mathbf{v}$ represents the velocity vector field of a steady state fluid, then the line integral (6.12) represents the *circulation* of the fluid around the curve. Indeed, $\mathbf{v} \cdot \mathbf{t}$ is proportional to the force exerted by the fluid in the direction of the curve, and so the circulation integral measures the average of the tangential fluid forces around the curve. Thus, for example, the rotational vector field (5.4) has a net circulation of $-\,2\pi$ around any circle centered at the origin. The minus sign tells us that the fluid is circulating in the clockwise direction — opposite to the direction in which we went around the circle.

A fluid flow is *irrotational* if the circulation is zero for all closed curves. An irrotational flow will not cause a paddle wheel to rotate — there will be just as much fluid pushing in one direction as in the opposite, and the net tangential forces will cancel each other out. The connection between circulation and the curl of the velocity vector field will be made evident shortly.

If the vector field happens to be the gradient of a scalar field, then we can readily evaluate its line integral.

Theorem 6.5. *If $\mathbf{v} = \nabla u$ is a gradient vector field, then its line integral*

$$\int_C \nabla u \cdot\, d\mathbf{x} = u(\mathbf{b}) - u(\mathbf{a}) \tag{6.15}$$

equals the difference between the potential function's values at the endpoints $\mathbf{a} = \mathbf{x}(a)$ and $\mathbf{b} = \mathbf{x}(b)$ of the curve C.

Corollary 6.6. *Let Ω be a connected domain. A scalar field has zero gradient, $\nabla u(x, y) = \mathbf{0}$, for all $(\,x, y\,)^T \in \Omega$, if and only if $u(x, y) \equiv c$ is constant on Ω.*

Proof: Indeed,, if $\mathbf{a}, \mathbf{b}$ are any two points in Ω, then, by connectivity, we can find a curve connecting them. Then (6.15) implies that $u(\mathbf{b}) = u(\mathbf{a})$, which shows that u is constant. *Q.E.D.*

Thus, the line integral of a gradient field is *independent of path*; its value does not depend on how you get from point $\mathbf{a}$ to point $\mathbf{b}$. In particular, if C is a closed curve, then

$$\oint_C \nabla u \cdot\, d\mathbf{x} = 0,$$

since the endpoints coincide: $\mathbf{a} = \mathbf{b}$. In fact, independence of path is both necessary and sufficient for the vector field to be a gradient.

Theorem 6.7. *Let* $\mathbf{v}$ *be a vector field defined on a domain* Ω*. Then the following are equivalent:*

(*a*) *The line integral* $\displaystyle\int_C \mathbf{v}\cdot d\mathbf{x}$ *is independent of path.*

(*b*) $\displaystyle\oint_C \mathbf{v}\cdot d\mathbf{x} = 0$ *for every closed curve* C*.*

(*c*) $\mathbf{v} = \nabla u$ *is the gradient of some potential function defined on* Ω*.*

In such cases, on any connected component, a potential function can be computed by integrating the vector field

$$u(\mathbf{x}) = \int_{\mathbf{a}}^{\mathbf{x}} \mathbf{v}\cdot d\mathbf{x}. \tag{6.16}$$

Here $\mathbf{a}$ is any fixed point (which defines the zero potential level), and we evaluate the line integral over any curve that connects $\mathbf{a}$ to $\mathbf{x}$; path-independence says that it does not matter which curve we use to get from $\mathbf{a}$ to $\mathbf{x}$. The proof that $\nabla u = \mathbf{v}$ is left as an exercise.

Example 6.8. The line integral

$$\int_C \mathbf{v}\cdot d\mathbf{x} = \int_C (x^2 - 3y)\,dx + (2-3x)\,dy$$

of the vector field $\mathbf{v} = \left(x^2 - 3y, 2 - 3x\right)^T$ is independent of path. Indeed, parametrizing a curve C by $(x(t), y(t))$, $a \le t \le b$, leads to

$$\begin{aligned}\int_C (x^2 - 3y)\,dx + (2-3x)\,dy &= \int_a^b \left[(x^2-3y)\frac{dx}{dt} + (2-3x)\frac{dy}{dt}\right] dt \\ &= \int_a^b \frac{d}{dt}\left(x^3 - 3xy + 2y\right) dt = \left(x^3 - 3xy + 2y\right)\Big|_{t=a}^{b}.\end{aligned}$$

The result only depends on the endpoints $\mathbf{a} = (\,x(a), y(a)\,)^T$, $\mathbf{b} = (\,x(b), y(b)\,)^T$, and not on the detailed shape of the curve. Integrating from $\mathbf{a} = \mathbf{0}$ to $\mathbf{b} = (x, y)$ produces the potential function

$$u(x,y) = x^3 - 3xy + 2y.$$

As guaranteed by (6.16), $\nabla u = \mathbf{v}$.

