Vector Calculus in Three Dimensions

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1. Introduction.

In these notes we review the fundamentals of three-dimensional vector calculus. We will be surveying calculus on curves, surfaces and solid bodies in three-dimensional space. The three methods of integration — line, surface and volume (triple) integrals — and the fundamental vector differential operators — gradient, curl and divergence — are intimately related. The differential operators and integrals underlie the multivariate versions of the fundamental theorem of calculus, known as Stokes' Theorem and the Divergence Theorem. A more detailed development can be found in any reasonable multi-variable calculus text, including [**1**, **6**, **9**].

2. Dot and Cross Product.

We begin by reviewing the basic algebraic operations between vectors in three-dimensional space $\mathbb{R}^3$; see [**10**] for details. We shall use column vector notation

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = (\, v_1, v_2, v_3 \,)^T \in \mathbb{R}^3.$$

The standard basis vectors of $\mathbb{R}^3$ are

$$\mathbf{e}_1 = \mathbf{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{e}_2 = \mathbf{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \mathbf{e}_3 = \mathbf{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{2.1}$$

We prefer the former notation, as it easily generalizes to n-dimensional space. Any vector

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = v_1\, \mathbf{e}_1 + v_2\, \mathbf{e}_2 + v_3\, \mathbf{e}_3$$

is a linear combination of the basis vectors. The coefficients he v_1, v_2, v_3 are the coordinates of the vector with respect to the standard basis.

Space comes equipped with an orientation — either right- or left-handed. One cannot alter[†] the orientation by physical motion, although looking in a mirror — or, mathematically, performing a reflection — reverses the orientation. The standard basis vectors are

[†] This assumes that space is identified with the three-dimensional Euclidean space $\mathbb{R}^3$, or, more generally, an oriented three-dimensional manifold, [**2**].

graphed with a right-hand orientation. When you point with your right hand, $\mathbf{e}_1$ lies in the direction of your index finger, $\mathbf{e}_2$ lies in the direction of your middle finger, and $\mathbf{e}_3$ is in the direction of your thumb. In general, a set of three linearly independent vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is said to have a *right-handed orientation* if they have the same orientation as the standard basis. It is not difficult to prove that this is the case if and only if the determinant of the 3×3 matrix whose columns are the given vectors is positive: $\det(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) > 0$. Interchanging the order of the vectors may switch their orientation; for example if $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are right-handed, then $\mathbf{v}_2, \mathbf{v}_1, \mathbf{v}_3$ is left-handed.

We will employ the Euclidean *dot product*[†]

$$\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + v_2 w_2 + v_3 w_3, \qquad \text{where} \qquad \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}, \qquad \mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}, \tag{2.2}$$

along with the Euclidean norm

$$\| \mathbf{v} \| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{v_1^2 + v_2^2 + v_3^2}\,. \tag{2.3}$$

The dot product is bilinear, symmetric: $\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$, and positive. The Cauchy–Schwarz inequality

$$| \mathbf{v} \cdot \mathbf{w} | \;\leq\; \| \mathbf{v} \| \, \| \mathbf{w} \|. \tag{2.4}$$

implies that the dot product can be used to measure the angle θ between the two vectors $\mathbf{v}$ and $\mathbf{w}$:

$$\mathbf{v} \cdot \mathbf{w} = \| \mathbf{v} \| \, \| \mathbf{w} \| \, \cos\theta. \tag{2.5}$$

Also of great importance — but particular to three-dimensional space — is the *cross product* between vectors. While the dot product produces a scalar, the three-dimensional cross product produces a vector, defined by the formula

$$\mathbf{v} \times \mathbf{w} = \begin{pmatrix} v_2 w_3 - v_3 w_2 \\ v_3 w_1 - v_1 w_3 \\ v_1 w_2 - v_2 w_1 \end{pmatrix} \qquad \text{where} \qquad \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}, \qquad \mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}, \tag{2.6}$$

We have chosen to employ the more modern wedge notation rather the more traditional cross symbol, $\mathbf{v} \times \mathbf{w}$, for this quantity. The cross product formula is most easily memorized as a formal 3×3 determinant

$$\begin{aligned} \mathbf{v} \times \mathbf{w} &= \det \begin{pmatrix} v_1 & w_1 & \mathbf{e}_1 \\ v_2 & w_2 & \mathbf{e}_2 \\ v_3 & w_3 & \mathbf{e}_3 \end{pmatrix} \\ &= (v_2 w_3 - v_3 w_2)\, \mathbf{e}_1 + (v_3 w_1 - v_1 w_3)\, \mathbf{e}_2 + (v_1 w_2 - v_2 w_1)\, \mathbf{e}_3, \end{aligned} \tag{2.7}$$

[†] Adapting these constructions to more general norms and inner products is an interesting exercise, but will not concern us here.

involving the standard basis vectors (2.1). We note that, like the dot product, the cross product is a bilinear function, meaning that

$$\begin{aligned}(c\,\mathbf{u} + d\,\mathbf{v}) \times \mathbf{w} &= c\,(\mathbf{u} \times \mathbf{w}) + d\,(\mathbf{v} \times \mathbf{w}),\\ \mathbf{u} \times (c\,\mathbf{v} + d\,\mathbf{w}) &= c\,(\mathbf{u} \times \mathbf{v}) + d\,(\mathbf{u} \times \mathbf{w}),\end{aligned} \tag{2.8}$$

for any vectors $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ and any scalars $c, d \in \mathbb{R}$. On the other hand, unlike the dot product, the cross product is an *anti-symmetric* quantity

$$\mathbf{v} \times \mathbf{w} = -\,\mathbf{w} \times \mathbf{v}, \tag{2.9}$$

which changes its sign when the two vectors are interchanged. In particular, the cross product of a vector with itself is automatically zero:

$$\mathbf{v} \times \mathbf{v} = \mathbf{0}.$$

Geometrically, the cross product vector $\mathbf{u} = \mathbf{v} \times \mathbf{w}$ is orthogonal to the two vectors $\mathbf{v}$ and $\mathbf{w}$:

$$\mathbf{v} \cdot (\mathbf{v} \times \mathbf{w}) = \mathbf{0} = \mathbf{w} \cdot (\mathbf{v} \times \mathbf{w}).$$

Thus, when $\mathbf{v}$ and $\mathbf{w}$ are linearly independent, their cross product $\mathbf{u} = \mathbf{v} \times \mathbf{w} \neq \mathbf{0}$ defines a normal direction to the plane spanned by $\mathbf{v}$ and $\mathbf{w}$. The direction of the cross product is fixed by the requirement that $\mathbf{v}, \mathbf{w}, \mathbf{u} = \mathbf{v} \times \mathbf{w}$ form a right-handed triple. The length of the cross product vector is equal to the area of the parallelogram defined by the two vectors, which is

$$\| \mathbf{v} \times \mathbf{w} \| = \| \mathbf{v} \|\, \| \mathbf{w} \|\, |\sin\theta\,| \tag{2.10}$$

where θ is than angle between the two vectors. Consequently, the cross product vector is zero, $\mathbf{v} \times \mathbf{w} = \mathbf{0}$, if and only if the two vectors are collinear (linearly dependent) and hence only span a line.

The *scalar triple product* $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$ between three vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ is defined as the dot product between the first vector with the cross product of the second and third vectors. The parenthesis is often omitted because there is only one way to make sense of $\mathbf{u} \cdot \mathbf{v} \times \mathbf{w}$. Combining (2.2), (2.7), shows that one can compute the triple product by the determinantal formula

$$\mathbf{u} \cdot \mathbf{v} \times \mathbf{w} = \det \begin{pmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{pmatrix}. \tag{2.11}$$

By the properties of the determinant, permuting the order of the vectors merely changes the sign of the triple product:

$$\mathbf{u} \cdot \mathbf{v} \times \mathbf{w} = -\,\mathbf{v} \cdot \mathbf{u} \times \mathbf{w} = +\,\mathbf{v} \cdot \mathbf{w} \times \mathbf{u} = \,\cdots\,.$$

The triple product vanishes, $\mathbf{u} \cdot \mathbf{v} \times \mathbf{w} = 0$, if and only if the three vectors are linearly dependent, i.e., coplanar or collinear. The triple product is positive, $\mathbf{u} \cdot \mathbf{v} \times \mathbf{w} > 0$ if and only if the three vectors form a right-handed basis. Its magnitude $|\,\mathbf{u} \cdot \mathbf{v} \times \mathbf{w}\,|$ measures the volume of the parallelepiped spanned by the three vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$.

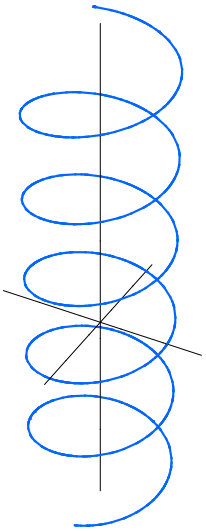

Figure 1. A Helix.

3. Curves.

A *space curve* $C \subset \mathbb{R}^3$ is parametrized by a vector-valued function

$$\mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} \in \mathbb{R}^3, \qquad a \leq t \leq b, \tag{3.1}$$

that depends upon a single parameter t that varies over some interval. We shall always assume that $\mathbf{x}(t)$ is continuously differentiable. The curve is *smooth* provided its *tangent vector* is continuous and everywhere *nonzero*:

$$\frac{d\mathbf{x}}{dt} = \dot{\mathbf{x}} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} \neq \mathbf{0}. \tag{3.2}$$

As in the planar situation, the smoothness condition (3.2) precludes the formulation of corners, cusps or other singularities in the curve.

Physically, we can think of a curve as the trajectory described by a particle moving in space. At each time t, the tangent vector $\dot{\mathbf{x}}(t)$ represents the instantaneous velocity of the particle. Thus, as long as the particle moves with nonzero speed, $\| \dot{\mathbf{x}} \| = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} > 0$, its trajectory is necessarily a smooth curve.

Example 3.1. A charged particle in a constant magnetic field moves along the curve

$$\mathbf{x}(t) = \begin{pmatrix} \rho \cos t \\ \rho \sin t \\ c\, t \end{pmatrix}, \tag{3.3}$$

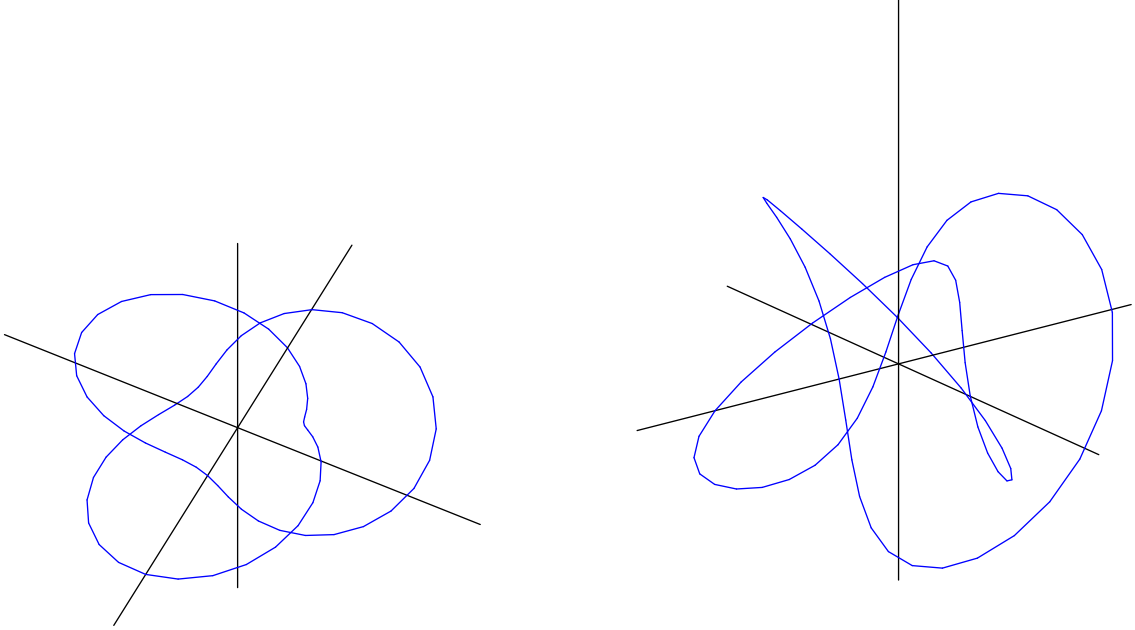

Figure 2. Two Views of a Trefoil Knot.

where $c > 0$ and $\rho > 0$ are positive constants. The curve describes a circular *helix* of *radius* ρ spiraling up the z axis. The parameter c determines the *pitch* of the helix, indicating how tightly its coils are wound; the smaller c is, the closer the winding. See Figure 1 for an illustration. DNA is, remarkably, formed in the shape of a (bent and twisted) double helix. The tangent to the helix at a point $\mathbf{x}(t)$ is the vector

$$\dot{\mathbf{x}}(t) = \begin{pmatrix} -\rho \sin t \\ \rho \cos t \\ c \end{pmatrix}.$$

Note that the speed of the particle,

$$\| \dot{\mathbf{x}} \| = \sqrt{\rho^2 \sin^2 t + \rho^2 \cos^2 t + c^2} = \sqrt{\rho^2 + c^2}\,, \tag{3.4}$$

remains constant, although the velocity vector $\dot{\mathbf{x}}$ twists around.

A curve is *simple* if it never crosses itself, and *closed* if its ends meet, $\mathbf{x}(a) = \mathbf{x}(b)$. In the plane, simple closed curves are all topologically equivalent, meaning one can be continuously deformed to the other. In space, this is no longer true. Closed curves can be knotted, and thus have nontrivial topology.

