Nonlinearity 5 (1992) 601–621. Printed in the UK

Equivalence of higher-order Lagrangians: III. New invariant differential equations

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Received 4 March 1991, in final form 10 October 1991
Accepted by J F Toland

Abstract. The solution to the equivalence problem for a higher-order Lagrangian leads to new differential equations which are invariantly associated with the variational functional. We derive explicit expressions for these equations in the case of second-order particle Lagrangians on the line under fibre-preserving, point and contact transformations. A geometrical interpretation of these equations based on the Poincaré–Cartan form is discussed.

AMS classification scheme numbers: 49F05, 53A55

1. Introduction

Higher-order variational problems have come to play an increasingly significant role in nonlinear science over the past few years. A striking illustration of this fact comes from the study of integrable nonlinear evolution equations such as the Korteweg–de Vries equation and Kadomtsev–Petviashvili (KP) equations which model nonlinear systems with dispersion. Indeed, these nonlinear partial differential equations (and many other soliton systems) are the Euler–Lagrange equations for particular higher-order Lagrangians; the infinite sets of conservation laws which account for their complete integrability arise from the analysis of these variational principles from the point of view of Noether's theorem, relating (generalized) variational symmetries to conservation laws [12]. These physically important results demonstrate that a better understanding of the invariant structures and invariant properties of higher-order Lagrangians, such as their scalar differential invariants, *invariant differential forms (e.g. the Poincaré–Cartan form* [6, 9]), *symmetries* and conservations laws—particularly those which do not manifest themselves in the much better understood first-order case—would be important not only from the point of view of mathematical analysis, but also in the realm of applications to the

§ Research supported in part by an NSERC grant.
|| Research supported in part by NSF grant DMS 89-01600.

physical sciences. Besides soliton systems, there are a number of other physically important examples of higher-order Lagrangians for which the study of their invariant properties may have significance, including problems from general relativity, such as the relativistic dynamics of charged particles, and spin–orbit coupling of gravitationally interacting bodies (cf. [4] and references therein), higher-order gauge theories [5], the second-order Polyakov string [13], and rod, plate and shell theories from elasticity [8]. In a later paper, we hope to explore some of these variational problems in more depth.

The invariants and symmetries associated with a problem (not only variational problems, but also differential equations, control systems, etc., cf. [6]) most naturally arise through the solution to the fundamental equivalence problem, which is to recognize when two such problems can be transformed into each other by a suitable change of variables. In the early years of this century, Elie Cartan developed a powerful method, now known as Cartan's method of equivalence, which can explicitly solve such equivalence problems. In particular, necessary sufficient conditions for equivalence can be given in terms of the fundamental invariants associated with the problem. Cartan applied this method to several specific equivalence problems from the calculus of variations, cf. [2, 3]. While almost all subsequent research has mostly dealt with first-order Lagrangians, the importance of higher-order Lagrangians in integrable nonlinear evolution equations and relativity led us to a more detailed study of the equivalence problem for higher-order Lagrangians from the point of view of Cartan's equivalence method, with particular emphasis on novel phenomena which do not occur in the first-order case.

This is the third in a series of papers devoted to this topic. In part I [11] we showed that the equivalence problem for an rth-order Lagrangian in any number of independent and dependent variables, with or without the addition of a divergence term, can be formulated as a Cartan equivalence problem [2, 6], on the rth-order jet bundle $J^r(\mathbb{R}^p, \mathbb{R}^q)$, i.e. the space coordinated by the independent variables, dependent variables and their derivatives up to order r, cf. [12]. This led us to discover that the Euler–Lagrange equation (which is of order $2r$ for a non-degenerate Lagrangian) is *not* the only differential equation associated invariantly to the variational functional, contrary to a commonly believed 'folklore' theorem. Indeed, we showed that, for a generic non-degenerate rth-order Lagrangian L, there exist mth-order differential equations invariantly associated with L for any $r < m < 2r$. In contrast to the Euler–Lagrange equation, these new invariant differential equations are all *nonlinear* in the Lagrangian. This property is confirmed by a theorem of Anderson [1], stating that the Euler–Lagrange equation is the *only* differential equation depending *linearly* on the Lagrangian which is invariantly associated to the Lagrangian. Thus, in the case of a higher-order Lagrangian, the Euler–Lagrange equation is but one element of a wide array of differential equations invariantly associated with the variational problem.

Our aim in this paper is to construct some explicit examples of these new invariant differential equations. We shall concentrate on the first non-trivial case, which is that of a non-degenerate second-order particle Lagrangian on the line, and consider fibre-preserving transformations, point transformations and, most generally, contact transformations in order. (See theorems 7–9 for the explicit form of the new invariant equations, and examples 10 and 11 for explicit formulae for a couple of particular variational problems of physical interest.) We shall first solve the equivalence problem for such Lagrangians under the pseudogroup of fibre-preserving transformations, using Cartan's method of equivalence [2, 6]. The *inductive method* introduced in [10] will then be used to solve the equivalence problem for these Lagrangians under the larger pseudogroup of point transformations and also to recover Cartan's solution [3] of the equivalence problem under the pseudogroup of contact transformations from our solution of the fibre-preserving equivalence problem. The inductive approach has the advantage of providing fairly compact, explicit expressions for the invariants and the invariant coframe given by Cartan in his intrinsic solution of the contact equivalence problem. The proofs are straightforward applications of the results of part I and of the method of equivalence, and we shall just outline the important steps in our presentation here. The final results will be stated in a form which will make them accessible to those readers who are not familiar with the method of equivalence.

Our knowledge of the explicit parametric forms of the invariant coframes for the three equivalence problems will enable us to construct a large number of 'derivative covariants' associated with a (suitably non-degenerate) second-order Lagrangian. As defined in part I, for a second-order Lagrangian, an nth-order derivative covariant is a function, depending on the Lagrangian and its derivatives, defined on the space $J^2(\mathbb{R}, \mathbb{R})$, but whose transformation rules are the same as those for the nth-order derivative of the dependent variable u under the given pseudogroup (fibre-preserving, point or contact transformations). (See (2.21) and the subsequent discussion for the simplest example.) Knowledge of the explicit parametric formulae for third-order derivative covariants will enable us to readily construct third-order ordinary differential equations which are invariantly associated with the second-order Lagrangian under the pseudogroups of fibre-preserving transformations, point transformations or, most restrictively, of contact transformations. Although the formulae for these invariant ordinary differential equations are derived under a non-degeneracy hypothesis, an argument utilizing analytic continuation will allow us to conclude that the equations are in fact invariant for all Lagrangians.

A particular third-order derivative covariant for the contact equivalence problem will be interpreted geometrically in terms of the fundamental integral invariant of the flow associated with the solutions of the Euler–Lagrange equation, namely the Poincaré–Cartan form of the Lagrangian [6, 9]. We produce an explicit invariant embedding ι of $J^2(\mathbb{R}, \mathbb{R})$, which is the jet bundle of definition of the Lagrangian, into $J^3(\mathbb{R}, \mathbb{R})$, which is the jet bundle on which the Poincaré–Cartan form lives, such that the pull-back of the Poincaré–Cartan form under ι is an *invariant* form on $J^2(\mathbb{R}, \mathbb{R})$. Similarly, higher-order derivative covariants enable us to geometrically relate general higher-order invariants of the variational problem, which live on $J^{2+k}(\mathbb{R}, \mathbb{R})$, $k > 0$, to second-order ones, living on $J^2(\mathbb{R}, \mathbb{R})$.

However, while we have a fairly satisfactory geometrical interpretation of our new invariant differential form in terms of the Poincaré–Cartan form, the problem of assigning a physical or mathematical interpretation to the new invariant differential equations remains open. From a mathematical point of view, these new equations should also play a significant role since the Poincaré–Cartan form is effectively used both in the formulation of Noether's theorem and in the field theory of strong minimizers. We also believe that the solutions of any differential equation invariantly associated with a physical variational problem should themselves be of physical interest. We are continuing our ongoing investigation into these tantalizing problems.

