MOVING FRAMES: A BRIEF SURVEY

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The aim of this note is to survey the recent literature on the new equivariant theory of moving frames developed by the author and Mark Fels[14,15]. The classical Cartan theory[11,18], as well as its more rigorous later revival[17,22], has a fairly limited range of geometrical applications. In contrast, the new equivariant theory can be systematically applied to completely general transformation groups, including infinite-dimensional Lie pseudo-groups. The full range of new applications is surprisingly broad, including complete classification of differential invariants and their syzygies, general equivalence and symmetry problems based on differential invariant and joint invariant signatures, classical invariant theory and algebra, computer vision and object recognition, the calculus of variations, Poisson geometry and solitons, and symmetry-based numerical approximation theory.

This note begins with a very brief outline of the key construction in the finite-dimensional Lie group context, illustrated by a very simple, classical example. The second part of the note lists all current references for the various applications. There are several more detailed surveys available[16,36,37,38,41]. A very elementary introduction can be found in Chapter 8 of my recent book[35]. The full details of the method can be found in the original paper with Fels[15]. Further important developments of the general construction can be found in the recent paper with Kogan[26]. All of my papers are available on my web site.

The Basic Construction: Let G be an r-dimensional Lie group acting smoothly on an m-dimensional manifold M. The crucial idea is to decouple the moving frame theory from reliance on any form of frame bundle. In other words, in general *Moving frames* $\neq$ *Frames*! A careful study of Cartan's analysis of projective curves[11], reveals that he was well aware of this distinction, that, unfortunately, was not properly appreciated by most subsequent developers of the method.

Definition 1 A *moving frame* is a smooth, G-equivariant map $\rho : M \to G$.

The group G acts on itself by left or right multiplication. If $\rho(z)$ is any right-equivariant moving frame then $\widetilde{\rho}(z) = \rho(z)^{-1}$ is left-equivariant and conversely. In geometrical situations, one can identify left-equivariant moving frames with the geometrical frame-based versions, but these identifications break down when dealing with more general group acctions.

Theorem 2 *A moving frame exists in a neighborhood of a point $z \in M$ if and only if G acts freely and regularly near z.*

Recall that G acts *freely* if the group element that fixes a point of M is the identity, i.e., $g \cdot z = z$ for some $z \in M$ if and only if $g = e$. This implies that the orbits all have the same dimension as G itself. *Regularity* requires that, in addition, each point $x \in M$ has a system of arbitrarily small neighborhoods whose intersection with each orbit is connected.

Of course, most interesting group actions are *not* free, and therefore do not admit moving frames in the sense of Definition 1. There are three basic methods for converting a non-free action into a free action. The first is to look at the product action of G on several copies of M, leading to joint invariants, also known as "semi-differential invariants" in the computer vision literature[12,32]. The second is to prolong the group action to jet space, which is the natural setting for the traditional moving frame theory, and leads to differential invariants. Combining the two methods of prolongation and product will lead to joint differential invariants. In applications of symmetry constructions to numerical approximations of derivatives and differential invariants, one requires a unification of these different actions into a new common framework, called multispace[40].

The practical construction of a moving frame is based on Cartan's method of *normalization*[11,23].

Theorem 3 *Let G act freely and regularly on M, and let $K \subset M$ be a (local) cross-section to the group orbits. Given $z \in M$, let $g = \rho(z)$ be the unique group element that maps z to the cross-section: $g \cdot z = \rho(z) \cdot z \in K$. Then $\rho : M \to G$ is a right moving frame.*

Given local coordinates $z = (z_1, \ldots, z_m)$ on M, let $w(g, z) = g \cdot z$ be the explicit formulae for the group transformations. The right moving frame $g = \rho(z)$ associated with a *coordinate cross-section* $K = \{ z_1 = c_1, \ldots, z_r = c_r \}$ is obtained by solving the *normalization equations*

$$w_1(g, z) = c_1, \qquad \ldots \qquad w_r(g, z) = c_r, \tag{1}$$

for the group parameters $g = (g_1, \ldots, g_r)$ in terms of the coordinates $z = (z_1, \ldots, z_m)$. Substituting the moving frame formulae into the remaining transformation rules leads to a complete system of invariants for the group action. These are, in fact, the local cross-section coordinates of the cross-section representative or *normal form* $k = \rho(z) \cdot z \in K$ of $z \in M$.