Example 3.2. The curve

$$\mathbf{x}(t) = \begin{pmatrix} (2 + \cos 3t) \cos 2t \\ (2 + \cos 3t) \sin 2t \\ \sin 3t \end{pmatrix} \qquad \text{for} \qquad 0 \le t \le 2\pi, \tag{3.5}$$

describes a closed curve that is in the shape of a trefoil knot, as depicted in Figure 2. The trefoil is a genuine knot, meaning it cannot be deformed into an unknotted circle without cutting and retying. (However, a rigorous proof of this fact is not easy.) The trefoil is the simplest of the "toroidal knots".

The study and classification of knots is a subject of great historical importance. Indeed, they were first considered from a mathematical viewpoint in the nineteenth century, when the English applied mathematician William Thompson (later Lord Kelvin) proposed a theory of atoms based on knots! In recent years, knot theory has witnessed a tremendous revival, owing to its great relevance to modern day mathematics and physics. We refer the interested reader to the advanced text [**7**] for details.

4. Line Integrals.

In this section, we discuss integrals along space curves.

Arc Length

The *length* of the space curve $\mathbf{x}(t)$ over the parameter range $a \le t \le b$ is computed by integrating the norm of its tangent vector:

$$\mathcal{L}(C) = \int_a^b \left\| \frac{d\mathbf{x}}{dt} \right\| dt = \int_a^b \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\; dt\,. \tag{4.1}$$

It is not hard to show that the length of the curve is independent of the parametrization — as it should be.

Starting at the endpoint $\mathbf{x}(a)$, the *arc length parameter* s is given by

$$s = \int_a^t \left\| \frac{d\mathbf{x}}{dt} \right\| dt \qquad \text{and so} \qquad ds = \| \dot{\mathbf{x}} \|\, dt = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\; dt. \tag{4.2}$$

The arc length s measures the distance along the curve starting from the initial point $\mathbf{x}(a)$. Thus, the length of the part of the curve between $s = \alpha$ and $s = \beta$ is exactly $\beta - \alpha$. It is often convenient to reparametrize the curve by its arc length, $\mathbf{x}(s)$. This has the same effect as moving along the curve at unit speed, since, by the chain rule,

$$\frac{d\mathbf{x}}{ds} = \frac{dx}{dt}\,\frac{dt}{ds} = \frac{\dot{\mathbf{x}}}{\| \dot{\mathbf{x}} \|}\,, \qquad \text{so that} \qquad \left\| \frac{d\mathbf{x}}{ds} \right\| = 1.$$

Therefore $d\mathbf{x}/ds$ is the unit tangent vector pointing in the direction of motion along the curve.

Example 4.1. The length of one turn of a helix (3.3) is, using (3.4),

$$\mathcal{L}(C) = \int_0^{2\pi} \left\| \frac{d\mathbf{x}}{dt} \right\| dt = \int_0^{2\pi} \sqrt{\rho^2 + c^2}\; dt = 2\,\pi\,\sqrt{\rho^2 + c^2}\,.$$

The arc length parameter, measured from the point $\mathbf{x}(0) = (\,r, 0, 0\,)^T$ is merely a rescaling,

$$s = \int_0^t \sqrt{\rho^2 + c^2}\; dt = \sqrt{\rho^2 + c^2}\; t,$$

of the original parameter t. When the helix is parametrized by arc length,

$$\mathbf{x}(s) = \left(\rho \cos \frac{s}{\sqrt{\rho^2 + c^2}}\,,\ \rho \sin \frac{s}{\sqrt{\rho^2 + c^2}}\,,\ \frac{c\,s}{\sqrt{\rho^2 + c^2}} \right)^T,$$

we move along it with unit speed. It now takes time $s = 2\,\pi\,\sqrt{\rho^2 + c^2}$ to complete on turn of the helix.

Example 4.2. To compute the length of the trefoil knot (3.5), we begin by computing the tangent vector

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} -2\,(2 + \cos 3t)\sin 2t - 3\,\sin 3t\,\cos 2t \\ 2\,(2 + \cos 3t)\cos 2t - 3\,\sin 3t\,\sin 2t \\ 3\,\cos 3t \end{pmatrix}.$$

After some algebra involving trigonometric identities, we find

$$\| \dot{\mathbf{x}} \| = \sqrt{27 + 16\,\cos 3t + 2\,\cos 6t}\,,$$

which is never 0. Unfortunately, the resulting arc length integral

$$\int_0^{2\,\pi} \| \dot{\mathbf{x}} \|\, dt = \int_0^{2\,\pi} \sqrt{27 + 16\,\cos 3t + 2\,\cos 6t}\; dt$$

cannot be completed in elementary terms. Numerical integration can be used to find the approximate value 31.8986 for the length of the knot.

The *arc length integral* of a scalar field $u(\mathbf{x}) = u(x, y, z)$ along a curve C is

$$\int_C u\, ds = \int_0^{\ell} u(\mathbf{x}(s))\, ds = \int_0^{\ell} u(x(s), y(s), z(s))\, ds, \tag{4.3}$$

where ℓ is the total length of the curve. For example, if $\rho(x, y, z)$ represents the density at position $\mathbf{x} = (x, y, z)$ of a wire bent in the shape of the curve C, then $\displaystyle\int_C \rho\, ds$ represents the total mass of the wire. In particular, the integral

$$\int_C ds = \int_0^{\ell} ds = \ell$$

recovers the length of the curve.

If it is not convenient to work directly with the arc length parametrization, we can still compute the arc length integral in terms of the original parametrization $\mathbf{x}(t)$ for $a \le t \le b$. Using the change of parameter formula (4.2), we find

$$\int_C u\, ds = \int_a^b u(\mathbf{x}(t))\, \| \dot{\mathbf{x}} \|\, dt = \int_a^b u(x(t), y(t), z(t))\, \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\, dt. \tag{4.4}$$

Example 4.3. The density of a wire that is wound in the shape of a helix is proportional to its height. Let us compute the mass of one full turn of the helical wire. Thus, the density is given by $\rho(x, y, z) = a\, z$, where a is the constant of proportionality, and we are assuming $z \ge 0$. Substituting into (4.4), the total mass of the wire is

$$\mathcal{L}(C) = \int_C a\, z\, ds = \int_0^{2\pi} a\, c\, t\, \sqrt{r^2 + c^2}\, dt = 2\,\pi^2\, a\, c\, \sqrt{r^2 + c^2}\,.$$

Line Integrals of Vector Fields

The *line integral* of a vector field $\mathbf{v}$ along a parametrized curve $\mathbf{x}(t)$ is obtained by integration of its tangential component with respect to the arc length. The tangential component of $\mathbf{v}$ is given by

$$\mathbf{v} \cdot \mathbf{t}, \qquad \text{where} \qquad \mathbf{t} = \frac{d\mathbf{x}}{ds}$$

is the unit tangent vector to the curve. Thus, the line integral of $\mathbf{v}$ is written as

$$\int_C \mathbf{v} \cdot d\mathbf{x} = \int_C v_1(x,y,z)\,dx + v_2(x,y,z)\,dy + v_3(x,y,z)\,dz = \int_C \mathbf{v} \cdot \mathbf{t}\; ds. \tag{4.5}$$

We can evaluate the line integral in terms of an arbitrary parametrization of the curve by the general formula

$$\begin{aligned} \int_C \mathbf{v} \cdot d\mathbf{x} &= \int_a^b \mathbf{v}(\mathbf{x}(t)) \cdot \frac{d\mathbf{x}}{dt}\; dt \\ &= \int_a^b \left[v_1(x(t),y(t),z(t))\,\frac{dx}{dt} + v_2(x(t),y(t),z(t))\,\frac{dy}{dt} + v_3(x(t),y(t),z(t))\,\frac{dz}{dt} \right] dt. \end{aligned} \tag{4.6}$$

Line integrals in three dimensions enjoy all of the properties of their two-dimensional siblings: Reversing the direction of parameterization along the curve changes the sign; also, the integral can be decomposed into sums over components:

$$\int_{-C} \mathbf{v} \cdot d\mathbf{x} = -\int_C \mathbf{v} \cdot d\mathbf{x}, \qquad \int_C \mathbf{v} \cdot d\mathbf{x} = \int_{C_1} \mathbf{v} \cdot d\mathbf{x} \;+\; \int_{C_2} \mathbf{v} \cdot d\mathbf{x}, \qquad C = C_1 \cup C_2. \tag{4.7}$$

If $\mathbf{f}(\mathbf{x})$ represents a force field, e.g., gravity, electromagnetic force, etc., then its line integral $\displaystyle\int_C \mathbf{f} \cdot d\mathbf{x}$ represents the *work* done by moving along the curve. As in two dimensions, work is independent of the parametrization of the curve, i.e., the particle's speed of traversal.

Example 4.4. Our goal is to move a mass through the force field $\mathbf{f} = (\,y, -x, 1\,)^T$ starting from the initial point $(\,1, 0, 1\,)^T$ and moving vertically to the final point $(\,1, 0, 2\,\pi\,)^T$. *Question*: does it require more work to move in a straight line $\mathbf{x}(t) = (\,1, 0, t\,)^T$ or along the spiral helix $\mathbf{x}(t) = (\,\cos t, \sin t, t\,)^T$, where, in both cases, $0 \le t \le 2\,\pi$? The work line integral has the form

$$\int_C \mathbf{f} \cdot d\mathbf{x} = \int_C y\,dx - x\,dy + dz = \int_0^{2\,\pi} \left[y\,\frac{dx}{dt} - x\,\frac{dy}{dt} + \frac{dz}{dt} \right] dt.$$

Along the straight line, the amount of work is

$$\int_C \mathbf{f} \cdot d\mathbf{x} = \int_0^{2\,\pi} dt = 2\,\pi.$$

As for the spiral helix,

$$\int_C \mathbf{f} \cdot d\mathbf{x} = \int_0^{2\pi} \big[-\sin^2 t - \cos^2 t + 1 \big]\, dt = 0.$$

Thus, although we travel a more roundabout route, it takes no work to move along the helix!

The reason for the second result is that the force vector field $\mathbf{f}$ is everywhere orthogonal to the tangent to the curve: $\mathbf{f} \cdot \mathbf{t} = 0$, and so there is no tangential force exerted upon the motion. In such cases, the work line integral

$$\int_C \mathbf{f} \cdot d\mathbf{x} = \int_C \mathbf{f} \cdot \mathbf{t}\, ds = 0$$

automatically vanishes. In other words, it takes no work whatsoever to move in any direction which is orthogonal to the given force vector.

5. Surfaces.

Curves are one-dimensional, and so can be traced out by a single parameter. Surfaces are two-dimensional, and hence require two distinct parameters. Thus, a *surface* $S \subset \mathbb{R}^3$ is parametrized by a vector-valued function

$$\mathbf{x}(p,q) = \big(x(p,q), y(p,q), z(p,q) \big)^T \tag{5.1}$$

that depends on two variables. As the parameters $(p,q) \in \Omega$ range over a prescribed plane domain $\Omega \subset \mathbb{R}^2$, the locus of points $\mathbf{x}(p,q)$ traces out the surface in space. The parameters are often thought of as defining a system of *local coordinates* on the curved surface.

We shall always assume that the surface is *simple*, meaning that it does not intersect itself, so $\mathbf{x}(p,q) = \mathbf{x}(\widetilde{p}, \widetilde{q})$ if and only if $p = \widetilde{p}$ and $q = \widetilde{q}$. In practice, this condition can be quite hard to check! The boundary

$$\partial S = \{\, \mathbf{x}(p,q) \mid \ (p,q) \in \partial\Omega \,\} \tag{5.2}$$

of a simple surface consists of one or more simple curves. If the underlying parameter domain Ω is bounded and simply connected, then $\partial\Omega$ is a simple closed plane curve, and so ∂S is also a simple closed curve.

Example 5.1. The simplest instance of a surface is a *graph* of a function. The parameters are the x, y coordinates, and the surface coincides with the portion of the graph of the function $z = u(x,y)$ that lies over a fixed domain $(x,y) \in \Omega \subset \mathbb{R}^2$. Thus, a graphical surface has the parametric form

$$\mathbf{x}(p,q) = \big(p, q, u(p,q) \big)^T, \qquad\qquad (p,q) \in \Omega.$$

Thus, the parametrization identifies $x = p$ and $y = q$, while $z = u(p,q) = u(x,y)$ represents the height of the surface above the point $(x,y) \in \Omega$.

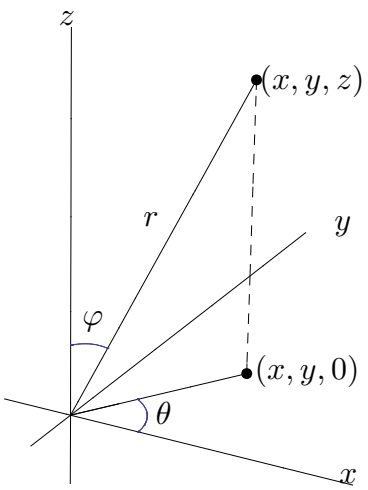

Figure 3. Spherical coordinates.

For example, the upper *hemisphere* S_r^+ of radius r centered at the origin can be parametrized as a graph

$$z = \sqrt{r^2 - x^2 - y^2}, \qquad x^2 + y^2 < r^2, \tag{5.3}$$

sitting over the disk $D_r = \{\, x^2 + y^2 < r^2 \,\}$ of radius r. The boundary of the hemisphere is the image of the circle $C_r = \partial D_r = \{\, x^2 + y^2 = r^2 \,\}$ of radius r, and is itself a circle of radius r sitting in the x, y plane: $\partial S_r^+ = \{\, x^2 + y^2 = r, z = 0 \,\}$.