2. Equivalence of second-order particle Lagrangians on the line

The basic methodology introduced in part I [11] for constructing the invariant differential equations associated with a variational problem requires the solution of the equivalence problem for the corresponding Lagrangian as obtained by applying Cartan's method of equivalence. In this paper, we specifically consider the case of a second-order particle Lagrangian on the line. Thus, we have a variational functional

$$\mathscr{L}[u]=\int_{\mathbb{R}} L(x, u, u', u'')\,\mathrm{d}x \tag{2.1}$$

where $u=f(x)$ is a scalar-valued function of the scalar variable x. The *Lagrangian form*

$$L(x, u, p, q)\,\mathrm{d}x \tag{2.2}$$

is a one-form defined on the second-order jet bundle $J^2=J^2(\mathbb{R}, \mathbb{R})$, which is endowed with local coordinates $x, u, p=u'$, and $q=u''$. (Since we are solely interested in the case of variational problems on the line, we shall denote the corresponding jet spaces $J^k(\mathbb{R}, \mathbb{R})$ by simply J^k throughout.) We shall assume that all the differential forms and maps we consider are smooth (C^∞), and restrict our attention to a domain $\Omega\subset J^2$ where the Lagrangian L satisfies the basic non-degeneracy conditions

$$\begin{aligned} &L(x, u, p, q)\neq 0\\ &L_q(x, u, p, q)\neq 0\\ &L_{qq}(x, u, p, q)\neq 0 \qquad (x, u, p, q)\in\Omega \end{aligned} \tag{2.3}$$

where the subscripts on L denote partial derivatives.

The Euler–Lagrange equation associated with L is

$$\boldsymbol{E}(L)=L_u-D_x(L_p)+D_x^2(L_q)=0 \tag{2.4}$$

where

$$D_x=\frac{\partial}{\partial x}+p\frac{\partial}{\partial u}+q\frac{\partial}{\partial p}+r\frac{\partial}{\partial q}+s\frac{\partial}{\partial r}+\cdots \tag{2.5}$$

denotes the usual total derivative operator, with $r=u'''$ and $s=u''''$. Under the non-degeneracy assumption (2.3), the Euler–Lagrange equation (2.4) is a fourth-order differential equation, so $\boldsymbol{E}(L)$ is defined on J^4. We recall the definition of the canonical 'momentum'

$$\boldsymbol{P}(L)=L_p-D_x(L_q) \tag{2.6}$$

(cf. [7]), which is defined on J^3, and appears as one of the coefficients of the Poincaré–Cartan form, cf. [9] and (3.30) below.

The equivalence problem for a variational functional given by (2.1) has been solved by Cartan [3] in the case of the pseudogroup of contact transformations, i.e. maps

$$\bar{x}=\varphi(x, u, p) \qquad \bar{u}=\psi(x, u, p) \qquad \bar{p}=\chi(x, u, p) \tag{2.7}$$

whose prolongation $\Psi\colon J^2\to J^2$ preserves the contact ideal:

$$\begin{aligned} \Psi^*(\mathrm{d}\bar{u}-\bar{p}\,\mathrm{d}\bar{x})&=\lambda_1(\mathrm{d}u-p\,\mathrm{d}x)\\ \Psi^*(\mathrm{d}\bar{p}-\bar{q}\,\mathrm{d}\bar{x})&=\lambda_2(\mathrm{d}u-p\,\mathrm{d}x)+\lambda_3(\mathrm{d}p-q\,\mathrm{d}x) \end{aligned} \tag{2.8}$$

where λ_1, λ_2 and λ_3 are real-valued functions on J^2. In this section, we shall also consider the more restrictive cases of equivalence under the pseudogroup of *fibre-preserving transformations*,

$$\bar{x} = \varphi(x) \qquad \bar{u} = \psi(x, u) \tag{2.9}$$

or the pseudogroup of *point transformations*

$$\bar{x} = \varphi(x, u) \qquad \bar{u} = \psi(x, u) \tag{2.10}$$

both of whose prolongations to J^2 also satisfy the contact conditions (2.8). Two Lagrangians are equivalent under a contact transformation (2.7), (2.8) if and only if

$$\Psi^*(\bar{L}\, \mathrm{d}\bar{x}) = L\, \mathrm{d}x + \alpha(\mathrm{d}p - q\, \mathrm{d}x) + \beta(\mathrm{d}u - p\, \mathrm{d}x) \tag{2.11}$$

where α and β are real-valued functions on J^2. For point transformations, $\alpha = 0$, while, for fibre-preserving transformations, $\alpha = \beta = 0$, and the Lagrangian form $L\, \mathrm{d}x$ is invariant.

We first need to formulate the equivalence problems for the variational problem (2.1) as Cartan equivalence problems, that is, as local equivalence problems for suitable G-structures. According to part I, to place the problem into a form amenable to Cartan's method, we should take the base coframe on J^2 given by the one-forms

$$\begin{aligned} \omega_1 &= \mathrm{d}u - p\, \mathrm{d}x & \omega_2 &= \frac{\mathrm{d}p - q\, \mathrm{d}x}{L} \\ \omega_3 &= L\, \mathrm{d}x & \omega_4 &= \frac{\mathrm{d}q}{L^2}. \end{aligned} \tag{2.12}$$

The relevant structure groups are given by the following subgroups of $\mathrm{GL}(4, \mathbb{R})$:

$$\begin{aligned} G_1 &= \left\{ \begin{pmatrix} a_1 & 0 & 0 & 0 \\ a_2 & a_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ a_5 & a_6 & a_7 & a_1 \end{pmatrix} : \ a_i \in \mathbb{R},\ a_1 \neq 0 \right\} \\ G_2 &= \left\{ \begin{pmatrix} b_1 & 0 & 0 & 0 \\ b_2 & b_1 & 0 & 0 \\ b_3 & 0 & 1 & 0 \\ b_5 & b_6 & b_7 & b_1 \end{pmatrix} : \ b_i \in \mathbb{R},\ b_1 \neq 0 \right\} \\ G_3 &= \left\{ \begin{pmatrix} c_1 & 0 & 0 & 0 \\ c_2 & c_1 & 0 & 0 \\ c_3 & c_4 & 1 & 0 \\ c_5 & c_6 & c_7 & c_1 \end{pmatrix} : \ c_i \in \mathbb{R},\ c_1 \neq 0 \right\}. \end{aligned} \tag{2.13}$$

The diagonal entries are equal due to the scaling of the coframe (2.12). According to part I (compare with (2.8) and (2.11)), the two equivalence problems can be reformulated as follows:

Theorem 1. Let $L(x, u, p, q)$ and $\bar{L}(\bar{x}, \bar{u}, \bar{p}, \bar{q})$ be non-degenerate second-order Lagrangians. Let $\omega = [\omega_1, \omega_2, \omega_3, \omega_4]^{\mathrm{T}}$ and $\bar{\omega} = [\bar{\omega}_1, \bar{\omega}_2, \bar{\omega}_3, \bar{\omega}_4]^{\mathrm{T}}$ be column vectors whose entries are the one-forms representing the base coframes given by (2.12). Then L and $\bar{L}$ are equivalent under a fibre-preserving ($\kappa = 1$), point ($\kappa = 2$), or contact ($\kappa = 3$) transformation, if and only if there is a (local) diffeomorphism $\Psi : J^2 \to J^2$, such that

$$\Psi^*(\bar{\omega}) = g \cdot \omega$$

where g is a G_κ-valued function on J^2, $\kappa = 1, 2, 3$.