On the other hand, the line integral

$$\int_C \mathbf{v}\cdot d\mathbf{x} = \int_C (x^3 - 2y)\,dx + x^2\,dy$$

of the vector field $\mathbf{v} = \left(x^3 - 2y, x^2\right)^T$ is not path-independent, and so $\mathbf{v}$ does not admit a potential function. Indeed, integrating from $(0,0)$ to $(1,1)$ along the straight line segment $\{\,(t,t)\,|\,0 \le t \le 1\,\}$, produces

$$\int_C (x^3 - 2y)\,dx + x^2\,dy = \int_0^1 \left(t^3 - 2t + t^2\right) dt = -\tfrac{5}{12}.$$

On the other hand, integrating along the parabola $\{ (t, t^2) \mid 0 \leq t \leq 1 \}$, yields a different value

$$\int_C (x^3 - 2y)\, dx + x^2\, dy = \int_0^1 \left(t^3 - 2t^2 + 2t^3 \right) dt = \tfrac{1}{12}.$$

If $\mathbf{v}$ represents a force field, then the line integral (6.12) represents the amount of *work* required to move along the given curve. Work is defined as force, or, more correctly, the tangential component of the force in the direction of motion, times distance. The line integral effectively totals up the infinitesimal contributions, the sum total representing the total amount of work expended in moving along the curve. Note that the work is independent of the parametrization of the curve. In other words (and, perhaps, counter-intuitively[†]), the amount of work expended doesn't depend upon how fast you move along the curve.

According to Theorem 6.7, the work does not depend on the route you use to get from one point to the other if and only if the force field admits a potential function: $\mathbf{v} = \nabla u$. Then, by (6.15), the work is just the difference in potential at the two points. In particular, for a gradient vector field there is no net work required to go around a closed path.

Flux

The second type of line integral is found by integrating the normal component of the vector field along the curve:

$$\int_C \mathbf{v} \cdot \mathbf{n}\, ds. \tag{6.17}$$

Using the formula (6.7) for the unit normal, we find that the inner product can be rewritten in the alternative form

$$\mathbf{v} \cdot \mathbf{n} = v_1 \frac{dy}{ds} - v_2 \frac{dx}{ds} = \mathbf{v}^\perp \cdot \mathbf{t}\,,$$

where $\mathbf{t} = d\mathbf{x}/ds$ is the unit tangent, while

$$\mathbf{v}^\perp = (\, -v_2, v_1 \,)^T \tag{6.18}$$

is a vector field that is everywhere orthogonal to the velocity vector field $\mathbf{v} = (\, v_1, v_2 \,)^T$. Thus, the normal line integral (6.17) can be rewritten as a tangential line integral

$$\int_C \mathbf{v} \cdot \mathbf{n}\, ds = \int_C v_1\, dy - v_2\, dx = \int_C \mathbf{v} \wedge d\mathbf{x} = \int_C \mathbf{v}^\perp \cdot d\mathbf{x} = \int_C \mathbf{v}^\perp \cdot \mathbf{t}\, ds. \tag{6.19}$$

If $\mathbf{v}$ represents the velocity vector field for a steady-state fluid flow, then the inner product $\mathbf{v} \cdot \mathbf{n}$ with the unit normal measures the *flux* of the fluid flow across the curve at the given point. The flux is positive if the fluid is moving in the normal direction $\mathbf{n}$ and

[†] The reason this doesn't agree with our intuition about work is that we are not taking frictional effects into account, and these are typically velocity-dependent.

negative if it is moving in the opposite direction. If the vector field admits a potential, $\mathbf{v} = \nabla u$, then the flux

$$\mathbf{v} \cdot \mathbf{n} = \nabla u \cdot \mathbf{n} = \frac{\partial u}{\partial \mathbf{n}} \tag{6.20}$$

equals its *normal derivative*, i.e., the directional derivative of the potential function u in the normal direction to the curve. The line integral $\displaystyle\int_C \mathbf{v} \cdot \mathbf{n}\, ds$ sums up the individual fluxes, and so represents the total flux across the curve, meaning the total volume of fluid that passes across the curve per unit time — in the direction assigned by the unit normal $\mathbf{n}$. In particular, if C is a simple closed curve and $\mathbf{n}$ is the outward normal, then the flux integral (6.17) measures the net outflow of fluid across C; if negative, it represents an inflow. The total flux is zero if and only if the total amount of fluid contained within the curve does not change. Thus, in the absence of sources or sinks, an incompressible fluid, such as water, will have zero net flux around any closed curve since the total amount of fluid within any given region cannot change.