Example 5.2. A *sphere* S_r of radius r can be explicitly parametrized by two angular variables φ, θ in the form

$$\mathbf{x}(\varphi, \theta) = (r \sin\varphi \cos\theta, r \sin\varphi \sin\theta, r \cos\varphi), \qquad 0 \le \theta < 2\pi, \quad 0 \le \varphi \le \pi. \tag{5.4}$$

The reader can easily check that $\| \mathbf{x} \|^2 = r^2$, as it should be. As illustrated in Figure 3, θ measures the *azimuthal angle* or *longitude*, while φ measures the *zenith angle* or *latitude*. Thus, the upper hemisphere S_r^+ is obtained by restricting the zenith parameter to the range $0 \le \varphi \le \frac{1}{2}\pi$. Each parameter value φ, θ corresponds to a unique point on the sphere, *except* when $\varphi = 0$ or π. All points $(\theta, 0)$ are mapped to the north pole $(\,0, 0, r\,)$, while all points (θ, π) are mapped to the south pole $(\,0, 0, -r\,)$. Away from the poles, the spherical angles provide *bona fide* coordinates on the sphere. Fortunately, the polar singularities do not interfere with the overall smoothness of the sphere. Nevertheless, one must always be careful at or near these two distinguished points.

Remark: In terrestrial cartography and navigation, the *latitude* is measured from the equator, and equals $\frac{1}{2}\pi - \varphi$, with positive values referring to the northern hemisphere and negative the southern hemisphere (or, vice versa, if you are an antipodean). The *longitude* is taken with respect to the prime meridian, through Greenwich, England, and equals $\pi - \theta$, with positive values referring to the western hemisphere. Of course, in practice, both are measured in degrees rather than radians. The curves $\{\, \varphi = c \,\}$ where the zenith angle takes

a prescribed constant value are the circular parallels of constant latitude — except for the north and south poles which are merely points. The equator is at $\varphi = \frac{1}{2}\,\pi$, while the tropics of Cancer and Capricorn are $23\frac{1}{2}^\circ \approx 0.41$ radians above and below the equator. The curves $\{\,\theta = c\,\}$ where the meridial angle is constant are the semi-circular meridians of constant longitude stretching from north to south pole. Note that $\theta = 0$ and $\theta = 2\,\pi$ describe the same meridian. In terrestrial navigation, latitude is the angle, in degrees, measured from the equator, while longitude is the angle measured from the Greenwich meridian.

Example 5.3. A *torus* is a surface of the form of an inner tube. One convenient parametrization of a particular toroidal surface is

$$\mathbf{x}(\psi,\theta) = (\,(2+\cos\psi)\cos\theta, (2+\cos\psi)\sin\theta, \sin\psi\,)^T \qquad \text{for} \qquad 0 \le \psi, \theta \le 2\,\pi. \tag{5.5}$$

Note that the parametrization is $2\,\pi$ periodic in both ψ and θ. If we introduce cylindrical coordinates

$$x = r\,\cos\theta, \qquad\qquad y = r\,\sin\theta, \qquad\qquad z,$$

then the torus is parametrized by

$$r = 2 + \cos\psi, \qquad\qquad z = \sin\psi.$$

Therefore, the relevant values of (r, z) all lie on the circle

$$(r-2)^2 + z^2 = 1 \tag{5.6}$$

of radius 1 centered at $(2, 0)$. As the polar angle θ increases from 0 to $2\,\pi$, the circle rotates around the z axis, and thereby sweeps out the torus.

Remark: The sphere and the torus are examples of *closed surfaces*. The requirements for a surface to be closed are that it be simple and bounded, and, moreover, have no boundary. In general, a subset $S \subset \mathbb{R}^3$ is *bounded* provided it does not stretch off infinitely far away. More precisely, boundedness is equivalent to the existence of a fixed number $R > 0$ which bounds the norm $\|\,\mathbf{x}\,\| < R$ of all points $\mathbf{x} \in S$.

Tangents to Surfaces

Consider a surface S parameterized by $\mathbf{x}(p, q)$ where $(p, q) \in \Omega$. Each parametrized curve $(p(t), q(t))$ in the parameter domain Ω will be mapped to a parametrized curve $C \subset S$ contained in the surface. The curve C is parametrized by the composite map

$$\mathbf{x}(t) = \mathbf{x}(p(t), q(t)) = (\,x(p(t), q(t)), y(p(t), q(t)), z(p(t), q(t))\,)^T.$$

The tangent vector

$$\frac{d\mathbf{x}}{dt} = \frac{\partial \mathbf{x}}{\partial p}\,\frac{dp}{dt} + \frac{\partial \mathbf{x}}{\partial q}\,\frac{dq}{dt} \tag{5.7}$$

to such a curve will be tangent to the surface. The set of all possible tangent vectors to curves passing through a given point in the surface traces out the *tangent plane* to the

surface at that point. Note that the tangent vector (5.7) is a linear combination of the two *basis tangent vectors*

$$\mathbf{x}_p = \frac{\partial \mathbf{x}}{\partial p} = \left(\frac{\partial x}{\partial p}, \frac{\partial y}{\partial p}, \frac{\partial z}{\partial p} \right)^T, \qquad \mathbf{x}_q = \frac{\partial \mathbf{x}}{\partial q} = \left(\frac{\partial x}{\partial q}, \frac{\partial y}{\partial q}, \frac{\partial z}{\partial q} \right)^T, \tag{5.8}$$

which therefore span the tangent plane to the surface at the point $\mathbf{x}(p,q) \in S$. The first basis vector is tangent to the curves where $q =$ constant, while the second is tangent to the curves where $p =$ constant.

Example 5.4. Consider the torus T parametrized as in (5.5). The basis tangent vectors are

$$\frac{\partial \mathbf{x}}{\partial \psi} = \begin{pmatrix} -(2+\cos\theta)\sin\psi \\ (2+\cos\theta)\cos\psi \\ 0 \end{pmatrix}, \qquad \frac{\partial \mathbf{x}}{\partial \theta} = \begin{pmatrix} -\sin\theta\,\cos\psi \\ -\sin\theta\,\sin\psi \\ \cos\theta \end{pmatrix}. \tag{5.9}$$

They serve to span the tangent plane to the torus at the point $\mathbf{x}(\theta,\psi)$. For example, at the point $\mathbf{x}(0,0) = (\,3,0,0\,)^T$ corresponding to the particular parameter values $\theta = \psi = 0$, the basis tangent vectors are

$$\mathbf{x}_\psi(0,0) = (\,0,3,0\,)^T = 3\,\mathbf{e}_2\,, \qquad \mathbf{x}_\theta(0,0) = (\,0,0,1\,)^T = \mathbf{e}_3\,,$$

and so the tangent plane at this particular point is the (y,z)–plane spanned by the standard basis vectors $\mathbf{e}_2, \mathbf{e}_3$.

The tangent to any curve contained within the torus at the given point will be a linear combination of these two vectors. For instance, the toroidal knot (3.5) corresponds to the straight line

$$\psi(t) = 2t, \qquad 0 \le t \le 2\pi, \qquad \theta(t) = 3t,$$

in the parameter space. Its tangent vector

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} -(4+2\cos 3t)\sin 2t - 3\,\sin 3t\,\cos 2t \\ (4+2\cos 3t)\cos 2t - 3\,\sin 3t\,\sin 2t \\ 3\,\cos 3t \end{pmatrix}$$

lies in the tangent plane to the torus at each point. In particular, at $t = 0$, the knot passes through the point $\mathbf{x}(0,0) = (\,3,0,0\,)^T$, and has tangent vector

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 0 \\ 6 \\ 3 \end{pmatrix} = 2\,\mathbf{x}_\psi(0,0) + 3\,\mathbf{x}_\theta(0,0) \qquad \text{since} \qquad \frac{d\psi}{dt} = 2,\ \frac{d\theta}{dt} = 3.$$

A point $\mathbf{x}(p,q) \in S$ on the surface is said to be *nonsingular* provided the basis tangent vectors $\mathbf{x}_p(p,q), \mathbf{x}_q(p,q)$ are linearly independent. Thus the point is nonsingular if and only if the tangent vectors span a full two-dimensional subspace of $\mathbb{R}^3$ — the tangent plane to the surface at the point. Nonsingularity ensures the smoothness of the surface at each point, which is a consequence of the general Implicit Function Theorem, [**12**]. Singular points, where the tangent vectors are linearly dependent, can take the form of corners,

cusps and folds in the surface. From now on, we shall always assume that our surface is *nonsingular* meaning every point is a nonsingular point.

Linear independence of the tangent vectors is equivalent to the requirement that their cross product is a nonzero vector:

$$\mathbf{N} = \frac{\partial \mathbf{x}}{\partial p} \times \frac{\partial \mathbf{x}}{\partial q} = \left(\frac{\partial(y,z)}{\partial(p,q)}, \frac{\partial(z,x)}{\partial(p,q)}, \frac{\partial(x,y)}{\partial(p,q)} \right)^T \neq \mathbf{0}. \tag{5.10}$$

In this formula, we have adopted the standard notation

$$\frac{\partial(x,y)}{\partial(p,q)} = \det \begin{pmatrix} x_p & x_q \\ y_p & y_q \end{pmatrix} = \frac{\partial x}{\partial p}\,\frac{\partial y}{\partial q} - \frac{\partial x}{\partial q}\,\frac{\partial y}{\partial p} \tag{5.11}$$

for the *Jacobian determinant* of the functions x, y with respect to the variables p, q. The cross-product vector $\mathbf{N}$ in (5.10) is orthogonal to both tangent vectors, and hence orthogonal to the entire tangent plane. Therefore, $\mathbf{N}$ defines a *normal* vector to the surface at the given (nonsingular) point.

Example 5.5. Consider a surface S parametrized as the graph of a function $z = u(x,y)$, and so, as in Example 5.1

$$\mathbf{x}(x,y) = \big(x, y, u(x,y) \big)^T, \qquad\qquad (x,y) \in \Omega.$$

The tangent vectors

$$\frac{\partial \mathbf{x}}{\partial x} = \left(1, 0, \frac{\partial u}{\partial x} \right)^T, \qquad\qquad \frac{\partial \mathbf{x}}{\partial y} = \left(0, 1, \frac{\partial u}{\partial y} \right)^T,$$

span the tangent plane sitting at the point $(x, y, u(x,y)$ on S. The normal vector is

$$\mathbf{N} = \frac{\partial \mathbf{x}}{\partial x} \times \frac{\partial \mathbf{x}}{\partial y} = \left(-\frac{\partial u}{\partial x}, -\frac{\partial u}{\partial y}, 1 \right)^T,$$

and points upwards. Note that every point on the graph is nonsingular.

The *unit normal* to the surface at the point is a unit vector orthogonal to the tangent plane, and hence given by

$$\mathbf{n} = \frac{\mathbf{N}}{\| \mathbf{N} \|} = \frac{\mathbf{x}_p \times \mathbf{x}_q}{\| \mathbf{x}_p \times \mathbf{x}_q \|}. \tag{5.12}$$

In general, the direction of the normal vector $\mathbf{N}$ depends upon the order of the two parameters p, q. Computing the cross product in the reverse order, $\mathbf{x}_q \times \mathbf{x}_p = -\mathbf{N}$, reverses the sign of the normal vector, and hence switches its direction. Thus, there are two possible unit normals to the surface at each point, namely $\mathbf{n}$ and $-\mathbf{n}$. For a closed surface, one normal points outwards and one points inwards.

When possible, a consistent (meaning continuously varying) choice of a unit normal serves to define an *orientation* of the surface. All closed surfaces, and most other surfaces can be oriented. The usual convention for closed surfaces is to choose the orientation defined by the outward normal. The simplest example of a non-orientable surface is the Möbius strip obtained by gluing together the ends of a twisted strip of paper.

Example 5.6. For the sphere of radius r parametrized by the spherical angles as in (5.4), the tangent vectors are

$$\frac{\partial \mathbf{x}}{\partial \varphi} = \begin{pmatrix} r\cos\varphi\cos\theta \\ r\sin\varphi\cos\theta \\ -r\,\sin\varphi \end{pmatrix}, \qquad \frac{\partial \mathbf{x}}{\partial \theta} = \begin{pmatrix} -r\sin\varphi\sin\theta \\ r\sin\varphi\cos\theta \\ 0 \end{pmatrix}.$$

These vectors are tangent to, respectively, the meridians of constant longitude, and the parallels of constant latitude. The normal vector is

$$\mathbf{N} = \frac{\partial \mathbf{x}}{\partial \varphi} \times \frac{\partial \mathbf{x}}{\partial \theta} = \begin{pmatrix} r^2\sin^2\varphi\cos\theta \\ r^2\sin^2\varphi\sin\theta \\ r^2\cos\varphi\sin\varphi \end{pmatrix} = r\sin\varphi\ \mathbf{x}. \tag{5.13}$$

Thus $\mathbf{N}$ is a non-zero multiple of the radial vector $\mathbf{x}$, except at the north or south poles when $\varphi = 0$ or π. This reconfirms our earlier observation that the poles are problematic points for the spherical angle parametrization. The unit normal

$$\mathbf{n} = \frac{\mathbf{N}}{\| \mathbf{N} \|} = \frac{\mathbf{x}}{r}$$

determined by the spherical coordinates φ, θ is the outward pointing normal. Reversing the order of the angles, θ, φ, would lead to the outwards normal $-\mathbf{n} = -\mathbf{x}/r$.

Remark: As we already saw in the example of the hemisphere, a given surface can be parametrized in many different ways. In general, to change parameters

$$p = g(\widetilde{p}, \widetilde{q}), \qquad q = h(\widetilde{p}, \widetilde{q}),$$

requires a smooth, invertible map between the two parameter domains $\widetilde{\Omega} \to \Omega$. Many interesting surfaces, particularly closed surfaces, cannot be parametrized in a single consistent manner that satisfies the smoothness constraint (5.10) on the entire surface. In such cases, one must assemble the surface out of pieces, each parametrized in the proper manner. The key problem in cartography is to find convenient parametrizations of the globe that do not significantly distort the geographical features of the planet.