Let us begin our discussion of the solution to these equivalence problems with the fibre-preserving problem. A straightforward application of Cartan's method of equivalence to the equivalence problem defined by the base coframe (2.12) and structure group G_1 in (2.13) will lead directly to a coframe which is *invariantly associated with* the Lagrangian L under the pseudogroup of all fibre-preserving transformations. Moreover, the associated structure equations will provide explicit scalar invariants, whose functional interrelationships provide the complete necessary and sufficient conditions for equivalence. To keep our presentation as short as possible, we will assume that the reader has a basic familiarity with the mechanics of the equivalence method of Cartan, as discussed in [6, 10]. Using the first structure group G_1 and base coframe (2.12) we introduce the 'lifted' coframe

$$\begin{aligned} &\xi_1 = a_1\omega_1 && \xi_2 = a_2\omega_1 + a_1\omega_2 \\ &\xi_3 = \omega_3 && \xi_4 = a_5\omega_1 + a_6\omega_2 + a_7\omega_3 + a_1\omega_4. \end{aligned} \tag{2.14}$$

The equivalence algorithm will, if possible, provide explicit formulae for the group parameters a_i in terms of the variables (x, u, p, q), the Lagrangian L and its derivatives, so that the resulting one-forms will be invariant under any prolonged fibre-preserving transformation. In the present problem, one of the forms, $\xi_3 = L\,\mathrm{d}x$, is already invariant but, to apply the full Cartan method, one needs to pin down the other three members of the full invariant coframe.

According to Cartan, the way to determine the desired formulae for the group parameters is to make use of the invariance of the exterior derivative d under pull-backs. Computing the differentials of the lifted one-forms (2.14) will lead us to invariant functions involving the variables, the Lagrangian and its derivatives, and the group parameters. We are free to normalize these invariants in a convenient manner, and thereby determine the required explicit formulaes for (some of the) group parameters. Iteration of this method, known as 'absorption of torsion and normalization of group parameters', will, in most cases, lead to the required formulae, and reduce the problem to that of the equivalence of coframes or $\{e\}$-structures, where the solution is well known, cf. [2, 6, 10].

Thus, we begin by computing the differentials of the one-forms (2.14); these are found to be of the form

$$\begin{aligned} &\mathrm{d}\xi_1 = \alpha_1 \wedge \xi_1 + \sigma_1 && \mathrm{d}\xi_2 = \alpha_2 \wedge \xi_1 + \alpha_1 \wedge \xi_2 + \sigma_2 \\ &\mathrm{d}\xi_3 = \sigma_3 && \mathrm{d}\xi_4 = \alpha_5 \wedge \xi_1 + \alpha_6 \wedge \xi_2 + \alpha_7 \wedge \xi_3 + \alpha_1 \wedge \xi_4 + \sigma_4 \end{aligned}$$

where α_1, α_2, α_5, α_6 and α_7 form a basis for the right-invariant one-forms on the Lie Group G_1 (the Maurer–Cartan forms), and where

$$\sigma_i = \sum_{j,k} \tau_{ijk}\xi_j \wedge \xi_k \qquad i = 1, \ldots, 4$$

are the corresponding 'torsion' terms. In the absorption part of Cartan's procedure, in order to deduce scalar invariants, we replace each one-form α_j by an expression of the form $\alpha_j + \sum z_{jk}\xi_k$, and choose the functions z_{jk} so as to make as many of the torsion coefficients τ_{ijk} vanish as possible. In our case, the unabsorbable torsion coefficients are the constants

$$\tau_{123} = -1 \qquad \tau_{124} = \tau_{134} = \tau_{312} = \tau_{314} = \tau_{324} = 0 \qquad \tau_{234} = 1$$

and the group-dependent invariants

$$\tau_{224} = -\tau_{334} = \frac{LL_q}{a_1} \qquad \tau_{313} = \frac{a_1^2 L_u - a_1 a_2 L L_p + (a_2 a_6 - a_1 a_5) L^2 L_q}{a_1^3 L}$$

$$\tau_{323} = \frac{a_1 L_p - a_6 L L_q}{a_1^2} \qquad \tau_{223} - \tau_{113} = \frac{\tilde{D}_x L}{L^2} + \frac{a_6 - 2a_2 - a_7 L L_q}{a_1}.$$

Here

$$\tilde{D}_x = \frac{\partial}{\partial x} + p\frac{\partial}{\partial u} + q\frac{\partial}{\partial p} \tag{2.15}$$

denotes the J^2 truncation of the total derivative operator (2.5). By our non-degeneracy assumption (2.3), L_q is not zero, so we can normalize these group-dependent invariants to take the constant values 1, 0, 0 and 0, respectively, by setting

$$a_1 = LL_q \qquad a_5 = \frac{L_u}{L} \qquad a_6 = L_p \qquad a_7 = \frac{L_p - 2a_2}{LL_q} + \frac{\tilde{D}_x L}{L^2}. \tag{2.16}$$

The normalizations (2.16) have the effect of reducing the original Lie group G_1 to a one-parameter subgroup, with a_2 the only remaining undetermined parameter.

In the second loop through the equivalence procedure, we substitute the expressions (2.16) for the normalized group parameters into the formulae for the lifted coframe (2.14), and recompute the differentials. The resulting non-constant unabsorbable torsion coefficients provide the two absolute invariants

$$\tau_{224} = \tau_{114} + 1 = -\frac{LL_{qq}}{L_q^2} \qquad \tau_{112} = \frac{L_p L_{qq} - L_q L_{pq}}{L_p^2} \tag{2.17}$$

and the group-dependent invariant

$$\tau_{113} = \tau_{223} = -\frac{a_2(2LL_{qq} + L_q^2) + L_q^2 \tilde{D}_x L_q - L_q L_{qq} \tilde{D}_x L - L_p L_q^2 - L L_p L_{qq}}{L L_q^3}.$$

At this point, the equivalence problem splits into two branches. If the additional 'non-degeneracy' condition

$$2LL_{qq} + L_q^2 \neq 0 \tag{2.18}$$

holds, then we are in the branch of 'generic' Lagrangians. The reader can check that condition (2.18) is invariant under fibre-preserving transformations. The Lagrangians for which the left-hand side of (2.18) vanishes identically, i.e. those of the form

$$L = (A(x, u, p)q + B(x, u, p))^{2/3} \tag{2.19}$$

where A and B are functions on J^1, lead to a prolongation in the implementation of Cartan's equivalence procedure. We shall leave these particular Lagrangians aside for the time being, and concentrate on the generic case. We thus assume (by possibly shrinking our domain) that (2.18) (and (2.3)) hold everywhere in Ω, and so we can normalize the torsion coefficients τ_{113} and τ_{223} to be zero by setting

$$a_2 = \frac{L_q L_{qq} \bar{D}_x L - L_q^2 \bar{D}_x L_q + L_p L_q^2 + L L_p L_{qq}}{2 L L_{qq} + L_q^2}. \tag{2.20}$$

We have now eliminated all the group parameters, so that an invariant coframe is determined from (2.14) with the group parameters taking the prescribed values (2.16) and (2.20). Before formulating the resulting theorem, we introduce some further notation that will effectively simplify the coframe, and lead eventually to the explicit formula for the promised invariant equation.

Consider the invariant form ξ_4. Inserting (2.16) and (2.20) into (2.14) shows that it has the explicit formula

$$\xi_4 = \frac{L_u(\mathrm{d}u - p\,\mathrm{d}x) + L_p(\mathrm{d}p - q\,\mathrm{d}x) + L_q(\mathrm{d}q - R_L\,\mathrm{d}x)}{L}$$

where the function

$$R(x, u, p, q) = R_L(x, u, p, q) \equiv \frac{-\bar{D}_x(L L_q^2) + L L_p L_q}{(L L_q^2)_q} \tag{2.21}$$

plays a particularly important role. (Note that the denominator of R does not vanish owing to the condition (2.18); indeed, according to (2.16) and (2.20), the group parameter a_7 has been normalized to be $a_7 = L_q R_L / L^2$.) Since ξ_4 is, by construction, an invariant one-form, according to proposition 16 of part I [11, p 386], the function R is a *third-order derivative covariant* which is invariantly associated (under fibre-preserving maps) with a non-degenerate second-order Lagrangian L. This means that R transforms in exactly the same way as the third-order derivative $r = u'''$ does under fibre-preserving transformations. In order to give this important concept a more concrete form, consider a fibre-preserving transformation (2.9). Its prolongation to J^3 has the form

$$\bar{p} = \chi(x, u, p) \qquad \bar{q} = \varpi(x, u, p, q) \qquad \bar{r} = \rho(x, u, p, q, r) \tag{2.22}$$

where the functions χ, ϖ and ρ can be explicitly written in terms of φ, ψ and their derivatives using the chain rule from calculus.