Example 6.9. For the radial vector field

$$\mathbf{v} = \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}, \qquad \text{we have} \qquad \mathbf{v}^\perp = \begin{pmatrix} -y \\ x \end{pmatrix}.$$

As we saw in Example 4.2, $\mathbf{v}$ represents the fluid flow due to a source at the origin. Thus, the resulting fluid flux across a circle C of radius r is computed using the line integral

$$\oint_C \mathbf{v} \cdot \mathbf{n}\, ds = \oint_C x\, dy - y\, dx = \int_0^{2\pi} r^2 \sin^2 t + r^2 \cos^2 t\, dt = 2\pi r^2.$$

Therefore, the source fluid flow has a net outflow of $2\pi r^2$ units across a circle of radius r. This is not an incompressible flow!

7. Double Integrals.

We assume that the student is familiar with the foundations of multiple integration, and merely review a few of the highlights in this section. Given a scalar function $u(x,y)$ defined on a domain Ω, its *double integral*

$$\iint_\Omega u(x,y)\, dx\, dy = \iint_\Omega u(\mathbf{x})\, d\mathbf{x} \tag{7.1}$$

is equal to the volume of the solid lying underneath the graph of u over Ω. If $u(x,y)$ represents the density at position $(x,y)^T$ in a plate having the shape of the domain Ω, then the double integral (7.1) measures the total mass of the plate. In particular,

$$\text{area}\ \Omega = \iint_\Omega dx\, dy$$

is equal to the *area* of the domain Ω.

In the particular case when

$$\Omega = \left\{ \varphi(x) < y < \psi(x), \quad a < x < b \right\} \tag{7.2}$$

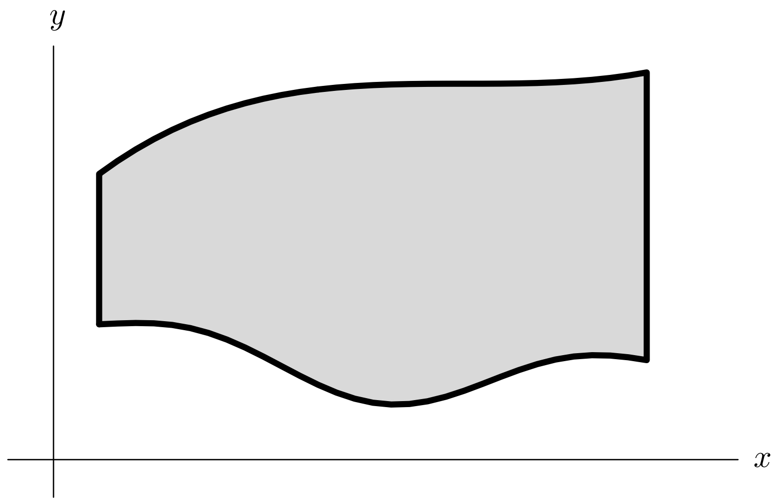


Figure 11. Double Integration Domain.

is given as the region lying between the graphs of two functions, as in Figure 11, then we can evaluate the double integral by repeated scalar integration,

$$\iint_\Omega u(x,y)\,dx\,dy = \int_a^b \left(\int_{\varphi(x)}^{\psi(x)} u(x,y)\,dy \right) dx, \tag{7.3}$$

in the two coordinate directions. Fubini's Theorem states that one can equally well evaluate the integral in the reverse order

$$\iint_\Omega u(x,y)\,dx\,dy = \int_c^d \left(\int_{\alpha(y)}^{\beta(y)} u(x,y)\,dx \right) dy \tag{7.4}$$

in the case

$$\Omega = \{\, \alpha(y) < x < \beta(y), \quad c < y < d \,\} \tag{7.5}$$

lies between the graphs of two functions of y.