A surface is *piecewise smooth* if it can be constructed by gluing together a finite number of smooth parts, joined along piecewise smooth curves. For example, a cube is a piecewise smooth surface, consisting of squares joined along straight line segments. We shall rely on the reader's intuition to formalize these ideas, leaving a rigorous development to a more comprehensive treatment of surface geometry, e.g., [**5**].

6. Surface Integrals.

As with spatial line integrals, there are two important types of surface integral. The first is the integration of a scalar field with respect to surface area. A typical application is to compute the area of a curved surface or the mass and center of mass of a curved shell of possibly variable density. The second type is the surface integral that computes the flux

associated with a vector field through an oriented surface. Applications appear in fluid mechanics, electromagnetism, thermodynamics, gravitation, and many other fields.

Surface Area

According to (2.10), the length of the cross product of two vectors measures the area of the parallelogram they span. This observation underlies the proof that the length of the normal vector to a surface (5.12), namely

$$\| \mathbf{N} \| = \| \mathbf{x}_p \times \mathbf{x}_q \|,$$

is a measure of the infinitesimal element of surface area, denoted

$$dS = \| \mathbf{N} \| \, dp \, dq = \| \mathbf{x}_p \times \mathbf{x}_q \| \, dp \, dq. \tag{6.1}$$

The total area of the surface is found by summing up these infinitesimal contributions, and is therefore given by the double integral

$$\begin{aligned} \text{area } S = \iint_S dS &= \iint_\Omega \| \mathbf{x}_p \times \mathbf{x}_q \| \, dp \, dq \\ &= \iint_\Omega \sqrt{\left(\frac{\partial(y,z)}{\partial(p,q)} \right)^2 + \left(\frac{\partial(z,x)}{\partial(p,q)} \right)^2 + \left(\frac{\partial(x,y)}{\partial(p,q)} \right)^2} \; dp \, dq. \end{aligned} \tag{6.2}$$

The surface's area does not depend upon the parametrization used to compute the integral. In particular, if the surface is parametrized by x, y as the graph $z = u(x,y)$ of a function over a domain $(x,y) \in \Omega$, then the surface area integral reduces to the familiar form

$$\text{area } S = \iint_S dS = \iint_\Omega \sqrt{1 + \left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)} \; dx \, dy. \tag{6.3}$$

A detailed justification of these formulae can be found in [**1**, **6**, **9**].

Example 6.1. The well-known formula for the surface area of a sphere is a simple consequence of the integral formula (6.2). Using the parametrization by spherical angles (5.4) and the formula (5.13) for the normal, we find

$$\text{area } S_r = \iint_{S_r} dS = \int_0^{2\pi} \int_0^{\pi} r^2 \, \sin \varphi \; d\varphi \, d\theta = 4 \, \pi \, r^2. \tag{6.4}$$

Fortunately, the problematic poles do not cause any difficulty in the computation, since they contribute nothing to the surface area integral.

Alternatively, we can compute the area of one hemisphere S_r^+ by realizing it as a graph

$$z = \sqrt{r^2 - x^2 - y^2} \qquad \text{for} \qquad x^2 + y^2 \leq 1,$$

over the disk of radius r, and so, by (6.3),

$$\begin{aligned} \text{area } S_r^+ &= \iint_\Omega \sqrt{1 + \frac{x^2}{r^2 - x^2 - y^2} + \frac{y^2}{r^2 - x^2 - y^2}} \; dx \, dy \\ &= \iint_\Omega \frac{r}{\sqrt{r^2 - x^2 - y^2}} \; dx \, dy = \int_0^r \int_0^{2\pi} \frac{r \, \rho}{\sqrt{r^2 - \rho^2}} \, d\theta \, d\rho = 2 \, \pi \, r^2, \end{aligned}$$

where we used polar coordinates $x = \rho \cos\theta, y = \rho \sin\theta$ to evaluate the final integral. The area of the entire sphere is twice the area of the hemisphere.

Example 6.2. Similarly, to compute the surface area of the torus T parametrized in (5.5), we use the tangent vectors in (5.9) to compute the normal to the torus:

$$\mathbf{N} = \mathbf{x}_\psi \times \mathbf{x}_\theta = \begin{pmatrix} (2+\cos\psi)\cos\psi\cos\theta \\ (2+\cos\psi)\cos\psi\sin\theta \\ (2+\cos\psi)\sin\psi \end{pmatrix}, \qquad \text{with} \qquad \| \mathbf{x}_\psi \times \mathbf{x}_\theta \| = 2 + \cos\psi.$$

Therefore,

$$\text{area } T = \int_0^{2\pi} \int_0^{2\pi} (2+\cos\psi)\, d\psi\, d\theta = 8\,\pi^2.$$

If $S \subset \mathbb{R}^3$ is a surface with finite area, the *mean* or *average* of a scalar function $f(x,y,z)$ over S is given by

$$\mathrm{M}_S[\,f\,] = \frac{1}{\text{area } S} \iint_S f\, dS. \tag{6.5}$$

For example, the mean of a function over a sphere $S_r = \{\, \| \mathbf{x} \| = r \,\}$ of radius r is explicitly given by

$$\mathrm{M}_{S_r}[\,f\,] = \frac{1}{4\,\pi\, r^2} \iint_{\| \mathbf{x} \| = r} f(\mathbf{x})\, dS = \frac{1}{4\,\pi} \int_0^{2\pi} \int_0^{\pi} F(r,\varphi,\theta) \sin\varphi\, d\varphi\, d\theta, \tag{6.6}$$

where $F(r,\varphi,\theta)$ is the spherical coordinate expression for the scalar function f. As usual, the mean lies between the maximum and minimum values of the function on the surface:

$$\min_S f \le \mathrm{M}_S[\,f\,] \le \max_S f.$$

In particular, the *center of mass* $\mathbf{C}$ of a surface (assuming it has constant density) is equal to the mean of the coordinate functions $\mathbf{x} = (\,x,y,z\,)^T$, so

$$\mathbf{C} = (\,\mathrm{M}_S[\,x\,], \mathrm{M}_S[\,y\,], \mathrm{M}_S[\,z\,]\,)^T = \frac{1}{\text{area } S} \left(\iint_S x\, dS, \iint_S y\, dS, \iint_S z\, dS \right)^T. \tag{6.7}$$

More generally, the integral of a scalar field $u(x,y,z)$ over the surface is given by

$$\iint_S u\, dS = \iint_\Omega u(x(p,q), y(p,q), z(p,q)) \, \| \mathbf{x}_p \times \mathbf{x}_q \| \, dp\, dq. \tag{6.8}$$

If S represents a thin curved shell, and $u = \rho(\mathbf{x})$ the density of the material at position $\mathbf{x} \in S$, then the surface integral (6.8) represents the total mass of the shell. For example, the integral of $u(x,y,z)$ over a hemisphere S_r^+ of radius r can be evaluated by either of the formulae

$$\begin{aligned} \iint_{S_r^+} u\, dS &= \int_0^{2\pi} \int_0^{\pi/2} u(r\cos\theta\sin\varphi, r\sin\theta\sin\varphi, r\cos\varphi)\; r^2 \sin\varphi\; d\varphi\, d\theta \\ &= \iint_{x^2+y^2\le r^2} \frac{r}{\sqrt{r^2-x^2-y^2}}\; u(x,y,\sqrt{r^2-x^2-y^2}\,)\; dx\, dy, \end{aligned} \tag{6.9}$$

depending upon whether one prefers spherical or graphical coordinates.

Flux Integrals

Now assume that S is an oriented surface with chosen unit normal $\mathbf{n}$. If $\mathbf{v} = (\, u, v, w \,)^T$ is a vector field, then the *surface integral*

$$\iint_S \mathbf{v} \cdot \mathbf{n} \, dS = \iint_\Omega \left(\mathbf{v} \cdot \mathbf{x}_p \times \mathbf{x}_q \right) \, dp \, dq = \iint_\Omega \det \begin{pmatrix} u & x_p & x_q \\ v & y_p & y_q \\ w & z_p & z_q \end{pmatrix} dp \, dq \tag{6.10}$$

of the normal component of $\mathbf{v}$ over the entire surface measures its *flux* through the surface. An alternative common notation for the flux integral is

$$\begin{aligned} \iint_S \mathbf{v} \cdot \mathbf{n} \, dS &= \iint_S u \, dy \, dz + v \, dz \, dx + w \, dx \, dy \\ &= \iint_\Omega \left(u(x,y,z) \, \frac{\partial(y,z)}{\partial(p,q)} + v(x,y,z) \, \frac{\partial(z,x)}{\partial(p,q)} + w(x,y,z) \, \frac{\partial(x,y)}{\partial(p,q)} \right) dx \, dy, \end{aligned} \tag{6.11}$$

Note how the Jacobian determinant notation (5.11) seamlessly interacts with the integration. In particular, if the surface is the graph of a function $z = h(x,y)$, then the surface integral reduces to the particularly simple form

$$\iint_S \mathbf{v} \cdot \mathbf{n} \, dS = \iint_\Omega \left(u(x,y,z) \, \frac{\partial z}{\partial x} + v(x,y,z) \, \frac{\partial z}{\partial y} + w(x,y,z) \right) dp \, dq \tag{6.12}$$

The flux surface integral relies upon the consistent choice of an orientation or unit normal on the surface. Thus, flux only makes sense through an oriented surface — it doesn't make sense to speak of "flux through a Möbius band". If we switch normals, using, say, the inward instead of the outward normal, then the surface integral changes sign — just like a line integral if we reverse the orientation of a curve. Similarly, if we decompose a surface into the union of two or more parts, with only their boundaries in common, then the surface integral similarly decomposes into a sum of surface integrals. Thus,

$$\begin{aligned} \iint_{-S} \mathbf{v} \cdot \mathbf{n} \, dS &= - \iint_S \mathbf{v} \cdot \mathbf{n} \, dS, \\ \iint_S \mathbf{v} \cdot \mathbf{n} \, dS &= \iint_{S_1} \mathbf{v} \cdot \mathbf{n} \, dS \; + \; \iint_{S_2} \mathbf{v} \cdot \mathbf{n} \, dS, \qquad S = S_1 \cup S_2. \end{aligned} \tag{6.13}$$

In the first formula, $-S$ denotes the surface S with the reverse orientation. In the second formula, S_1 and S_2 are only allowed to intersect along their boundaries; moreover, they must be oriented in the same manner as S, i.e., have the same unit normal direction.

Example 6.3. Let S denote the triangular surface given by that portion of the plane $x + y + z = 1$ that lies inside the positive orthant $\{ x \geq 0, \; y \geq 0, \; z \geq 0 \}$. The flux of the vector field $\mathbf{v} = (\, y, x\,z, 0 \,)^T$ through S equals the surface integral

$$\iint_S y \, dy \, dz + x \, z \, dz \, dx,$$

where we orient S by choosing the upwards pointing normal. To compute, we note that S can be identified as the graph of the function $z = 1 - x - y$ lying over the triangle $T = \{\, 0 \le x \le 1, 0 \le y \le 1 - x \,\}$. Therefore, by (6.11),

$$\iint_S y\,dy\,dz + x\,z\,dz\,dx = \iint_T \left[y\,\frac{\partial(y, 1-x-y)}{\partial(x,y)} + x\,(1-x-y)\,\frac{\partial(1-x-y,x)}{\partial(x,y)} \right] dx\,dy$$
$$= \int_0^1 \int_0^{1-x} (1-x)\,(y+x)\,dy\,dx = \int_0^1 \left(\tfrac{1}{2} + \tfrac{1}{2}\,x - \tfrac{1}{2}\,x^2 + \tfrac{1}{2}\,x^3 \right) dx = \tfrac{17}{24}.$$

If $\mathbf{v}$ represents the velocity vector field for a steady state fluid flow, then its flux integral (6.10) tells us the total volume of fluid passing through S per unit time. Indeed, at each point on S, the volume fluid that flows across a small part the surface in unit time will fill a thin cylinder whose base is the surface area element dS and whose height $\mathbf{v} \cdot \mathbf{n}$ is the normal component of the fluid velocity $\mathbf{v}$. Summing (integrating) all these flux cylinder volumes over the surface results in the flux integral. The choice of orientation or unit normal specifies the convention for measuring the direction of positive flux through the surface. If S is a closed surface, and we choose $\mathbf{n}$ to be the unit outward normal, then the flux integral (6.10) represents the net amount of fluid flowing *out* of the solid region bounded by S per unit time.

Example 6.4. The vector field $\mathbf{v} = (\,0, 0, 1\,)^T$ represents a fluid moving with constant velocity in the vertical direction. Let us compute the fluid flux through a hemisphere

$$S_r^+ = \left\{ \, z = \sqrt{r^2 - x^2 - y^2} \;\middle|\; x^2 + y^2 \le 1 \, \right\},$$

sitting over the disk D_r of radius r in the x, y plane. The flux integral over S_r^+ is computed using (6.12), so

$$\iint_{S_r^+} \mathbf{v} \cdot \mathbf{n}\; dS = \iint_{S_r^+} dx \times dy = \iint_{D_r} dx\,dy = \pi\, r^2.$$

The resulting double integral is just the area of the disk. Indeed, in this case, the value of the flux integral is the *same* for all surfaces $z = h(x, y)$ sitting over the disk D_r.