Proposition 2. If a fibre-preserving transformation (2.9) maps the Lagrangian L to the Lagrangian $\bar{L}$, then the corresponding functions R_L and $\bar{R}_{\bar{L}}$ are related by the same transformation rule,

$$\bar{R}_{\bar{L}}(\bar{x}, \bar{u}, \bar{p}, \bar{q}) = \rho(x, u, p, q, R_L(x, u, p, q)) \tag{2.23}$$

as the third-order derivatives r and $\bar{r}$ as given by (2.22).

The proof of proposition 2 follows from the invariance of ξ_4 and proposition 16 in part I, although it can also be proved directly by a long calculation using the explicit formula for R and for ρ. Since the transformation rules for R and the third-order derivative r are the same, we can invariantly replace third-order derivatives r in any expression by the derivative covariant R; the resulting expression, which only involves second-order derivatives, obeys the same transformation rules as the original one. For example, the third-order truncation of the total derivative operator can be so changed to yield an operator

$$D_x^* = \tilde{D}_x + R\frac{\partial}{\partial q} = \frac{\partial}{\partial x} + p\frac{\partial}{\partial u} + q\frac{\partial}{\partial p} + R\frac{\partial}{\partial q} \tag{2.24}$$

which is a vector field on J^2. The transformation rules for D_x^* are the same as those for D_x, and hence permit us to readily construct a whole hierarchy of derivative covariants.

Proposition 3. If $Q_n(x, u, p, q)$ is any nth-order derivative covariant associated with a non-degenerate Lagrangian L, then $Q_{n+1} = D_x^*(Q_n)$ is an $(n+1)$th-order derivative covariant.

In particular, we can start the sequence of derivative covariants with u itself, which is trivially a zeroth-order derivative covariant:

$$u \qquad p = D_x^*(u) \qquad q = D_x^*(p) \qquad R = D_x^*(q) \qquad S = D_x^*(R) \ldots . \tag{2.25}$$

The nth term $U^{(n)} = (D_x^*)^n u$ in this sequence is an nth-order derivative covariant: a function just defined on J^2, but which transforms in exactly the same way that the nth derivative $u^{(n)}$ does, i.e. an equation analogous to (2.23) holds.

Replacing higher-order derivatives of $u^{(n)}$ by the corresponding derivative covariants $U^{(n)}$ in the hierarchy (2.25) allows us to replace any function defined on J^n, $n > 2$, by a counterpart defined on J^2 depending only on at most second-order derivatives of u, but which transforms in exactly the same way as the original function does under the pseudogroup of fibre-preserving transformations of the Lagrangian. Important examples are the modified J^2 Euler–Lagrange expression

$$\boldsymbol{E}^*(L) = L_u - D_x^*(L_p) + D_x^{*2}(L_q) \tag{2.26}$$

which is obtained from the usual Euler–Lagrange equation (2.4) by replacing r by R and s by S, as given in (2.25), wherever they occur. Furthermore, we have already encountered the modified J^2 momentum

$$\boldsymbol{P}^*(L) \equiv L_p - D_x^*(L_q) \tag{2.27}$$

(cf. (2.6)); indeed, a short calculation using (2.21) and (2.24) shows that $a_2 = \boldsymbol{P}^*(L)$ is an equivalent form for the normalized value (2.20) of the group parameter a_2.

With these in hand, we can at last state our solution to the fibre-preserving Lagrangian equivalence problem.

Theorem 4. Let $L(x, u, p, q)$ be a non-degenerate Lagrangian satisfying the additional condition (2.18). The coframe on J^2 given by

$$\begin{aligned}
\xi_1 &= LL_q(\mathrm{d}u - p\,\mathrm{d}x)\\
\xi_2 &= \boldsymbol{P}^*(L)(\mathrm{d}u - p\,\mathrm{d}x) + L_q(\mathrm{d}p - q\,\mathrm{d}x)\\
\xi_3 &= L\,\mathrm{d}x\\
\xi_4 &= \frac{L_u(\mathrm{d}u - p\,\mathrm{d}x) + L_p(\mathrm{d}p - q\,\mathrm{d}x) + L_q(\mathrm{d}q - R_L\,\mathrm{d}x)}{L}
\end{aligned} \tag{2.28}$$

where R_L and $\boldsymbol{P}^*(L)$ are given by (2.21) and (2.27), is invariantly associated with L under fibre-preserving transformations. That is, a diffeomorphism $\Psi: J^2 \to J^2$ solves the fibre-preserving equivalence problem if and only if it satisfies

$$\Psi^*(\xi_i) = \xi_i \qquad i = 1, \ldots, 4. \tag{2.29}$$

In other words, under the nondegeneracy assumptions (2.3) and (2.18), the coframe (2.28) determines an $\{e\}$*-structure* on J^2 whose automorphisms are in one-to-one correspondence with the fibre-preserving symmetries (self-equivalences) of the Lagrangian L. (There also exist corresponding invariant coframes for Lagrangians of the form (2.19) which require prolongation, but we will not need their explicit form in this paper.)

The equivalence problem for $\{e\}$-structures has a well-known solution [6, 14] based on the study of its structure functions obtained by differentiating the invariant coframe. In our case, the structure equations for the coframe (2.28) have the form

$$\begin{aligned}
\mathrm{d}\xi_1 &= I_1\, \xi_1 \wedge \xi_2 - (I_2 + 1)\xi_1 \wedge \xi_4 - \xi_2 \wedge \xi_3 \\
\mathrm{d}\xi_2 &= I_3\, \xi_1 \wedge \xi_2 + I_4\, \xi_1 \wedge \xi_3 + I_5\, \xi_1 \wedge \xi_4 - I_2\, \xi_2 \wedge \xi_4 + \xi_3 \wedge \xi_4 \\
\mathrm{d}\xi_3 &= -\xi_3 \wedge \xi_4 \\
\mathrm{d}\xi_4 &= I_6\, \xi_1 \wedge \xi_3 + I_7\, \xi_2 \wedge \xi_3 + I_8\, \xi_3 \wedge \xi_4.
\end{aligned} \tag{2.30}$$

The coefficients of the invariant two-forms on the right-hand sides of the structure equations are known as the *structure functions* for the $\{e\}$-structure given by (2.28), and can be written concisely using the covariant derivatives associated with the given coframe. These are defined by the formula

$$\mathrm{d}F = \sum_{i=1}^{4} F_{,\xi_i}\xi_i \qquad \text{for any } F: J^2 \to \mathbb{R}. \tag{2.31}$$

Explicitly,

$$\begin{aligned}
F_{,\xi_1} &= \frac{1}{LL_q^3}\left(L_q \frac{\partial(L, F)}{\partial(q, u)} - \mathbf{P}^*(L) \frac{\partial(L, F)}{\partial(q, p)}\right) \\
F_{,\xi_2} &= \frac{1}{L_q^2} \frac{\partial(L, F)}{\partial(q, p)} \\
F_{,\xi_3} &= \frac{D_x^* F}{L} \\
F_{,\xi_4} &= \frac{L}{L_q} F_q.
\end{aligned} \tag{2.32}$$