Example 7.1. Compute the volume of the solid lying under the positive part of the paraboloid $z = 1 - x^2 - y^2$. Note that $z > 0$ if and only if $x^2 + y^2 < 1$, and hence we should evaluate the double integral

$$\iint_\Omega (1 - x^2 - y^2)\,dx\,dy$$

over the unit disk $\Omega = \{\, x^2 + y^2 < 1 \,\}$. We may represent the disk in the form (7.2), so that

$$\Omega = \{\, -\sqrt{1-x^2} < y < \sqrt{1-x^2}, \quad -1 < x < 1 \,\}.$$

Therefore, we evaluate the volume by repeated integration

$$\iint_\Omega [\,1 - x^2 - y^2\,]\,dx\,dy = \int_{-1}^{1} \left[\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (1 - x^2 - y^2)\,dy \right] dx$$

$$= \int_{-1}^{1} \left[\left(y - x^2 y - \tfrac{1}{3} y^3 \right) \Big|_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \right] dx = \int_{-1}^{1} \tfrac{4}{3} (1 - x^2)^{3/2}\,dx = \tfrac{1}{2}\,\pi.$$

The final integral is most easily effected via a trigonometric substitution.

Alternatively, and much easier, one can use polar coordinates to evaluate the integral. The unit disk takes the form $D = \{ 0 \le r < 1, 0 \le \theta < 2\pi \}$, and so

$$\iint_D (1 - x^2 - y^2)\, dx\, dy = \iint_D (1 - r^2)\, r\, dr\, d\theta = \int_0^1 \left(\int_0^{2\pi} (r - r^3)\, d\theta \right) dr = \tfrac{1}{2}\pi.$$

We are using the standard formula

$$dx\, dy = r\, dr\, d\theta \tag{7.6}$$

for the area element in polar coordinates, [**1**, **4**].

The polar integration formula (7.6) is a consequence of the general change of variables formula for double integrals. If

$$x = x(s,t), \qquad\qquad y = y(s,t),$$

is an invertible change of variables that maps $(s,t)^T \in D$ to $(x,y)^T \in \Omega$, then

$$\iint_\Omega u(x,y)\, dx\, dy = \iint_D stU(s,t) \left| \frac{\partial(x,y)}{\partial(s,t)} \right|. \tag{7.7}$$

Here $U(s,t) = u(x(s,t), y(s,t))$ denotes the function when rewritten in the new variables, while

$$\frac{\partial(x,y)}{\partial(s,t)} = \det \begin{pmatrix} x_s & x_t \\ y_s & y_t \end{pmatrix} = \frac{\partial x}{\partial s}\frac{\partial y}{\partial t} - \frac{\partial x}{\partial t}\frac{\partial y}{\partial s} \tag{7.8}$$

is the *Jacobian determinant* of the functions x, y with respect to the variables s, t, which measures the local change in area under the map.

In the event that the domain of integration is more complicated than either (7.2) or (7.5), then one performs "surgery" by chopping up the domain

$$\Omega = \Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_k$$

into smaller pieces. The pieces Ω_i are not allowed to overlap, and so have at most their boundary curves in common. The double integral

$$\iint_\Omega u(x,y)\, dx\, dy = \iint_{\Omega_1} u(x,y)\, dx\, dy + \ \cdots \ + \iint_{\Omega_k} u(x,y)\, dx\, dy \tag{7.9}$$

can then be evaluated as a sum of the double integrals over the individual pieces.

8. Green's Theorem.

For double integrals, the role of the Fundamental Theorem of Calculus is played by *Green's Theorem*. The Fundamental Theorem relates an integral over an interval $I = [a,b]$ to an evaluation at the boundary $\partial I = \{a, b\}$, which consists of the two endpoints of the interval. In a similar manner, Green's Theorem relates certain double integrals over a planar domain Ω to line integrals around its boundary curve(s) $\partial\Omega$.

Theorem 8.1. *Let $\mathbf{v}(\mathbf{x})$ be a smooth vector field defined on a bounded domain $\Omega \subset \mathbb{R}^2$. Then the line integral of $\mathbf{v}$ around the boundary $\partial\Omega$ equals the double integral of the curl of $\mathbf{v}$ over the domain. This result can be written in either of the equivalent forms*

$$\iint_\Omega \nabla \wedge \mathbf{v}\, d\mathbf{x} = \oint_{\partial\Omega} \mathbf{v} \cdot d\mathbf{x}, \qquad \iint_\Omega \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} \right) dx\, dy = \oint_{\partial\Omega} v_1\, dx + v_2\, dy\,. \tag{8.1}$$

Green's Theorem was first formulated in 1828 by the English mathematician and miller George Green, and, contemporaneously, by the Russian mathematician Mikhail Ostrogradski.