This example provides a particular case of a surface-independent flux integral, which are defined in analogy with the path-independent line integrals that we encountered earlier. In general, a flux integral is called *surface-independent* if

$$\iint_{S_1} \mathbf{v} \cdot \mathbf{n}\; dS = \iint_{S_2} \mathbf{v} \cdot \mathbf{n}\; dS \tag{6.14}$$

whenever the surfaces S_1 and S_2 have a common boundary $\partial S_1 = \partial S_2$. In other words, the value of the integral depends only upon the boundary of the surface. The veracity of (6.14) requires that the surfaces be oriented in the "same manner". For instance, if they do not cross, then the combined surface $S = S_1 \cup S_2$ is closed, and one uses the outward pointing normal on one surface and the inward pointing normal on the other. In more complex situations, one checks that the two surfaces induce the same orientation on their

common boundary. (We defer a discussion of the boundary orientation until later.) Finally, applying (6.13) to the closed surface $S = S_1 \cup S_2$ and using the prescribed orientations, we deduce an alternative characterization of surface-independent vector fields.

Proposition 6.5. *A vector field leads to a surface-independent flux integral if and only if*

$$\iint_S \mathbf{v} \cdot \mathbf{n} \, dS = 0 \tag{6.15}$$

for every closed surface S contained in the domain of definition of $\mathbf{v}$.

A fluid is *incompressible* when its volume is unaltered by the flow. Therefore, in the absence of sources or sinks, there cannot be any net inflow or outflow across a simple closed surface bounding a region occupied by the fluid. Thus, the flux integral over a closed surface must vanish: $\iint_S \mathbf{v} \cdot \mathbf{n} \, dS = 0$. Proposition 6.5 implies that the fluid velocity vector field defines a surface-independent flux integral. Thus, the flux of an incompressible fluid flow through any surface depends only on the (oriented) boundary curve of the surface!

7. Volume Integrals.

Volume or triple integrals take place over domains $\Omega \subset \mathbb{R}^3$ representing solid three-dimensional bodies. A simple example of such a domain is a *ball*

$$\mathrm{B}_r(\mathbf{a}) = \{ \, \mathbf{x} \mid \, \| \, \mathbf{x} - \mathbf{a} \, \| < r \, \} \tag{7.1}$$

of radius $r > 0$ centered at a point $\mathbf{a} \in \mathbb{R}^3$. Other examples of domains include solid cubes, solid cylinders, solid tetrahedra, solid tori (doughnuts and bagels), solid cones, etc.

In general, a subset $\Omega \subset \mathbb{R}^3$ is *open* if, for every point $\mathbf{x} \in \Omega$, a small open ball $\mathrm{B}_\varepsilon(\mathbf{a}) \subset \Omega$ centered at $\mathbf{a}$ of radius $\varepsilon = \varepsilon(\mathbf{a}) > 0$, which may depend upon $\mathbf{a}$, is also contained in Ω. In particular, the ball (7.1) is open. The *boundary* $\partial\Omega$ of an open subset Ω consists of all limit points which are not in the subset. Thus, the boundary of the open ball $\mathrm{B}_r(\mathbf{a})$ is the sphere $\mathrm{S}_r(\mathbf{a}) = \{ \, \| \, \mathbf{x} - \mathbf{a} \, \| = r \, \}$ of radius r centered at the point a. An open subset is called a *domain* if its boundary $\partial\Omega$ consists of one or more simple, piecewise smooth surfaces. We are allowing corners and edges in the bounding surfaces, so that an open cube will be a perfectly valid domain.

A subset $\Omega \subset \mathbb{R}^3$ is *bounded* provided it fits inside a sphere of some (possibly large) radius. For example, the solid ball $\mathrm{B}_r = \{ \, \| \, \mathbf{x} \, \| < R \}$ is bounded, while its exterior $\mathrm{E}_r = \{ \, \| \, \mathbf{x} \, \| > R \}$ is an unbounded domain. The sphere $S_R = \{ \, \| \, \mathbf{x} \, \| = R \}$ is the common boundary of the two domains: $S_R = \partial B_r = \partial E_R$. Indeed, any simple closed surface separates $\mathbb{R}^3$ into two domains that have a common boundary — its *interior*, which is bounded, and its unbounded *exterior*.

The boundary of a bounded domain consists of one or more closed surfaces. For instance, the solid annular domain

$$A_{r,R} = \big\{ \, 0 < r < \| \, \mathbf{x} \, \| < R \, \big\} \tag{7.2}$$

consisting of all points lying between two concentric spheres of respective radii r and R has boundary given by the two spheres: $\partial A_{r,R} = S_r \cup S_R$. On the other hand, setting $r = 0$ in (7.2) leads to a *punctured ball* of radius R whose center point has been removed. A punctured ball is *not* a domain, since the center point is part of the boundary, but is not a *bona fide* surface.

If the domain $\Omega \subset \mathbb{R}^3$ represents a solid body, and the scalar field $\rho(x, y, z)$ represents its density at a point $(x, y, z) \in \Omega$, then the triple integral

$$\iiint_\Omega \rho(x, y, z)\, dx\, dy\, dz \tag{7.3}$$

equals the total mass of the body. In particular, the volume of Ω is equal to

$$\operatorname{vol} \Omega = \iiint_\Omega dx\, dy\, dz. \tag{7.4}$$

Triple integrals can be directly evaluated when the domain has the particular form

$$\Omega = \big\{\, \xi(x, y) < z < \eta(x, y), \quad \varphi(x) < y < \varphi(x), \quad a < x < b \,\big\} \tag{7.5}$$

where the z coordinate lies between two graphical surfaces sitting over a common domain in the (x, y)–plane that is itself of the form of (7.5) used to evaluate double integrals. In such cases we can evaluate the triple integral by iterated integration first with respect to z, then with respect to y and, finally, with respect to x:

$$\iiint_\Omega u(x, y, z)\, dx\, dy\, dz = \int_a^b \left(\int_{\varphi(x)}^{\psi(x)} \left(\int_{\xi(x,y)}^{\eta(x,y)} u(x, y, z)\, dz \right) dy \right) dx. \tag{7.6}$$

A similar result holds for other orderings of the coordinates.

Fubini's Theorem, [**11**, **12**], assures us that the result of iterated integration does not depend upon the order in which the variables are integrated. Of course, the domain must be of the requisite type in order to write the volume integral as repeated single integrals. More general triple integrals can be evaluated by chopping the domain up into disjoint pieces that have the proper form.

Example 7.1. The volume of a solid ball B_R of radius R can be computed as follows. We express the domain of integration $x^2 + y^2 + z^2 < R^2$ in the form

$$-R < x < R, \quad -\sqrt{R^2 - x^2} < y < \sqrt{R^2 - x^2}\,, \quad -\sqrt{R^2 - x^2 - y^2} < z < \sqrt{R^2 - x^2 - y^2}\,.$$

Therefore, in accordance with (7.6),

$$\begin{aligned}
\iiint_{B_R} dx\,dy\,dz &= \int_{-R}^{R} \left(\int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} \left(\int_{-\sqrt{R^2-x^2-y^2}}^{\sqrt{R^2-x^2-y^2}} dz \right) dy \right) dx \\
&= \int_{-R}^{R} \left(\int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} 2\sqrt{R^2-x^2-y^2}\,dy \right) dx \\
&= \int_{-R}^{R} \left(y\sqrt{R^2-x^2-y^2} + (R^2-x^2)\sin^{-1}\frac{y}{\sqrt{R^2-x^2}} \right) \Bigg|_{y=-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} dx \\
&= \int_{-R}^{R} \pi(R^2-x^2)\,dx = \pi\left(R^2x - \frac{x^3}{3} \right) \Bigg|_{x=-R}^{R} = \frac{4}{3}\pi R^3,
\end{aligned}$$

recovering the standard formula, as it should.

Change of Variables

Sometimes, an inspired change of variables can be used to simplify a volume integral. If

$$x = f(p,q,r), \qquad y = g(p,q,r), \qquad z = h(p,q,r), \tag{7.7}$$

is an invertible change of variables — meaning that each point (x,y,z) corresponds to a unique point (p,q,r) — then

$$\iiint_{\Omega} u(x,y,z)\,dx\,dy\,dz = \iiint_{D} U(p,q,r) \left| \frac{\partial(x,y,z)}{\partial(p,q,r)} \right| dp\,dq\,dr. \tag{7.8}$$

Here

$$U(p,q,r) = u(x(p,q,r), y(p,q,r), z(p,q,r))$$

is the expression for the integrand in the new coordinates, while D is the domain consisting of all points (p,q,r) that map to points $(x,y,z) \in \Omega$ in the original domain. Invertibility requires that each point in D corresponds to a unique point in Ω. The change in volume is governed by the absolute value of the three-dimensional *Jacobian determinant*

$$\frac{\partial(x,y,z)}{\partial(p,q,r)} = \det \begin{pmatrix} x_p & x_q & x_r \\ y_p & y_q & y_r \\ z_p & z_q & z_r \end{pmatrix} = \mathbf{x}_p \cdot \mathbf{x}_q \times \mathbf{x}_r \tag{7.9}$$

for the change of variables. The identification of the vector triple product (7.9) with an (infinitesimal) volume element lies behind the justification of the change of variables formula; see [**1**, **6**, **9**] for a detailed proof.

By far, the two most important cases are cylindrical and spherical coordinates. *Cylindrical coordinates* correspond to replacing the x and y coordinates by their polar counterparts, while retaining the vertical z coordinate unchanged. Thus, the change of coordinates has the form

$$x = r\,\cos\theta, \qquad y = r\,\sin\theta, \qquad z = z. \tag{7.10}$$

The Jacobian determinant for cylindrical coordinates is

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \det \begin{pmatrix} x_r & x_\theta & x_z \\ y_r & y_\theta & y_z \\ z_r & z_\theta & z_z \end{pmatrix} = \det \begin{pmatrix} \cos\theta & -r\,\sin\theta & 0 \\ \sin\theta & r\,\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = r. \tag{7.11}$$

Therefore, the general change of variables formula (7.8) tells us the formula for a triple integral in cylindrical coordinates:

$$\iiint f(x,y,z)\,dx\,dy\,dz = \iiint f(r\,\cos\theta, r\,\sin\theta, z)\,r\,dr\,d\theta\,dz. \tag{7.12}$$

Example 7.2. For example, consider an ice cream cone

$$C_h = \big\{\, x^2 + y^2 < z^2, \;\; 0 < z < h \,\big\} = \big\{\, r < z, \;\; 0 < z < h \,\big\}$$

of height h. To compute its volume, we express the domain in terms of the cylindrical coordinates, leading to

$$\iiint_{C_h} dx\,dy\,dz = \int_0^h \int_0^{2\,\pi} \int_0^z r\,dr\,d\theta\,dz = \int_0^h \pi\,z^2\,dz = \tfrac{1}{3}\,\pi\,h^3.$$

Spherical coordinates are denoted by r, φ, θ, where

$$x = r\,\sin\varphi\,\cos\theta, \qquad y = r\,\sin\varphi\,\sin\theta, \qquad z = r\,\cos\varphi. \tag{7.13}$$

Here $r = \| \mathbf{x} \| = \sqrt{x^2 + y^2 + z^2}$ represents the radius, $0 \le \varphi \le \pi$ is the zenith angle or latitude, while $0 \le \theta < 2\,\pi$ is the azimuthal angle or longitude. The reader may recall that we already encountered these coordinates in our parametrization (5.4) of the sphere. It is important to distinguish between the spherical r, θ and the cylindrical r, θ — even though the same symbols are used, they represent *different* quantities.

Warning: In many books, particularly those in physics, the roles of θ and φ are *reversed*, leading to much confusion when perusing the literature. We prefer the mathematical convention as the azimuthal angle θ agrees with its cylindrical counterpart. You need to be very careful to determine which convention is being used when consulting any reference!

A short computation proves that the spherical coordinate Jacobian determinant is

$$\begin{aligned} \frac{\partial(x,y,z)}{\partial(r,\varphi,\theta)} &= \det \begin{pmatrix} x_r & x_\varphi & x_\theta \\ y_r & y_\varphi & y_\theta \\ z_r & z_\varphi & z_\theta \end{pmatrix} \\ &= \det \begin{pmatrix} \sin\varphi\,\cos\theta & r\,\cos\varphi\,\cos\theta & -r\,\sin\varphi\,\sin\theta \\ \sin\varphi\,\sin\theta & r\,\cos\varphi\,\sin\theta & r\,\sin\varphi\,\cos\theta \\ \cos\varphi & -r\,\sin\varphi & 0 \end{pmatrix} = r^2\,\sin\varphi. \end{aligned} \tag{7.14}$$

Therefore, a triple integral is evaluated in spherical coordinates according to the formula

$$\iiint f(x,y,z)\,dx\,dy\,dz = \iiint F(r,\varphi,\theta)\,r^2\,\sin\varphi\;\,dr\,d\varphi\,d\theta, \tag{7.15}$$

where we rewrite the integrand

$$F(r, \varphi, \theta) = f(r \sin\varphi \cos\theta, r \sin\varphi \sin\theta, r \cos\varphi) \tag{7.16}$$

as a function of the spherical coordinates.

Example 7.3. The integration required in Example 7.1 to compute the volume of a ball B_R of radius R can be considerably simplified by switching over to spherical coordinates. The ball is given by $B_R = \big\{\, 0 \le r < R, 0 \le \varphi \le \pi, 0 \le \theta < 2\pi \,\big\}$. Thus, using (7.15), we compute

$$\iiint_{B_R} dx\, dy\, dz = \int_0^R \int_0^\pi \int_0^{2\pi} r^2 \sin\varphi \; d\theta\, d\varphi\, dr = \int_0^R 4\pi r^2\, dr = \tfrac{4}{3}\pi R^3. \tag{7.17}$$

The reader may note that the next-to-last integrand represents the surface area of the sphere of radius R. Thus, we are, in effect, computing the volume by summing up (i.e., integrating) the surface areas of concentric thin spherical shells.