(Here $\partial(L, F)/\partial(q, p) = L_q F_p - L_p F_q$ denotes the usual Jacobian derivative.) The fundamental invariants can then be shown to be expressed by the following formulae:

$$
\begin{aligned}
I_1 &= -(\log|LL_q|)_{,\xi_2} = \frac{L_pL_{qq} - L_qL_{pq}}{L_q^3} \\
I_2 &= (\log|LL_q|)_{,\xi_4} - 1 = \frac{LL_{qq}}{L_q^2} \\
I_3 &= -\frac{1}{L}\left(\frac{\boldsymbol{P}^*(L)}{L_q}\right)_{,\xi_2} + (\log L_q)_{,\xi_1} \\
I_4 &= -\frac{1}{L}\left(\frac{\boldsymbol{P}^*(L)}{L_q}\right)_{,\xi_3} + \frac{\boldsymbol{P}^*(L)^2 - \boldsymbol{P}^*(L)L_q + L_pL_q}{L^2L_q^2} = \frac{\boldsymbol{E}^*(L)}{L^2L_q} \\
I_5 &= -\frac{1}{L}\left(\frac{\boldsymbol{P}^*(L)}{L_q}\right)_{,\xi_4} \\
I_6 &= -\frac{L_{,\xi_3,\xi_1}}{L} \\
I_7 &= -\frac{L_{,\xi_3,\xi_2}}{L} \\
I_8 &= \frac{L_{,\xi_3,\xi_4}}{L}.
\end{aligned}
\tag{2.33}
$$

The most interesting invariant is I_4, which provides an 'invariant J^2 version of the Euler–Lagrange equation associated with L'.

The invariants I_j and their covariant derivatives $I_{j,\xi_k} I_{j,\xi_k,\xi_l}$, etc., give the basic differential invariants of the Lagrangian L under fibre-preserving transformations, and are used to effect the complete necessary and sufficient conditions for equivalence. We refer the interested reader to [6, 10, 14] for a general discussion. In particular, if $m = m(L)$ denotes the number of functionally independent differential invariants, then the symmetry group of the variational problem, meaning the group of all fibre-preserving maps which leaves it unchanged, is a (local) Lie group of dimension $4 - m$. (The particular Lagrangians (2.19) requiring prolongation can, in special cases—e.g. A and B constant—admit a five-dimensional symmetry group, which is the largest dimension allowed for a non-degenerate second-order Lagrangian.) Finally, note that the non-degeneracy condition (2.18) can be stated in invariant form as just $I_2 \neq -1/2$.

Turning to the other two equivalence problems, we will use the inductive method of [10] to construct a coframe invariantly associated with the Lagrangian L under point or contact transformations. In the case of point transformations, since the structure group G_1 of the fibre-preserving problem is a subgroup of the structure group G_2 of the point transformation problem, instead of working with the lifted coframe based on the original base coframe (2.9) we should rather 'lift' the fibre-preserving invariant coframe (2.25). Therefore, we introduce the adapted lifted coframe

$$
\begin{aligned}
\eta_1 &= b_1\xi_1 & \eta_2 &= b_2\xi_1 + b_1\xi_2 \\
\eta_3 &= b_3\xi_1 + \xi_3 & \eta_4 &= b_5\xi_1 + b_6\xi_2 + b_7\xi_3 + b_1\xi_4.
\end{aligned}
\tag{2.34}
$$

We now apply the Cartan algorithm to the equivalence problem determined by the lifted one-forms (2.34). Thus, we begin by computing the differentials

$$d\eta_1 = \beta_1 \wedge \eta_1 + \hat{\sigma}_1 \qquad d\eta_2 = \beta_2 \wedge \eta_1 + \beta_1 \wedge \eta_2 + \hat{\sigma}_2$$
$$d\eta_3 = \beta_3 \wedge \eta_1 + \hat{\sigma}_3$$
$$d\eta_4 = \beta_5 \wedge \eta_1 + \beta_6 \wedge \eta_2 + \beta_7 \wedge \eta_3 + \beta_1 \wedge \eta_4 + \hat{\sigma}_4$$

where β_1, β_2, β_3, β_5, β_6 and β_7 form a basis for the right-invariant one-forms on the Lie group G_2, and where the new torsion components have the form

$$\hat{\sigma}_i = \sum_{j,k} \hat{\tau}_{ijk} \eta_j \wedge \eta_k \qquad i = 1, \ldots, 4.$$

We now briefly indicate the unabsorbable torsion components, the chosen normalizations, and the group reductions resulting from the normalizations, arising in each loop of the equivalence method for this problem.

Loop 1.

$$\hat{\tau}_{334} = \hat{\tau}_{114} - \hat{\tau}_{224} = 1 \qquad b_1 = 1$$
$$\hat{\tau}_{323} = 0 \qquad b_3 = -b_6$$
$$\hat{\tau}_{113} = \hat{\tau}_{223} \qquad b_7 = b_6 - 2b_2.$$

Loop 2. At this stage, an additional non-degeneracy condition must be imposed on the Lagrangian in order not to have to prolong the problem, namely

$$(1 + 2I_2)(2 + I_2) \neq 0 \qquad \text{i.e. } (LL_{qq} + 2L_q^2)(2LL_{qq} + L_q^2) \neq 0. \tag{2.35}$$

(Of course, one of these factors has already been assumed to be non-zero according to (2.18).) We note that this condition is invariant under the pseudogroup of point transformations. The Lagrangians for which the left-hand side of (2.35) vanishes identically, which require prolongation, include our earlier special Lagrangians (2.19), as well as Lagrangians of the form

$$L = (A(x, u, p)q + B(x, u, p))^{1/3} \tag{2.36}$$

where A and B are functions on J^1. Note that a Lagrangian of the form (2.36) can be transformed into one of the form (2.19) under the 'hodograph transformation', which interchanges independent variable x and dependent variable u. Under the condition (2.35), the following unabsorbable torsion terms can be normalized:

$$\hat{\tau}_{112} = 0 \qquad b_6 = -I_9$$
$$\hat{\tau}_{113} = 0 \qquad b_2 = -(1 + I_2)I_{10}$$

where I_9, I_{10} are the additional fibre-preserving invariants

$$I_9 = \frac{I_1}{2 + I_2} \qquad I_{10} = \frac{I_9}{1 + 2I_2} = \frac{I_1}{(2 + I_2)(1 + 2I_2)}. \tag{2.37}$$

Loop 3. Finally, we normalize

$$\hat{\tau}_{313} = 0 \qquad b_5 = I_{11} \equiv I_{10} I_{9,\xi_4} - I_{9,\xi_3} + (1 + I_2) I_9 I_{10}. \tag{2.38}$$

This completes the equivalence procedure for a Lagrangian satisfying the strengthened non-degeneracy condition (2.35). We summarize the result as follows.

Theorem 5. Let L be a non-degenerate Lagrangian satisfying condition (2.35). The coframe on J^2 given by

$$\begin{aligned} \eta_1 &= \xi_1 \\ \eta_2 &= -(1+I_2)I_{10}\,\xi_1 + \xi_2 \\ \eta_3 &= I_9\,\xi_1 + \xi_3 \\ \eta_4 &= I_{11}\,\xi_1 - I_9\,\xi_2 + I_{10}\,\xi_3 + \xi_4 \end{aligned} \tag{2.39}$$

is invariantly associated with L under the pseudogroup of point transformations, that is, a diffeomorphism Ψ: $J^2 \to J^2$ solves the point transformation equivalence problem if and only if it satisfies

$$\Psi^*(\bar{\eta}_i) = \eta_i \qquad i = 1, \ldots, 4.$$

It is now straightforward to write the explicit structure equations and scalar invariants associated with this coframe, and thereby deduce analogous necessary and sufficient conditions for equivalence of Lagrangians under a point transformation. However, this will not be essential for our purposes, and so we omit the complicated formulae here.