Theorem 4 *If $g = \rho(z)$ is the moving frame solution to the normalization equations* (1), *then the functions*

$$I_1(z) = w_{r+1}(\rho(z), z), \qquad \ldots \qquad I_{m-r}(z) = w_m(\rho(z), z), \tag{2}$$

form a complete system of functionally independent invariants.

Example 5 Let us illustrate the theory with a very simple, well-known example: curves in the Euclidean plane. The orientation-preserving Euclidean group SE(2) acts on $M = \mathbb{R}^2$, mapping a point $z = (x, u)$ to

$$y = x\cos\theta - u\sin\theta + a, \qquad v = x\sin\theta + u\cos\theta + b. \tag{3}$$

the action is not free, and so to construct a moving frame we prolong to the jet space. (Alternatively, one could "prolong" by taking Cartesian products.) For a parametrized curve $z(t) = (x(t), u(t))$, the prolonged group transformations

$$v_y = \frac{dv}{dy} = \frac{x_t\sin\theta + u_t\cos\theta}{x_t\cos\theta - u_t\sin\theta}\,, \qquad v_{yy} = \frac{d^2v}{dy^2} = \frac{x_t u_{tt} - x_{tt}u_t}{(x_t\cos\theta - u_t\sin\theta)^3}\,, \tag{4}$$

and so on, are found by successively applying implicit differentiation operator

$$D_y = \frac{1}{x_t\cos\theta - u_t\sin\theta}\,D_t \tag{5}$$

to v. The classical Euclidean moving frame for planar curves[18], follows from the cross-section normalizations

$$y = 0, \qquad v = 0, \qquad v_y = 0. \tag{6}$$

Solving for the group parameters $g = (\theta, a, b)$ leads to the right-equivariant moving frame

$$\theta = -\tan^{-1}\frac{u_t}{x_t}\,, \qquad a = -\,\frac{x x_t + u u_t}{\sqrt{x_t^2 + u_t^2}}\,, \qquad b = \frac{x u_t - u x_t}{\sqrt{x_t^2 + u_t^2}}\,. \tag{7}$$

The inverse group transformation $g^{-1} = (\widetilde{\theta}, \widetilde{a}, \widetilde{b})$ is the classical left moving frame[11,18]: one identifies the translation component $(\widetilde{a}, \widetilde{b}) = (x, u) = z$ as the point on the curve, while the columns of the rotation matrix $R_{\widetilde{\theta}} = (\mathbf{t}, \mathbf{n})$ are the unit tangent and unit normal vectors. Substituting the moving frame normalizations (7) into the prolonged transformation formulae (4), results in the fundamental differential invariants

$$v_{yy} \longmapsto \kappa = \frac{x_t u_{tt} - x_{tt}u_t}{(x_t^2 + u_t^2)^{3/2}}\,, \qquad v_{yyy} \longmapsto \frac{d\kappa}{ds}\,, \qquad v_{yyyy} \longmapsto \frac{d^2\kappa}{ds^2} + 3\kappa^3, \tag{8}$$

where $D_s = (x_t^2 + u_t^2)^{-1/2}\,D_t$ is the arc length derivative — which is itself found by substituting the moving frame formulae (7) into the implicit differentiation operator (5). A complete system of differential invariants for the planar Euclidean group is provided by the curvature and its successive derivatives with respect to arc length: $\kappa, \kappa_s, \kappa_{ss}, \ldots$.

The one caveat is that the first prolongation of SE(2) is only locally free on J^1 since a 180° rotation has trivial first prolongation. The even derivatives of κ with respect to s change sign under a 180° rotation, and so only their absolute values are fully invariant. The ambiguity can be removed by including the second order constraint $v_{yy} > 0$ in the derivation of the moving frame. Extending the analysis to the full Euclidean group E(2) adds in a second sign ambiguity which can only be resolved at third order[39].