Remark: Sometimes, we will be sloppy and use the same letter for a function in an alternative coordinate system. Thus, we may use $f(r, \varphi, \theta)$ to represent the spherical coordinate form (7.16) of a function $f(x, y, z)$. Technically, this is not correct! However, the clarity and intuition sometimes outweighs the pedantic use of a new letter each time we change coordinates. Moreover, in geometry and modern physical theories, [**2**], the symbol "f" represents an intrinsic scalar field, and $f(x, y, z)$ and $f(r, \varphi, \theta)$ merely its incarnations in two different coordinate charts on $\mathbb{R}^3$. Hopefully, this will be clear from the context.

8. Gradient, Divergence, and Curl.

There are three important vector differential operators that play a ubiquitous role in three-dimensional vector calculus, known as the gradient, divergence and curl.

The Gradient

We begin with the three-dimensional version of the *gradient* operator

$$\nabla u = \begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix}. \tag{8.1}$$

The gradient defines a linear operator that maps a scalar function $u(x, y, z)$ to the vector field whose components are its partial derivatives with respect to the Cartesian coordinates.

If $\mathbf{x}(t) = \big(\, x(t), y(t), z(t) \,\big)^T$ is any parametrized curve, then the rate of change in the function u as we move along the curve is given by the inner product

$$\frac{d}{dt}\, u(x(t), y(t), z(t)) = \frac{\partial u}{\partial x}\,\frac{dx}{dt} + \frac{\partial u}{\partial y}\,\frac{dy}{dt} + \frac{\partial u}{\partial z}\,\frac{dz}{dt} = \nabla u \cdot \dot{\mathbf{x}} \tag{8.2}$$

between the gradient and the tangent vector to the curve. Therefore, as we reasoned earlier in the planar case, the gradient ∇u points in the direction of steepest increase in the function u, while its negative $-\nabla u$ points in the direction of steepest decrease. For

example, if $u(x, y, z)$ represents the temperature at a point (x, y, z) in space, then ∇u points in the direction in which temperature is getting the hottest, while $-\nabla u$ points in the direction it gets the coldest. Therefore, if one wants to cool down as rapidly as possible, one should move in the direction of $-\nabla u$ at each instant, which is the direction of the flow of heat energy. Thus, the path $\mathbf{x}(t)$ to be followed for the fastest cool down will be a solution to the gradient flow equations

$$\dot{\mathbf{x}} = -\nabla u, \tag{8.3}$$

or, explicitly,

$$\frac{dx}{dt} = -\frac{\partial u}{\partial x}(x, y, z), \qquad \frac{dy}{dt} = -\frac{\partial u}{\partial y}(x, y, z), \qquad \frac{dz}{dt} = -\frac{\partial u}{\partial z}(x, y, z).$$

A solution $\mathbf{x}(t)$ to such a system of ordinary differential equations will experience continuously decreasing temperature. One can use such gradient flows to locate and numerically approximate the minima of functions, [**4**].

The set of all points where a scalar field $u(x, y, z)$ has a given value,

$$u(x, y, z) = c \tag{8.4}$$

for some fixed constant c, is known as a *level set* of u. If u measures temperature, then its level sets are the isothermal surfaces of equal temperature. If u is sufficiently smooth, most of its level sets are smooth surfaces. In fact, if $\nabla u \neq 0$ at a point, then one can prove that all nearby level sets are smooth surfaces near the point in question. This important fact is a consequence of the general Implicit Function Theorem, [**12**]. Thus, if $\nabla u \neq 0$ at all points on a level set, then the level set is a smooth surface, and, if bounded, a simple closed surface. (On the other hand, finding an explicit parametrization of a level set may be quite difficult!)

Theorem 8.1. *If nonzero, the gradient vector $\nabla u \neq \mathbf{0}$ defines the normal direction to the level set $\{\, u = c \,\}$ at each point.*

Proof: Indeed, suppose $\mathbf{x}(t)$ is any curve contained in the level set, so that

$$u(x(t), y(t), z(t)) = c \qquad \text{for all} \qquad t.$$

Since c is constant, the derivative with respect to t is zero, and hence, by (8.2),

$$\frac{d}{dt}\, u(x(t), y(t), z(t)) = \nabla u \cdot \dot{\mathbf{x}} = 0,$$

which implies that the gradient vector ∇u is orthogonal to the tangent vector $\dot{\mathbf{x}}$ to the curve. Since this holds for all such curves contained within the level set, the gradient must be orthogonal to the entire tangent plane at the point, and hence, if nonzero, defines a normal direction to the level surface. *Q.E.D.*

Physically, Theorem 8.1 tells us that the direction of steepest increase in temperature is perpendicular to the isothermal surfaces at each point. Consequently, the solutions to the gradient flow equations (8.3) form an orthogonal system of curves to the level set surfaces of u, and one should follow these curves to minimize the temperature as rapidly as possible. Similarly, in a steady state fluid flow, the fluid potential is represented by a scalar field $\varphi(x, y, z)$. Its gradient $\mathbf{v} = \nabla\varphi$ determines the fluid velocity at each point. The streamlines followed by the fluid particles are the solutions to the gradient flow equations $\dot{\mathbf{x}} = \mathbf{v} = \nabla\varphi$, while the level sets of φ are the equipotential surfaces. Thus, fluid particles flow in a direction orthogonal to the equipotential surfaces.

Example 8.2. The level sets of the radial function $u = x^2+y^2+z^2$ are the concentric spheres centered at the origin. Its gradient $\nabla u = (\,2\,x, 2\,y, 2\,z\,)^T = 2\,\mathbf{x}$ points in the radial direction, orthogonal to each spherical level set. Note that $\nabla u = \mathbf{0}$ only at the origin, which is a level set, but not a smooth surface.

The radial vector also specifies the direction of fastest increase (decrease) in the function u. Indeed, the solution to the associated gradient flow system (8.3), namely

$$\dot{\mathbf{x}} = -\,2\,\mathbf{x} \qquad \text{is} \qquad \mathbf{x}(t) = \mathbf{x}_0\, e^{-2\,t},$$

where $\mathbf{x}_0 = \mathbf{x}(0)$ is the initial position. Therefore, to decrease the function u as rapidly as possible, one should follow a radial ray into the origin.

Example 8.3. An implicit equation for the torus (5.5) is obtained by replacing $r = \sqrt{x^2 + y^2}$ in (5.6). In this manner, we are led to consider the level sets of the function

$$u(x, y, z) = x^2 + y^2 + z^2 - 4\sqrt{x^2 + y^2} \;=\; c, \tag{8.5}$$

with the particular value $c = -3$ corresponding to (5.5). The gradient of the function is

$$\nabla u(x, y, z) = \left(2\,x - \frac{4\,x}{\sqrt{x^2 + y^2}},\; 2\,y - \frac{4\,y}{\sqrt{x^2 + y^2}},\; 2\,z \right)^T, \tag{8.6}$$

which is well-define except on the z axis, where $x = y = 0$. Note that $\nabla F \neq \mathbf{0}$ unless $z = 0$ and $x^2 + y^2 = 4$. Therefore, the level sets of u are smooth, toroidal surfaces except for z axis and the circle of radius 2 in the (x, y) plane.

Divergence and Curl

The second important vector differential operator is the *divergence*,

$$\operatorname{div} \mathbf{v} = \nabla \cdot \mathbf{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}\ . \tag{8.7}$$

The divergence maps a vector field $\mathbf{v} = (\,v_1, v_2, v_3\,)^T$ to a scalar field $\mathbf{f} = \nabla \cdot \mathbf{v}$. For example, the radial vector field $\mathbf{v} = (\,x, y, z\,)^T$ has constant divergence $\nabla \cdot \mathbf{v} = 3$.

In fluid mechanics, the divergence measures the local, instantaneous change in the volume of a fluid packet as it moves. Thus, a steady state fluid flow is *incompressible*, with

unchanging volume, if and only if its velocity vector field is divergence-free: $\nabla \cdot \mathbf{v} \equiv 0$. The connection between incompressibility and the earlier zero-flux condition will be addressed in the Divergence Theorem 9.6 below.

As in the two-dimensional situation, the composition of divergence and gradient produces the Laplacian operator:

$$\nabla \cdot \nabla u = \Delta u = u_{xx} + u_{yy} + u_{zz}. \tag{8.8}$$

Indeed, as we shall see, except for the missing minus sign and the all-important boundary conditions, this is effectively the same as the self-adjoint form of the three-dimensional Laplacian:

$$\nabla^* \circ \nabla u = -\nabla \cdot \nabla u = -\Delta u.$$

The third important vector differential operator is the *curl*, which, in three dimensions, maps vector fields to vector fields. It is most easily memorized in the form of a (formal) 3×3 determinant

$$\operatorname{curl} \mathbf{v} = \nabla \times \mathbf{v} = \begin{pmatrix} \dfrac{\partial v_3}{\partial y} - \dfrac{\partial v_2}{\partial z} \\ \dfrac{\partial v_1}{\partial z} - \dfrac{\partial v_3}{\partial x} \\ \dfrac{\partial v_2}{\partial x} - \dfrac{\partial v_1}{\partial y} \end{pmatrix} = \det \begin{pmatrix} \partial_x & v_1 & \mathbf{e}_1 \\ \partial_y & v_2 & \mathbf{e}_2 \\ \partial_z & v_3 & \mathbf{e}_3 \end{pmatrix}, \tag{8.9}$$

in analogy with the determinantal form (2.6) of the cross product. For instance, the radial vector field $\mathbf{v} = (\, x, y, z \,)^T$ has zero curl:

$$\nabla \times \mathbf{v} = \det \begin{pmatrix} \partial_x & x & \mathbf{e}_1 \\ \partial_y & y & \mathbf{e}_2 \\ \partial_z & z & \mathbf{e}_3 \end{pmatrix} = \mathbf{0}.$$

This is indicative of the lack or any rotational effect of the induced flow.

If $\mathbf{v}$ represents the velocity vector field of a steady state fluid flow, its curl $\nabla \times \mathbf{v}$ measures the instantaneous rotation of the fluid flow at a point, and is known as the *vorticity* of the flow. When non-zero, the direction of the vorticity vector represents the axis of rotation, while its magnitude $\| \nabla \times \mathbf{v} \|$ measures the instantaneous angular velocity of the swirling flow. Physically, if we place a microscopic turbine in the fluid so that its shaft points in the direction specified by a unit vector $\mathbf{n}$, then its rate of spin will be proportional to component of the vorticity vector $\nabla \times \mathbf{v}$ in the direction of its shaft. This is equal to the dot product

$$\mathbf{n} \cdot (\nabla \times \mathbf{v}) = \| \nabla \times \mathbf{v} \| \, \cos \varphi,$$

where φ is the angle between $\mathbf{n}$ and the curl vector. Therefore, the maximal rate of spin will occur when $\varphi = 0$, and so the shaft of the turbine lines up with the direction of the vorticity vector $\nabla \times \mathbf{v}$. In this orientation, the angular velocity of the turbine will be proportional to its magnitude $\| \nabla \times \mathbf{v} \|$. On the other hand, if the axis of the turbine is

orthogonal to the direction of the vorticity, then it will not rotate. If $\nabla \times \mathbf{v} \equiv \mathbf{0}$, then there is no net motion of a turbine, not matter which orientation it is placed in the fluid flow. Thus, a flow with zero curl is *irrotational*. The precise connection between this definition and the earlier zero circulation condition will be explained shortly.

Example 8.4. Consider a helical fluid flow with velocity vector

$$\mathbf{v} = (\, -y, x, 1\,)^T .$$

Integrating the ordinary differential equations $\dot{\mathbf{x}} = \mathbf{v}$, namely

$$\dot{x} = -y, \qquad \dot{y} = x, \qquad \dot{z} = 1,$$

with initial conditions $x(0) = x_0$, $y(0) = y_0$, $z(0) = z_0$ gives the flow

$$x(t) = x_0 \cos t - y_0 \sin t, \qquad y(t) = x_0 \sin t + y_0 \cos t, \qquad z(t) = z_0 + t. \quad (8.10)$$

Therefore, the fluid particles move along helices spiraling up the z axis.

The divergence of the vector field $\mathbf{v}$ is

$$\nabla \cdot \mathbf{v} = \frac{\partial}{\partial x}(-y) + \frac{\partial}{\partial y} x + \frac{\partial}{\partial z} 1 = 0,$$

and hence the flow is incompressible. Indeed, any fluid packet will spiral up the z axis unchanged in shape, and so its volume does not change.

The vorticity or curl of the velocity is

$$\nabla \times \mathbf{v} = \begin{pmatrix} \dfrac{\partial}{\partial y} 1 - \dfrac{\partial}{\partial z} x \\ \dfrac{\partial}{\partial z}(-y) - \dfrac{\partial}{\partial x} 1 \\ \dfrac{\partial}{\partial x} x - \dfrac{\partial}{\partial y}(-y) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix},$$

which points along the z-axis. This reflects the fact that the flow is spiraling up the z-axis. If a turbine is placed in the fluid at an angle φ with the z-axis, then its rate of rotation will be proportional to $2 \cos \varphi$.

Example 8.5. Any planar vector field $\mathbf{v} = (\, v_1(x,y), v_2(x,y)\,)^T$ can be identified with a three-dimensional vector field

$$\mathbf{v} = (\, v_1(x,y), v_2(x,y), 0\,)^T$$

that has no vertical component. If $\mathbf{v}$ represents a fluid velocity, then the fluid particles remain on horizontal planes $\{\, z = c\,\}$, and the individual planar flows are identical. Its three-dimensional curl

$$\nabla \times \mathbf{v} = \left(0, 0, \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} \right)^T$$

is a purely vertical vector field.