A similar induction starting again with the fibre-preserving invariant coframe produces the invariant coframe for the contact equivalence problem. (One could also start with the point transformation invariant coframe instead). Rather than belabour the issue any further, we just state the final result. Define the fibre-preserving invariant

$$I_{12} \equiv (I_{2,\xi_4} + 2I_2^2 + 2I_2)^2 + 2I_2^3 = \frac{L^2(LL_{qqq} + 3L_qL_{qq})^2 + 2L^3L_{qq}^3}{L_q^6}. \tag{2.40}$$

To avoid prolongation, we must require the non-degeneracy condition

$$I_{12} \neq 0; \tag{2.41}$$

again, this condition is invariant, this time under all contact transformations. (This is precisely the same as Cartan's condition $k^2 \neq \varepsilon/2$, cf. [3, p 1350].) The Lagrangians for which the invariant I_{12} vanishes identically, and the equivalence problem must be prolonged, all have the form

$$L(x, u, p, q) = (A(x, u, p)q + B(x, u, p))^{2/3}(C(x, u, p)q + D(x, u, p))^{1/3} \tag{2.42}$$

where A, B, C and D are arbitrary functions on J^1. As shown by Cartan, these can all be transformed into our earlier class (2.19) by a suitable contact transformations. Assuming (2.41), we define the further fibre-preserving invariants

$$\begin{aligned} I_{13} &= \frac{(I_{2,\xi_2} - I_{2,\xi_3} - 2I_1I_2)I_{2,\xi_4} - 2I_2I_{2,\xi_3} + (2I_2 - I_2^2)I_{2,\xi_2} - 4(I_2^3 + I_2^2)I_1}{2I_{12}} \\ I_{14} &= \frac{(3I_2 + 2I_2^2 + I_{2,\xi_4})I_{2,\xi_3} - I_2I_{2,\xi_2} + 2I_1I_2^2}{I_{12}} \\ I_{15} &= I_{14,\xi_3} - I_{13}I_{14,\xi_4} + I_{14}^2 + I_5I_{13} - I_4. \end{aligned} \tag{2.43}$$

Theorem 6. Let L be a non-degenerate Lagrangian satisfying the condition $I_{12} \neq 0$. The coframe on J^2 given by

$$\begin{aligned}
\zeta_1 &= \kappa\sqrt{|I_2|}\,\xi_1 \\
\zeta_2 &= \kappa\sqrt{|I_2|}\,(I_{13}\,\xi_1 + \xi_2) \\
\zeta_3 &= I_2 I_{14}\,\xi_1 + \xi_2 + \xi_3 \\
\zeta_4 &= \kappa\sqrt{|I_2|}\,(I_{15}\,\xi_1 + (2I_{13} + I_{14})\xi_2 + I_{14}\,\xi_3 + \xi_4)
\end{aligned} \tag{2.44}$$

where $\kappa = \pm 1$ is an unspecified sign, is invariant under contact transformations. That is, a diffeomorphism Ψ: $J^2 \to J^2$ solves the contact equivalence problem if and only if

$$\Psi^*(\bar{\zeta}_i) = \zeta_i \qquad i = 1, \ldots, 4.$$

(See [10] for a discussion of the ambiguous signs which arise in equivalence problems.)

The coframe (2.44) determines an $\{e\}$-structure whose automorphisms are in one-to-one correspondence with the contact symmetries of the Lagrangian. It is precisely the $\{e\}$-structure found by Cartan in his paper (see equation (A) on p 1348 of [3]). We refer the reader to Cartan for a detailed study of this $\{e\}$-structure and geometric interpretations of the invariants, as well as a discussion of the cases requiring prolongation.

3. New invariant differential equations

We showed in part I that, given an rth-order Lagrangian in p independent and q dependent variables, for any $m > r$ there exist mth-order (systems of) differential equations invariantly associated with L. The only examples known prior to this are the Euler–Lagrange equation for the critical point of the variational functional associated with L (which is an equation of order $2r$ for a non-degenerate Lagrangian) and its covariant derivatives. Thus, any invariant equation of order $m < 2r$ associated with a non-degenerate rth-order Lagrangian is a new differential equation endowed with an intrinsic meaning for the variational problem.

One easy way to form invariant differential equations of order r is by setting any of the invariant functions for the equivalence problem equal to a constant. Thus, for the fibre-preserving Lagrangian equivalence problem, any equation of the form

$$F(I_j, I_{j,\xi_k}, I_{j,\xi_k,\xi_l}, \ldots) = C \tag{3.1}$$

is an invariant second-order differential equation which is endowed with an intrinsic meaning for any suitably non-degenerate ('generic') variational problem. In this section, we go beyond this simple approach, and construct further explicit examples of higher-order invariant equations, including a *third-order* ordinary differential equation which is associated in a contact-invariant way with *any* second-order particle Lagrangian on the line. These new invariant equations are much less obvious than those of the form (3.1), and have the advantages of (i) taking an explicit polynomial form, and (ii) therefore being invariant for *all* Lagrangians, not just the generic ones. They can also be written in an explicit solved form for the highest-order derivative. Finally, we give a geometrical interpretation of the contact-invariant third-order equation in terms of the Poincaré–Cartan form associated with the Lagrangian.

In general, we say that a third-order differential equation

$$F_L(x, u, p, q, r) = 0 \tag{3.2}$$

is invariantly associated with the Lagrangian L (under fibre-preserving maps) if the prolongation Ψ of any fibre-preserving transformation (2.9) maps the solutions of (3.2) to the solutions of the corresponding equation $\bar{F}_{\bar{L}} = 0$ for the transformed Lagrangian $\bar{L}$, *mutatis mutandis* for point or contact transformations. According to part I, any such invariant equation (3.2) gives rise to a third-order derivative covariant when solved for the third-order derivative

$$r = \hat{R}(x, u, p, q) \tag{3.3}$$

conversely, any third-order derivative covariant $\hat{R}$ leads to an invariant differential equation (3.3).

Theorem 7. Let $L(x, u, p, q)$ be any analytic second-order Lagrangian on the line. The ordinary differential equation

$$D_x(LL_q^2) = LL_pL_q \tag{3.4}$$

which is at most third order, is invariantly associated to L under fibre-preserving transformations.

Proof. Assume first that the non-degeneracy conditions (2.3) and (2.18) hold. Using the explicit formula (2.21) for the resulting third-order derivative covariant R_L, we find that (3.3) with $\hat{R} = R_L$ simplifies to (3.4) once denominators are cleared. Proposition 2 then immediately implies the invariance of (3.3), and hence of the simplified version (3.4).

To complete the argument for non-degenerate Lagrangians, we need to look at the case when $2LL_{qq} + L_q^2$ vanishes identically, which means that the Lagrangian has the form $L = (Aq + B)^{2/3}$, cf. (2.19). In this case, the ordinary differential equation (3.4) is

$$A(qA_p + 2pA + 2A_x - B_p) = 0. \tag{3.5}$$

Note especially that this is a *second-order* equation. To show the invariance of (3.5), we use a direct calculation since we do not have the invariant coframe (2.28) at our disposal in this case. Consider a fibre-preserving map given by (2.9). Transforming the Lagrangian, we find that it has the same form (2.19), but with new coefficients $\bar{A}$ and $\bar{B}$ given by

$$A = \Phi^*(\bar{A}) = \bar{A}\left(\varphi, \psi, \frac{\psi_x + p\psi_u}{\varphi_x}\right)\frac{\psi_u}{\sqrt{\varphi_x}} \tag{3.6}$$

$$B = \Phi^*(\bar{B}) = \bar{B}\left(\varphi, \psi, \frac{\psi_x + p\psi_u}{\varphi_x}\right)\varphi_x^{3/2} + \bar{A}\left(\varphi, \psi, \frac{\psi_x + p\psi_u}{\varphi_x}\right)$$
$$\times \frac{\psi_{xx}\varphi_x - \psi_x\varphi_{xx} + 2p\psi_{xu}\varphi_x - p\psi_u\varphi_{xx} + p^2\psi_{uu}\varphi_x}{\varphi_x^{3/2}}. \tag{3.7}$$

If we set

$$F(A, B) = qA_p + 2pA + 2A_x - B_p$$

then it follows from (3.6), (3.7) and the chain rule that

$$F(A, B) = \psi_u\sqrt{\varphi_x}\,\Phi^*(F(\bar{A}, \bar{B}))$$

which proves the invariance of (3.5) when $A \neq 0$. The special branch of singular solutions satisfying $A = 0$ is also invariant, which follows directly from (3.6). This completes the proof.