We now survey of the current applications of this basic construction.

Classification of Differential Invariants and Syzygies: The moving frame method was used to completely solve the main classification problems for differential invariants[15]. The recurrence formulae relating the differentiated invariants and the normalized invariants, as in (8), are constructed by purely infinitesimal methods, using only linear algebra and differentiation. The recurrence formulae lead to a complete solution to the problem of classifying syzygies (functional relations) among differential invariants. The moving frame construction was used to clarify the singularities and geometric structure of prolonged group actions on submanifolds[37]. These ideas were extended[26,27] to construct a group-invariant version of the full variational bicomplex[1,2,42].

Inductive Construction: Kogan[24,25] establishes a useful inductive method for building a moving frame for a large group based on a moving frame for a subgroup. The inductive algorithm leads to the general formulae relating the differential invariants of groups and their subgroups.

Joint Invariants and Joint Differential Invariants: The moving frame method provides a direct route to the classification of joint invariants and joint differential invariants[15,39]. Further developments appear in Boutin's thesis[5,6].

Equivalence, Symmetry and Rigidity: The fundamental differential invariants, as specified by the recurrence formulae, serve to parametrize the signature manifold associated with a given submanifold. For example the Euclidean signature of a plane curve is the curve parametrized by the first two differential invariants κ, κ_s. The signature completely solves the basic equivalence problem: Two submanifolds be mapped to each other by a group transformation if and only if they have the same signature[15,10,35]. Extensions to noise-resistant joint invariant signatures are extensively developed[39]. Applications include general rigidity theorems for submanifolds under group actions[15].

Calculus of Variations: Most modern physical theories begin by postulating a symmetry group and then formulating field equations based on a group-invariant variational principle. As first recognized by Sophus Lie[29], every invariant variational problem can be written in terms of the differential invariants of the symmetry group. The associated Euler-Lagrange equations inherit the symmetry group of the variational problem, and so can also be written in terms of the differential invariants. The moving frame constructions were applied to establish a general group-invariant formula that enables one to directly construct the Euler-Lagrange equations from the invariant form of the variational problem[26,27]. These results are based on the invariant variational bicomplex construction and the resulting recurrence formulae. An alternative foundation of the subject, based on a new approach to symmetry reduction of exterior differential systems and variational problems, can be found in Itskov[20].

Classical Invariant Theory: The moving frame theory was applied to produce new, practical algorithms for solving the basic symmetry and equivalence problems of univariate polynomials (binary forms) that form the foundation of classical invariant theory[35,3,24]. An early version of the required signature was based on a fortuitous connection with a Cartan equivalence problem in the calculus of variations[33,34]. Extensions to polynomials in several variables can be found in Kogan's thesis[24].

Poisson Geometry and Solitons: Moving frames have been used to classify the differential invariants of projective curves and surfaces, and applied to generate integrable Poisson flows in soliton theory[31]. A similar construction for space curves under the conformal group appears in Marí Beffa[30].

Computer Vision: Earlier work on applications of the Cartan moving frame theory can be found in Faugeras[13]. The general characterization of submanifolds via their differential invariant signatures was applied to the problem of object recognition and symmetry detection in digital images[10]. Boutin[5,7] applies moving frame methods to the problems of polygon recognition and symmetry detection. Extensions to projective actions appear in the recent thesis of Hann[19].

Numerical Methods and Geometric Integration: The approximation of higher order differential invariants by joint invariants underlies the formulation of fully invariant finite difference numerical schemes[9,10,4,5]. Applications of moving frames to the construction of invariant numerical algorithms and the theory of geometric integration[8,21] are under development[40,28].

Infinite-dimensional Pseudo-groups: The moving frame algorithm has been extended to several examples of infinite-dimensional pseudo-group actions[14]. However, a full, rigorous foundation for the theory has yet to be completed. Once completed, the theory will produce pseudo-group versions of all of the preceding applications.

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