Interconnections and Connectedness

The three basic vector differential operators — gradient, curl and divergence — are intimately inter-related. The proof of the key identities relies on the equality of mixed partial derivatives, which in turn requires that the functions involved are sufficiently smooth. We leave the explicit verification of the key result to the reader.

Proposition 8.6. *If u is a smooth scalar field, then $\nabla \times \nabla u \equiv \mathbf{0}$. If $\mathbf{v}$ is a smooth vector field, then $\nabla \cdot (\nabla \times \mathbf{v}) \equiv 0$.*

Therefore, the curl of any gradient vector field is automatically zero. As a consequence, all gradient vector fields represent irrotational flows. Also, the divergence of any vector field that is a curl is also automatically zero. Thus, all curl vector fields represent incompressible flows. On the other hand, the divergence of a gradient vector field is the Laplacian of the underlying potential, as we previously noted, and hence is zero if and only if the potential is a harmonic function.

The converse statements are almost true. As in the two-dimensional case, the precise statement of this result depends upon the topology of the underlying domain. In two dimensions, we only had to worry about whether or not the domain contained any holes, i.e., whether or not the domain was simply connected. Similar concerns arise in three dimensions. Moreover, there are two possible classes of "holes" in a solid domain — called tunnels and voids – and so there are two different types of connectivity. For lack of a better terminology, we introduce the following definition.

Definition 8.7. A domain $\Omega \subset \mathbb{R}^3$ is said to be

(a) 0–*connected* or *pathwise connected* if there is a curve $C \subset \Omega$ connecting any two points $\mathbf{x}_0, \mathbf{x}_1 \in \Omega$, so that[†] $\partial C = \{ \mathbf{x}_0, \mathbf{x}_1 \}$.

(b) 1–*connected* if every unknotted simple closed curve $C \subset \Omega$ is the boundary, $C = \partial S$ of an oriented surface $S \subset \Omega$.

(c) 2–*connected* if every simple closed surface $S \subset \Omega$ is the boundary, $S = \partial D$ of a subdomain $D \subset \Omega$.

Remark: The unknotted condition is to avoid considering "wild" curves that fail to bound any oriented surface $S \subset \mathbb{R}^3$ whatsoever.

For example, $\mathbb{R}^3$ is both 0, 1 and 2–connected, as are all solid balls, cubes, tetrahedra, solid cylinders, and so on. A disjoint union of balls is not 0–connected, although it does remain both 1 and 2–connected. The domain $\Omega = \big\{ \, 0 \le r < \sqrt{x^2 + y^2} < R \, \big\}$ lying between two cylinders is not 1–connected since it has a "one-dimensional" hole drilled through it. Indeed, if $C \subset \Omega$ is any closed curve that encircles the inner cylinder, then every bounding surface S with $\partial S = C$ must pass across the inner cylinder and hence will not lie entirely within the domain. On the other hand, this cylindrical domain Ω is both 0 and 2–connected — even an annular surface that encircles the inner cylinder will bound a solid annular domain contained inside Ω. Similarly, the domain $\Omega = \{ \, 0 \le r < \| \, \mathbf{x} \, \| < R \, \}$ between two

[†] We use the notation ∂C to denote the endpoints of a curve C.

concentric spheres is 0 and 1–connected, but not 2–connected owing to the spherical cavity inside. Any closed curve $C \subset \Omega$ will bound a surface $S \subset \Omega$; for instance, a circle going around the equator of the inner sphere will still bound a hemispherical surface that does not pass through the spherical cavity. On the other hand, a sphere that lies between the inner and outer spheres will not bound a solid domain contained within the domain. A full discussion of the topology underlying the various types of connectivity, the nature of tunnels and voids or cavities, and their connection with the existence of scalar and vector potentials, must be deferred to a more advanced course in algebraic topology, [**3**].

We can now state the basic theorem relating the connectivity of domains to the kernels of the fundamental vector differential operators.

Theorem 8.8. *Let $\Omega \subset \mathbb{R}^3$ be a domain.*

(a) *If Ω is 0–connected, then a scalar field $u(x, y, z)$ defined on all of Ω has vanishing gradient, $\nabla u \equiv \mathbf{0}$, if and only if $u(x, y, z) =$* constant.

(b) *If Ω is 1–connected, then a vector field $\mathbf{v}(x, y, z)$ defined on all of Ω has vanishing curl, $\nabla \times \mathbf{v} \equiv \mathbf{0}$, if and only if there is a scalar field φ, known as a* scalar potential *for $\mathbf{v}$, such that $\mathbf{v} = \nabla\varphi$.*

(c) *If Ω is 2–connected, then a vector field $\mathbf{v}(x, y, z)$ defined on all of Ω has vanishing divergence, $\nabla \cdot \mathbf{v} \equiv 0$, if and only if there is a vector field $\mathbf{w}$, known as a* vector potential *for $\mathbf{v}$, such that $\mathbf{v} = \nabla \times \mathbf{w}$.*

If $\mathbf{v}$ represents the velocity vector field of a steady-state fluid flow, then the curl-free condition $\nabla \times \mathbf{v} \equiv \mathbf{0}$ corresponds to an irrotational flow. Thus, on a 2–connected domain, every irrotational flow field $\mathbf{v}$ has a scalar potential φ with $\nabla\varphi = \mathbf{v}$. The divergence-free condition $\nabla \cdot \mathbf{v} \equiv \mathbf{0}$ corresponds to an incompressible flow. If the domain is 1–connected, every incompressible flow field $\mathbf{v}$ has a vector potential $\mathbf{w}$ that satisfies $\nabla \times \mathbf{w} = \mathbf{v}$. The vector potential can be viewed as the three-dimensional analog of the stream function for planar flows. If the fluid is both irrotational and incompressible, then its scalar potential satisfies

$$0 = \nabla \cdot \mathbf{v} = \nabla \cdot \nabla\varphi = \Delta\varphi,$$

which is Laplace's equation! Thus, just as in the two-dimensional case, the scalar potential to an irrotational, incompressible fluid flow is a harmonic function. This fact is used in modeling many problems arising in physical fluids, including water waves, [**8**]. Unfortunately, in three dimensions there is no counterpart of complex function theory to represent the solutions of the Laplace equation, or to connect the vector and scalar potentials.

Example 8.9. The vector field

$$\mathbf{v} = (\, -y, x, 1\,)^T$$

that generates the helical flow (8.10) satisfies $\nabla \cdot \mathbf{v} = 0$, and so is divergence-free, reconfirming our observation that the flow is incompressible. Since $\mathbf{v}$ is defined on all of $\mathbb{R}^3$, Theorem 8.8 assures us that there is a vector potential $\mathbf{w}$ that satisfies $\nabla \times \mathbf{w} = \mathbf{v}$. One candidate for the vector potential is

$$\mathbf{w} = \big(\, y, 0, \tfrac{1}{2}\,x^2 + \tfrac{1}{2}\,y^2\,\big)^T.$$

The helical flow is not irrotational, and so it does not admit a scalar potential.

Remark: The construction of a vector potential is not entirely straightforward, but we will not dwell on this problem. Unlike a scalar potential which, when it exists, is uniquely defined up to a constant, there is, in fact, quite a bit of ambiguity in a vector potential. Adding in *any* gradient,

$$\widetilde{\mathbf{w}} = \mathbf{w} + \nabla\varphi$$

will give an equally valid vector potential. Indeed, using Proposition 8.6, we have

$$\nabla \times \widetilde{\mathbf{w}} = \nabla \times \mathbf{w} + \nabla \times \nabla\varphi = \nabla \times \mathbf{w}.$$

Thus, any vector field of the form

$$\mathbf{w} = \left(y + \frac{\partial \varphi}{\partial x}\,,\ \frac{\partial \varphi}{\partial y}\,,\ \frac{x^2}{2} + \frac{y^2}{2} + \frac{\partial \varphi}{\partial z} \right)^T,$$

where $\varphi(x, y, z)$ is an *arbitrary* function, is also a valid vector potential for the helical vector field $\mathbf{v} = (\,-y, x, 1\,)^T$.

9. The Fundamental Integration Theorems.

In three-dimensional vector calculus there are 3 fundamental differential operators — gradient, curl and divergence. There are also 3 types of integration — line, surface and volume integrals. And, not coincidentally, there are 3 basic theorems that generalize the Fundamental Theorem of Calculus to line, surface and volume integrals in three-dimensional space. In all three results, the integral of some differentiated quantity over a curve, surface, or domain is related to an integral of the quantity over its boundary. The first theorem relates the line integral of a gradient over a curve to the values of the function at the boundary or endpoints of the curve. Stokes' Theorem relates the surface integral of the curl of a vector field to the line integral of the vector field around the boundary curve of the surface. Finally, the Divergence Theorem, also known as Gauss' Theorem, relates the volume integral of the divergence of a vector field to the surface integral of that vector field over the boundary of the domain.

The Fundamental Theorem for Line Integrals

We begin with the Fundamental Theorem for line integrals.

Theorem 9.1. *Let $C \subset \mathbb{R}^3$ be a curve that starts at the endpoint $\mathbf{a}$ and goes to the endpoint $\mathbf{b}$. Then the line integral of a gradient of a function along C is given by*

$$\int_C \nabla u \cdot\, d\mathbf{x} = u(\mathbf{b}) - u(\mathbf{a}). \tag{9.1}$$

Since its value only depends upon the endpoints, the line integral of a gradient is independent of path. In particular, if C is a closed curve, then $\mathbf{a} = \mathbf{b}$, and so the endpoint contributions cancel out:

$$\oint_C \nabla u \cdot\, d\mathbf{x} = 0.$$

Conversely, if $\mathbf{v}$ is any vector field with the property that its integral around any closed curve vanishes,

$$\oint_C \mathbf{v} \cdot d\mathbf{x} = 0, \tag{9.2}$$

then $\mathbf{v} = \nabla\varphi$ admits a potential. Indeed, as long as the domain is 0–connected, one can construct a potential $\varphi(\mathbf{x})$ by integrating over any convenient curve C connecting a fixed point $\mathbf{a} \in \Omega$ to the point $\mathbf{x}$

$$\varphi(\mathbf{x}) = \int_{\mathbf{a}}^{\mathbf{x}} \mathbf{v} \cdot d\mathbf{x}.$$

If $\mathbf{v}$ represents the velocity vector field of a three-dimensional steady state fluid flow, then its line integral around a closed curve C, namely

$$\oint_C \mathbf{v} \cdot d\mathbf{x} = \oint_C \mathbf{v} \cdot \mathbf{t}\, ds$$

is the integral of the tangential component of the velocity vector field. This represents the circulation of the fluid around the curve C. In particular, if the circulation line integral is 0 for every closed curve, then the fluid flow will be *irrotational* because $\nabla \times \mathbf{v} = \nabla \times \nabla\varphi \equiv \mathbf{0}$.

Stokes' Theorem

The second of the three fundamental integration theorems is known as *Stokes' Theorem*. This important result relates the circulation line integral of a vector field around a closed curve with the integral of its curl over any bounding surface. Stokes' Theorem first appeared in an 1850 letter from Lord Kelvin (William Thompson) written to George Stokes, who made it into an undergraduate exam question for the Smith Prize at Cambridge University in England.

Theorem 9.2. *Let $S \subset \mathbb{R}^3$ be an oriented, bounded surface whose boundary ∂S consists of one or more piecewise smooth simple closed curves. Let $\mathbf{v}$ be a smooth vector field defined on S. Then*

$$\oint_{\partial S} \mathbf{v} \cdot d\mathbf{x} = \iint_S (\nabla \times \mathbf{v}) \cdot \mathbf{n}\, dS. \tag{9.3}$$

To make sense of Stokes' formula (9.3), we need to assign a consistent orientation to the surface — meaning a choice of unit normal $\mathbf{n}$ — and to its boundary curve — meaning a direction to go around it. The proper choice is described by the following *left hand rule*: If we walk along the boundary ∂S with the normal vector $\mathbf{n}$ on S pointing upwards, then the surface should be on our left hand side. For example, if $S \subset \{\, z = 0 \,\}$ is a planar domain and we choose the upwards normal $\mathbf{n} = (\, 0, 0, 1 \,)^T$, then C should be oriented in the usual, counterclockwise direction. Indeed, in this case, Stokes' Theorem 9.2 reduces to Green's Theorem!

Stokes' formula (9.3) can be rewritten using the alternative notations (4.5), (6.11), for surface and line integrals in the form

$$\oint_{\partial S} u\,dx + v\,dy + w\,dz = \iint_S \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) dy\,dz + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) dz\,dx + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy. \tag{9.4}$$

Recall that a closed surface is one without boundary: $\partial S = \varnothing$. In this case, the left hand side of Stokes' formula (9.3) is zero, and we find that integrals of curls vanish on closed surfaces.

Proposition 9.3. *If the vector field* $\mathbf{v} = \nabla \times \mathbf{w}$ *is a curl, then* $\displaystyle\iint_S \mathbf{v}\cdot\mathbf{n}\,dS = 0$ *for every closed surface* S.

Thus, every curl vector field defines a surface-independent integral.