Example 8.2. Let us apply Green's Theorem 8.1 to the particular vector field $\mathbf{v} = (\,0, x\,)^T$. Since $\nabla \wedge \mathbf{v} \equiv 1$, we find

$$\oint_{\partial\Omega} x\, dy = \iint_\Omega \, dx\, dy = \text{area}\, \Omega. \tag{8.2}$$

This means that we can compute the area of a planar domain by computing the indicated line integral around its boundary! For example, to compute the area of a disk D_r of radius r, we parametrize its bounding circle C_r by $(\, r\cos t, r \sin t\,)^T$ for $0 \le t \le 2\pi$, and compute

$$\text{area}\, D_r = \oint_{C_r} x\, dy = \int_0^{2\pi} r^2 \cos^2 t\, dt = \pi r^2.$$

If we interpret $\mathbf{v}$ as the velocity vector field associated with a steady state fluid flow, then the right hand side of formula (8.1) represents the circulation of the fluid around the boundary of the domain Ω. Green's Theorem implies that the double integral of the curl of the velocity vector must equal this circulation line integral.

If we divide the double integral in (8.1) by the area of the domain,

$$\frac{1}{\text{area}\, \Omega} \iint_\Omega \nabla \wedge \mathbf{v}\, d\mathbf{x} = \mathrm{M}_\Omega \left[\, \nabla \wedge \mathbf{v}\, \right],$$

we obtain the mean of the curl $\nabla \wedge \mathbf{v}$ of the vector field over the domain. In particular, if the domain Ω is very small, then $\nabla \wedge \mathbf{v}$ does not vary much, and so its value at any point in the domain is more or less equal to the mean. On the other hand, the right hand side of (8.1) represents the circulation around the boundary $\partial\Omega$. Thus, we conclude that the curl $\nabla \wedge \mathbf{v}$ of the velocity vector field represents an "infinitesimal circulation" at the point it is evaluated. In particular, the fluid is irrotational, with no net circulation around any curve, if and only if $\nabla \wedge \mathbf{v} \equiv 0$ everywhere. Under the assumption that its domain of definition is simply connected, Theorem 6.7 tell us that this is equivalent to the existence of a velocity potential u with $\nabla u = \mathbf{v}$.

Theorem 8.3. *A vector field $\mathbf{v}$ defined on a simply connected domain $\Omega \subset \mathbb{R}^2$ admits a potential, $\mathbf{v} = \nabla \varphi$ for some $\varphi \colon \Omega \to \mathbb{R}$ if and only if $\nabla \wedge \mathbf{v} \equiv 0$.*

We can also apply Green's Theorem 8.1 to flux line integrals of the form (6.17). Using the identification (6.19) followed by (8.1), we find that

$$\oint_{\partial\Omega} \mathbf{v}\cdot\mathbf{n}\,ds = \oint_{\partial\Omega} \mathbf{v}^{\perp}\cdot d\mathbf{x} = \iint_{\Omega} \nabla\wedge\mathbf{v}^{\perp}\,dx\,dy.$$

However, note that the curl of the orthogonal vector field (6.18), namely

$$\nabla\wedge\mathbf{v}^{\perp} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} = \nabla\cdot\mathbf{v}, \tag{8.3}$$

coincides with the *divergence* of the original velocity field. Combining these together, we have proved the divergence or flux form of Green's Theorem:

$$\iint_{\Omega} \nabla\cdot\mathbf{v}\,dx\,dy = \oint_{\partial\Omega} \mathbf{v}\cdot\mathbf{n}\,ds. \tag{8.4}$$

As before, Ω is a bounded domain, and $\mathbf{n}$ is the unit outward normal to its boundary $\partial\Omega$.

In the fluid flow interpretation, the right hand side of (8.4) represents the net fluid flux out of the region Ω. Thus, the double integral of the divergence of the flow vector must equal this net change in area. Thus, in the absence of sources or sinks, the divergence of the velocity vector field, $\nabla\cdot\mathbf{v}$ will represent the local change in area of the fluid at each point. In particular, if the fluid is incompressible if and only if $\nabla\cdot\mathbf{v}\equiv 0$ everywhere.

An ideal fluid flow is both incompressible, $\nabla\cdot\mathbf{v} = 0$, and irrotational, $\nabla\wedge\mathbf{v} = \mathbf{0}$. Assuming its domain is simply connected, we introduce velocity potential $u(x,y)$, so that $\nabla u = \mathbf{v}$. Therefore,

$$0 = \nabla\cdot\mathbf{v} = \nabla\cdot\nabla u = u_{xx} + u_{yy}. \tag{8.5}$$

Therefore, the velocity potential for an incompressible, irrotational fluid flow is a harmonic function, i.e., it satisfies the Laplace equation. Water waves are typically modeled in this manner, and so many problems in fluid mechanics rely on the solution to Laplace's equation.

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