However, there is another way to prove directly the invariance of (3.4) without any further calculation. We already know that (3.4) is invariant for a generic Lagrangian, i.e. one that satisfies (2.3) and (2.18). To prove that it must also therefore hold for special Lagrangians, including those of the form (2.19) and the degenerate Lagrangians $Aq + B$, we argue as follows. Note first that (3.4) is an equation which depends on L and its derivatives in a polynomial manner. Therefore, it holds on an open subset of the space coordinated by these derivatives, and so, by analytic continuation, must hold for all Lagrangians. More explicitly, suppose we have a fibre-preserving transformation (2.9). Let L and $\bar{L}$ be related Lagrangians, so, by (2.11),

$$L(x, u, p, q) = \varphi_x(x)\bar{L}(\varphi(x), \psi(x, u), \chi(x, u, p), \varpi(x, u, p, q)) \quad (3.8)$$

using the notation in (2.22) for the prolongation of the transformation to J^2, with χ, ϖ depending on p, q, and the derivatives of φ and ψ. Since (3.4) is invariant, we know that there is an identity of the form

$$D_x(LL_q^2) - LL_pL_q = \Lambda(D_{\bar{x}}(\bar{L}\bar{L}_{\bar{q}}^2) - \bar{L}\bar{L}_{\bar{p}}\bar{L}_{\bar{q}}) \quad (3.9)$$

where Λ is some as yet undetermined factor, which holds for any pair of generic Lagrangians L and $\bar{L}$ related by (3.8). If we now substitute (3.8) into this formula, the result is an identity involving (i) the coordinates x, u, p and q (ii) the Lagrangian $\bar{L}$ and its derivatives with respect to $\bar{x}$, $\bar{u}$, $\bar{p}$ and $\bar{q}$, all evaluated at (2.9) and (2.22), and (iii) the functions φ and ψ and their derivatives with respect to x and u; moreover, this identity is known to hold for generic Lagrangians $\bar{L}$. The proportionality factor Λ is not hard to determine explicitly. Since, by the chain rule,

$$\bar{q} = \frac{\psi_u}{\varphi_x^2} q + \cdots \qquad \bar{r} = \frac{\psi_u}{\varphi_x^3} r + \cdots$$

(the ellipses indicating terms involving lower-order derivatives of u), (3.8) implies

$$\frac{\partial^n L}{\partial q^n} = \frac{\psi_u^n}{\varphi_x^{2n-1}} \frac{\partial^n \bar{L}}{\partial \bar{q}^n}.$$

Therefore, the coefficient of r in $D_x(LL_q^2) - LL_pL_q$ is

$$2LL_qL_{qq} + L_q^3 = (2\bar{L}\bar{L}_{\bar{q}}\bar{L}_{\bar{q}\bar{q}} + \bar{L}_{\bar{q}}^3)\frac{\psi_u^3}{\varphi_x^3}$$

whereas the coefficient of $r = \psi_u \bar{r}/\varphi_x^3$ in $D_{\bar{x}}(\bar{L}\bar{L}_{\bar{q}}^2) - \bar{L}\bar{L}_{\bar{p}}\bar{L}_{\bar{q}}$ is

$$(2\bar{L}\bar{L}_{\bar{q}}\bar{L}_{\bar{q}\bar{q}} + \bar{L}_{\bar{q}}^3)\frac{\varphi_x^3}{\psi_u}.$$

Therefore

$$\Lambda = \frac{\psi_u^2}{\varphi_x^6}.$$

In particular, Λ only depends on φ and ψ, and hence (3.9) is a polynomial in the derivatives of $\bar{L}$. Therefore, using analytic continuation, it is easy to see that if this identity is true for a generic function $\bar{L}$, it must also hold for an arbitrary function $\bar{L}$. In particular, (3.9) will hold for the functions which fail to satisfy (2.3) and (2.18), either at a single point (a 'singularity') or in an entire subdomain, and hence (3.4) is an invariant differential equation for these Lagrangians also. Thus, our previous explicit verification for the particular Lagrangians (2.19) was not really necessary (although it is reassuring). Indeed, the same argument shows that (3.4) will remain invariant even for singular Lagrangians which fail to satisfy (2.3). This completes the proof of the theorem.

The function R given in (2.21) is not the only third-order derivative covariant which can be constructed using these methods. Note that if $\hat{I}$ is any invariant for the fibre-preserving equivalence problem, then the one-form

$$\hat{I} + \xi_4 \equiv \frac{L_q}{L}\left[\mathrm{d}q - \left(R - \frac{L^2}{L_q}\hat{I}\right)\mathrm{d}x\right] \qquad \mathrm{mod}\{\mathrm{d}u - p\,\mathrm{d}x,\, \mathrm{d}p - q\,\mathrm{d}x\} \tag{3.10}$$

is also an invariant form and, hence, by the result of proposition 16 in part I, the function

$$\hat{R} = R - \frac{L^2}{L_q}\hat{I} \tag{3.11}$$

is also a third-order derivative covariant. Thus, each choice of $\hat{I}$ leads to another invariantly defined third-order differential equation of the form (3.3). In fact, as we shall see below, third-order derivative covariants for the point and contact equivalence problems are constructed from the derivative covariant R in precisely this fashion, using particular choices of $\hat{I}$.

Theorem 8. The third-order ordinary differential equation

$$(LL_{qq} + 2L_q^2)D_x(LL_q^2) = LL_q^2(LL_q)_p \tag{3.12}$$

is invariantly associated with any second-order Lagrangian $L(x, u, p, q)$ under point transformations.

Proof. From theorem 5, we know that the one-form

$$\eta_4 \equiv \frac{L_q}{L}\left[\mathrm{d}q - \left(R - \frac{L^2}{L_q}I_{10}\right)\mathrm{d}x\right] \qquad \mathrm{mod}\{\mathrm{d}u - p\,\mathrm{d}x,\, \mathrm{d}p - q\,\mathrm{d}x\}$$

is invariant under arbitrary point transformations. Therefore, the function

$$\tilde{R} = R - \frac{L^2}{L_q}I_{10} \tag{3.13}$$

which is of the form (3.11), is a third-order derivative covariant for the point transformation equivalence problem. The corresponding invariant third-order ordinary differential equation $r = \tilde{R}$ can be written explicitly using (2.21), (2.33) and (2.37) for R and I_{10}. This proves that (3.12) is invariant for a generic Lagrangian satisfying (2.3) and (2.35). The extension to completely general Lagrangians proceeds as in the proof of theorem 7.

Theorem 9. The third-order ordinary differential equation

$$D_x(L^9L_q^2L_{qq}^4) + L^8L_qL_{qq}^2L_{qqq}D_x(L^2L_{qq}) = L^5L_qL_{qq}^3(L^5L_{qq})_p \tag{3.14}$$

is invariantly associated with any second-order Lagrangian L under contact transformations.

Proof. The proof follows that of theorem 8. The invariant coframe element

$$\zeta_4 \equiv \kappa \frac{L_q}{L} \sqrt{|I_2|} \left[\mathrm{d}q - \left(R - \frac{L^2}{L_q} I_{14} \right) \mathrm{d}x \right] \mathrm{mod}\{\mathrm{d}u - p\,\mathrm{d}x, \mathrm{d}p - q\,\mathrm{d}x\}$$

as given by (2.44) produces the third-order derivative covariant

$$R^* = R - \frac{L^2}{L_q} I_{14} \tag{3.15}$$

for the contact equivalence problem. Using the formulae (2.21), (2.33), (2.40) and (2.43) for R and I_{14}, we find (3.14) to be the explicit form of the contact invariant equation $r = R^*$.