Example 9.4. Let $S = \{\, x + y + z = 1,\ x > 0,\ y > 0,\ z > 0 \,\}$ denote the triangular surface considered in Example 6.3. Its boundary $\partial S = L_x \cup L_y \cup L_z$ is a triangle composed of three line segments

$$\begin{aligned} L_x &= \{\, x = 0,\ y + z = 1,\ y \geq 0,\ z \geq 0 \,\},\\ L_y &= \{\, y = 0,\ x + z = 1,\ x \geq 0,\ z \geq 0 \,\},\\ L_z &= \{\, z = 0,\ x + y = 1,\ x \geq 0,\ y \geq 0 \,\}. \end{aligned}$$

To compute the line integral

$$\oint_{\partial S} \mathbf{v}\cdot d\mathbf{x} = \oint_{\partial S} y^2\,dx + x\,z^2\,dy$$

of the vector field $\mathbf{v} = \big(\, y^2, x\,z^2, 0 \,\big)^T$, we could proceed directly, but this would require evaluating three separate integrals over the three sides of the triangle. As an alternative, we can use Stokes formula (9.3), and compute the integral of its curl $\nabla\times\mathbf{v} = (\, 2\,y, 2\,x\,z, 0\,)^T$ over the triangle, which is

$$\oint_{\partial S} \mathbf{v}\cdot d\mathbf{x} = \iint_S (\nabla\times\mathbf{v})\cdot\mathbf{n}\,dS = \iint_S 2\,y\,dy\,dz + 2\,x\,z\,dz\,dx = \frac{17}{12}\,,$$

where this particular surface integral was already computed in Example 6.3.

We remark that Stokes' Theorem 9.2 is consistent with Theorem 8.8. Suppose that $\mathbf{v}$ is a curl-free vector field, so $\nabla\times\mathbf{v} = \mathbf{0}$, which is defined on a 1–connected domain $\Omega \subset \mathbb{R}^3$. Since every simple (unknotted) closed curve $C \subset \Omega$ bounds a surface, $C = \partial S$, with $S \subset \Omega$ also contained inside the domain, then, Stokes' formula (9.3) implies

$$\oint_C \mathbf{v}\cdot d\mathbf{x} = \iint_S (\nabla\times\mathbf{v})\cdot\mathbf{n}\,dS = 0.$$

Since this happens for *every*[†] $C \subset \Omega$, then the path-independence condition (9.2) is satisfied, and hence $\mathbf{v} = \nabla \varphi$ admits a potential.

Example 9.5. The Newtonian gravitational force field

$$\mathbf{v}(\mathbf{x}) = \frac{\mathbf{x}}{\| x \|^3} = \frac{(x, y, z)^T}{(x^2 + y^2 + z^2)^{3/2}}$$

is well defined on $\Omega = \mathbb{R}^3 \setminus \{\mathbf{0}\}$, and is divergence-free: $\operatorname{div} \mathbf{v} \equiv 0$. Nevertheless, this vector field does not admit a vector potential. Indeed, on the sphere $S_a = \{ \| \mathbf{x} \| = a \}$ of radius a, the unit normal vector at a point $\mathbf{x} \in S_a$ is $\mathbf{n} = \mathbf{x}/\| \mathbf{x} \|$. Therefore,

$$\iint_{S_a} \mathbf{v} \cdot \mathbf{n} \, dS = \iint_{S_a} \frac{\mathbf{x}}{\| \mathbf{x} \|^3} \cdot \frac{\mathbf{x}}{\| \mathbf{x} \|} \, dS = \iint_{S_a} \frac{1}{\| \mathbf{x} \|^2} \, dS = \frac{1}{a^2} \iint_{S_a} dS = 4\pi,$$

since S_a has surface area $4\pi a^2$. Note that this result is independent of the radius of the sphere. If $\mathbf{v} = \nabla \times \mathbf{w}$, this would contradict Proposition 9.3.

The problem is, of course, that the domain Ω is not 2–connected, and so Theorem 8.8 does not apply. However, it would apply to the vector field $\mathbf{v}$ on any 2–connected subdomain, for example the domain $\widetilde{\Omega} = \mathbb{R}^3 \setminus \{ x = y = 0, z \leq 0 \}$ obtained by omitting the negative z-axis.

We further note that $\mathbf{v}$ is curl free: $\nabla \times \mathbf{v} \equiv \mathbf{0}$. Since the domain of definition Ω is 1–connected, Theorem 8.8 tells us that $\mathbf{v}$ admits a scalar potential — the Newtonian gravitational potential. Indeed, $\nabla\big(\| \mathbf{x} \|^{-1} \big) = \mathbf{v}$, as the reader can check.

The Divergence Theorem

The last of the three fundamental integral theorems is the *Divergence Theorem*, also known as *Gauss' Theorem*. This result relates a surface flux integral over a closed surface to a volume integral over the domain it bounds.

Theorem 9.6. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain whose boundary $\partial\Omega$ consists of one or more piecewise smooth simple closed surfaces. Let $\mathbf{n}$ denote the unit* outward *normal to the boundary of Ω. Let $\mathbf{v}$ be a smooth vector field defined on Ω and continuous up to its boundary. Then*

$$\iint_{\partial\Omega} \mathbf{v} \cdot \mathbf{n} \, dS = \iiint_{\Omega} \nabla \cdot \mathbf{v} \, dx \, dy \, dz. \tag{9.5}$$

In terms of the alternative notation (6.11) for surface integrals, the divergence formula (9.5) can be rewritten in the form

$$\iint_S u \, dy \, dz + v \, dz \, dx + w \, dx \, dy = \iiint_{\Omega} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) dx \, dy \, dz. \tag{9.6}$$

[†] It suffices to know this for unknotted curves to conclude it for arbitrary closed curves.

Example 9.7. Let us compute the surface integral

$$\iint_S x\,y\,dz\,dx + z\,dx\,dy$$

of the vector field $\mathbf{v} = (\,0, x\,y, z\,)^T$ over the sphere $S = \{\,\|\,\mathbf{x}\,\| = 1\,\}$ of radius 1. A direct evaluation in either graphical or spherical coordinates is not so pleasant. But the divergence formula (9.6) immediately gives

$$\begin{aligned}\iint_S x\,y\,dz\,dx + z\,dx\,dy &= \iiint_\Omega \left(\frac{\partial(x\,y)}{\partial y} + \frac{\partial z}{\partial z}\right) dx\,dy\,dz\\ &= \iiint_\Omega (x+1)\,dx\,dy\,dz = \iiint_\Omega x\,dx\,dy\,dz + \iiint_\Omega dx\,dy\,dz = \tfrac{4}{3}\,\pi,\end{aligned}$$

where $\Omega = \{\,\|\,\mathbf{x}\,\| < 1\,\}$ is the unit ball with boundary $\partial\Omega = S$. The final two integrals are, respectively, the x coordinate of the center of mass of the sphere multiplied by its volume, which is clearly 0, plus the volume of the spherical ball.

Example 9.8. Suppose $\mathbf{v}(t, \mathbf{x})$ is the velocity vector field of a time-dependent fluid flow. Let $\rho(t, \mathbf{x})$ represent the density of the fluid at time t and position $\mathbf{x}$. Then the surface flux integral $\displaystyle\iint_S (\rho\,\mathbf{v})\cdot\mathbf{n}\,dS$ represents the mass flux of fluid through the surface $S \subset \mathbb{R}^3$. In particular, if $S = \partial\Omega$ represents a closed surface bounding a domain Ω, then, by the Divergence Theorem 9.6,

$$\iint_{\partial\Omega} (\rho\,\mathbf{v})\cdot\mathbf{n}\,dS = \iiint_\Omega \nabla\cdot(\rho\,\mathbf{v})\,dx\,dy\,dz$$

represents the net mass flux out of the domain Ω at time t. On the other hand, this must equal the rate of change of mass in the domain, namely

$$-\,\frac{\partial}{\partial t}\iiint_\Omega \rho\,dx\,dy\,dz = -\iiint_\Omega \frac{\partial\rho}{\partial t}\,dx\,dy\,dz,$$

the minus sign coming from the fact that we are measuring net mass loss due to outflow. Equating these two, we discover that

$$\iiint_\Omega \left(\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\,\mathbf{v})\right) dx\,dy\,dz = 0$$

for *every* domain occupied by the fluid. Since the domain is arbitrary, this can only happen if the integrand vanishes, and hence

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\,\mathbf{v}) = 0. \tag{9.7}$$

The latter is the basic *continuity equation* of fluid mechanics, which takes the form of a conservation law.

For a steady state fluid flow, the left hand side of the divergence formula (9.5) measures the fluid flux through the boundary of the domain $\partial\Omega$, while the left hand side integrates the divergence over the domain Ω. As a consequence, the divergence must represent the net local change in fluid volume at a point under the flow. In particular, if $\nabla\mathbf{v} = 0$, then there is no net flux, and the fluid flow is incompressible.

The Divergence Theorem 9.6 is also consistent with Theorem 8.8. Let $\mathbf{v}$ is a divergence-free vector field, $\nabla \cdot \mathbf{v} = 0$, defined on a 2–connected domain $\Omega \subset \mathbb{R}^3$. Every simple closed surface $S \subset \Omega$ bounds a subdomain, so $S = \partial D$, with $D \subset \Omega$ also contained inside the domain of definition of $\mathbf{v}$. Then, by the divergence formula (9.5),

$$\iint_S \mathbf{v} \cdot \mathbf{n} \; dS = \iiint_\Omega \nabla \cdot \mathbf{v} \; dx\, dy\, dz = 0.$$

Therefore, by Theorem 8.8, $\mathbf{v} = \nabla \times \mathbf{w}$ admits a vector potential.

Remark: The proof of all three of the fundamental integral theorems, can, in fact, be reduced to the Fundamental Theorem of (one-variable) Calculus. They are, in fact, all special cases of the general Stokes' Theorem, which forms the foundation of the profound theory of integration on manifolds, [**3**, **6**, **13**]. Stokes' Theorem has deep and beautiful connections with topology — and is of fundamental importance in modern mathematics and physics. However, the full ramifications lie beyond the scope of this introductory text.

10. Electromagnetic Theory.

One of the primary motivations for the nineteenth development of three-dimensional vector calculus was the rapidly developing fields of electricity and magnetism. The discoveries of the fundamental laws of electromagnetic theory culminated in Maxwell's equations that govern the dynamical propagation of electromagnetic waves, which include light waves.

The electric field $\mathbf{E}$ is curl-free: $\nabla \times \mathbf{E} = 0$, while the magnetic field $\mathbf{H}$ is divergence free: $\nabla \cdot \mathbf{H} = 0$. Apply Theorem 8.8 to construct the scalar electric potential ϕ with $\nabla\phi = \mathbf{E}$, and the vector magnetic potential $\mathbf{A}$ with $\nabla \times \mathbf{A} = \mathbf{H}$. Gauss' Law says that the divergence of the dielectric field is the charge density, and so $\nabla \cdot \mathbf{D} = \rho$. The dielectric filed is related to the electric field by the dielectric constant ε, with $\mathbf{D} = \varepsilon\, \mathbf{E}$. The net result is the three-dimensional Poisson equation

$$-\nabla \cdot (\varepsilon\, \nabla\phi) = \rho.$$

Faraday's Law states that the circulation of the electric intensity vector around any closed curve is equal to the rate of change of the magnetic flux

$$F = \iint_S \mathbf{B} \cdot \mathbf{n}\, dS$$

through any surface bounded by C, and so

$$\oint_C E \cdot d\mathbf{x} = \frac{\partial}{\partial t} \iint_S \mathbf{B} \cdot \mathbf{n}\, dS.$$

Applying Stokes' Theorem 9.2 leads to

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}.$$

Ampère's law states that the circulation integral of the magnetic flux vector around a closed curve is equal to the rate of change of the current

$$I = \iint_S \mathbf{J} \cdot \mathbf{n}\, dS$$

flowing through any surface bounded by C, and so

$$\oint_C E \cdot d\mathbf{x} = \frac{\partial}{\partial t} \iint_S \mathbf{B} \cdot \mathbf{n}\, dS.$$

Here $\mathbf{J}$ represents the current density. Applying Stokes' Theorem 9.2 leads to

$$\mathbf{J} = \nabla \times \mathbf{H}.$$

Maxwell was the first to realize that

$$\mathbf{J} = \sigma\, \mathbf{E} + \varepsilon\, \frac{\partial \mathbf{E}}{\partial t}\,,$$

where σ is the conductivity of the medium and ε is permittivity.

Gauss' law for electric fields asserts that the flux integral of the electric flux density $\mathbf{D}$ over any closed surface is equal to the total electric charge it encloses, and so

$$\iint_S \mathbf{D} \cdot \mathbf{n}\, dS = \iiint_\Omega Q,$$

where Q represents the charge density. Applying the Divergence Theorem 9.6 yields

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon}\,.$$

Gauss' law for magnetic fields asserts that the flux integral of the magnetic flux density $\mathbf{B}$ over any closed surface is equal to zero

$$\iint_S \mathbf{D} \cdot \mathbf{n}\, dS = 0.$$

There are no magnetic charges. Applying the Divergence Theorem 9.6 yields

$$\nabla \cdot \mathbf{B} = 0\,.$$

Maxwell's equations take the full form

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon}\,, \qquad \nabla \cdot \mathbf{B} = 0, \qquad \frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}, \qquad \frac{\partial \mathbf{E}}{\partial t} = \frac{1}{\mu\, \varepsilon} \nabla \times \mathbf{B} + \frac{\mathbf{j}}{\varepsilon}. \tag{10.1}$$

If the source terms $\rho, \mathbf{j}$ are absent, then we use the identity

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \Delta \mathbf{E}. \tag{10.2}$$

Since both electric and magnetic fields are divergence-free, this implies that

$$\frac{\partial^2 \mathbf{E}}{\partial t^2} = c^2\, \Delta \mathbf{E}, \qquad \frac{\partial^2 \mathbf{B}}{\partial t^2} = c^2\, \Delta \mathbf{B}, \qquad \text{where} \qquad c^2 = \frac{1}{\mu\, \varepsilon}\,. \tag{10.3}$$

Therefore, each component satisfies the scalar wave equation with velocity $1/\sqrt{\mu\, \varepsilon}$.

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