As in (3.11), we can add in invariant multiples of the coframe element ζ_3 to produce yet more contact-invariant third-order derivative covariants, and yet more invariantly defined third-order differential equations. These all have the form

$$r = \hat{R}^* = R^* - \kappa \frac{L^2}{L_q} \sqrt{|I_2|}\, \hat{I}^* \tag{3.16}$$

where $\hat{I}^*$ is any invariant of the contact equivalence problem. Similar remarks hold for the point transformation case.

Example 10. Consider the Lagrangian

$$L(x, u, u', u'') = \tfrac{1}{2}u''^2 + f(u) \tag{3.17}$$

on a domain where $\partial f/\partial u \neq 0$. The Euler–Lagrange equation is given by

$$u'''' + f_u(u) = 0 \tag{3.18}$$

while the third-order invariant equation (3.4) is given by

$$D_x[(\tfrac{1}{2}u''^2 + f(u))u''^2] = 0 \tag{3.19}$$

or, in detail,

$$(u''^2 + f(u))u''' + \tfrac{1}{2}u'u''f_u(u) = 0. \tag{3.20}$$

We can, of course, integrate (3.19) once, leading to the second-order equation

$$(\tfrac{1}{2}u''^2 + f(u))u''^2 = k \tag{3.21}$$

for some constant k. Any common solution to the two equations (3.18) and (3.19) must satisfy the restriction

$$(u''^2 + f(u))\sqrt{2(c - f(u))} + \tfrac{1}{2}u'u''f_u(u) = 0$$

for some constant c. This shows that, generically, the invariant equation (3.4) will have solutions which are not solutions to the Euler–Lagrange equation. We remark that since (3.21) is autonomous, it can, by standard Lie methods [12], be reduced to a first-order differential equation,

$$(\tfrac{1}{2}p^2p'^2 + f(u))p^2p'^2 = k$$

for the derivative variable $p = u'$ as a function of u, $p = p(u)$, the solution of which allows that to the original equation to be reconstructed by quadrature.

Similarly, the equation (3.12) invariant under arbitrary point transformations takes the form

$$(\tfrac{5}{2}u''^2 + f(u))D_x[(\tfrac{1}{2}u''^2 + f(u))u''^2] = 0 \tag{3.22}$$

which has two distinct families of solutions:

$$\tfrac{5}{2}u''^2 + f(u) = 0 \qquad \text{or} \qquad (\tfrac{1}{2}u''^2 + f(u))u''^2 = k.$$

Finally, the equation (3.14) invariant under arbitrary contact transformations takes the form

$$D_x[(\tfrac{1}{2}u''^2 + f(u))^9 u''^2] = 0 \tag{3.23}$$

which can also be integrated once:

$$(\tfrac{1}{2}u''^2 + f(u))u''^{2/9} = k.$$

Note the striking similarity of the fibre-preserving equation (3.19) and the contact invariant equation (3.23), the only difference being the exponent. This leads to the question as to whether other exponents besides 1 and 9 lead to invariant equations, but, as far as we can see, only these two exponents are admissible.

Example 11. The Korteweg–de Vries equation of soliton fame is a Hamiltonian system [12], but is not an Euler–Lagrange equation. However, it can be readily converted into one, cf. [15], by going to the potential form

$$u_{xt} - 12u_x u_{xx} - u_{xxxx} = 0.$$

Its stationary (time-independent) solutions are given as solutions to the fourth-order ordinary differential equation

$$u'''' + 12u'u'' = 0 \tag{3.24}$$

which is the Euler–Lagrange equation for the Lagrangian

$$L(x, u, u', u'') = \tfrac{1}{2}u''^2 - 2u'^3. \tag{3.25}$$

The third-order fibre-preserving invariant equation (3.4) has two branches:

$$u'' = 0 \qquad \text{or} \qquad (2u''^2 + 4u'^3)u''' + 3u'^2u''^2 - 12u'^5 = 0. \tag{3.26}$$

The point transformation invariant equation (3.12) also has two branches:

$$u'' = 0 \qquad \text{or} \qquad (5u''^2 + 4u'^3)(u''^2 + 2u'^3)u''' + 12u'^2u''^4 = 0. \tag{3.27}$$

The contact invariant equation (3.14) has three branches of solutions:

$$\begin{aligned} &u''^2 + 4u'^3 = 0 \qquad \text{or} \qquad u'' = 0 \qquad \text{or} \\ &(10u''^2 + 4u'^3)u''' + 39u'^2u''^2 - 60u'^5 = 0. \end{aligned} \tag{3.28}$$

We have not been able to solve these complicated third-order ordinary differential equations (although, since they are autonomous, they can be reduced to second-order equations). In particular, their connection with soliton theory remains unexplored.

It is a tantalizing problem to find the significance of these new invariant differential equations. One particular geometrical interpretation which we now present is suggested by the results of part II of this series [9], where we showed that the Poincaré–Cartan from Θ_C of an rth-order particle Lagrangian on the line, which is a one-form on J^{2r-1}, arises naturally as part of the invariant coframe associated with L by solving the equivalence problem on any jet bundle J^{r+k} under contact transformations, *provided* $k \geqslant r-1$. Since we know from part I that there is a one-to-one correspondence between the solutions of the equivalence problem on J^r and J^{r+l}, $l \geqslant 0$, one expects that, by choosing an appropriate embedding $\iota : J^r \to J^{2r-1}$, the pull-back $\iota^*\Theta_C$ of the Poincaré–Cartan form to J^r should be an *invariant* linear combination of the elements of the invariant coframe associated with L on J^r. It is at the level of this embedding that the new invariant differential equations find a geometrical interpretation.

We now make this explicit in the case of a second-order particle Lagrangian $L(x, u, p, q)$ under contact transformations. From (2.44), (2.43) and (2.28) we obtain the following expressions for the one-form ζ_3 on J^2 associated invariantly with L (satisfying $I_{12} \neq 0$) under contact transformations:

$$\zeta_3 = L\, dx + (P^*(L) + LL_q I_2 I_{14})(du - p\, dx) + L_q(dp - q\, dx). \tag{3.29}$$

The Poincaré–Cartan form is the one-form

$$\Theta_C = L\, dx + P(L)(du - p\, dx) + L_q(dp - q\, dx) \tag{3.30}$$

where $P(L)$ is the (ordinary third-order) momentum given by (2.6).

Theorem 12. The contact invariant embedding $\iota : J^2 \to J^3$ given by

$$\iota(x, u, p, q) = (x, u, p, q, R^*(x, u, p, q)) \tag{3.31}$$

where R^* is given by (3.15), pulls back the Poincaré–Cartan from Θ_C on J^3 to the contact invariant one-form ζ_3 on J^2:

$$\iota^*\Theta_C = \zeta_3.$$

Proof. From (3.29), (3.30) and the form of the embedding ι, it is immediately clear that all we have to prove is the following equality:

$$\iota^*(P(L)) = P^*(L) + LL_q I_2 I_{14}.$$

Now, from (2.6), (2.5) and (2.15),

$$P(L) = L_p - \tilde{D}_x L_q - rL_{qq}$$

hence, according to (3.31), (3.15) and (2.27),

$$\iota^*(P(L)) = L_p - \tilde{D}_x L_q - R^* L_{qq} = P^*(L) + \frac{L^2 L_{qq}}{L_q} I_{14}.$$

The proof is completed by recalling that $I_2 = LL_{qq}/L_q^2$, cf. (2.33).

In order words, theorem 12 states that we can lower the order of the derivates in the Poincaré–Cartan form so as to obtain an invariant form on the jet bundle on which the Lagrangian itself is defined by requiring that all the derivatives of order higher than two which appear in the Poincaré–Cartan form are replaced by the corresponding derivative covariants.

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