Recursive Moving Frames for Lie Pseudo-Groups

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Abstract

This paper introduces a new, fully recursive algorithm for computing moving frames and differential invariants of Lie pseudo-group actions. The recursive method avoids unwieldy symbolic expressions that complicate the treatment of large scale applications of the equivariant moving frame method. The development leads to novel results on partial moving frames, structure equations, and new differential operators underlying the moving frame construction. In particular, our methods produce a streamlined computational algorithm for determining moving frames and differential invariants of finite-dimensional Lie group actions.

1 Introduction

The theory of equivariant moving frames, first formulated for finite-dimensional Lie group actions in [17] and then extended to infinite-dimensional Lie pseudo-groups in [56], is a new theoretical formulation of the classical moving frame method most closely associated with Élie Cartan, [10,21]. The wide range of new applications emerging from this new foundation of moving frames underscores its significance. Equivariant moving frames are used for object recognition and symmetry detection in image processing, [8, 23, 33], classical invariant theory, [5, 31, 49, 60, 71], symmetries of differential equations, [14, 45, 46], group foliations of partial differential equations for constructing invariant, partially invariant, and non-invariant solutions, [68], invariant

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variational problems and invariant geometric submanifold flows, [32, 52, 67], invariant reformulations of Noether's two theorems relating symmetries of variational problems and conservation laws of their Euler–Lagrange equations, [18–20], Poisson geometry and integrable systems, [43], turbulence modeling in geophysical fluid dynamics, [6], and many other areas of mathematics and its applications. They have further been employed to construct joint invariants, joint differential invariants, and invariant numerical schemes, [6, 29, 50, 51, 61], Laplace invariants of differential operators, [63], invariants and covariants of Killing tensors arising in general relativity, [16, 44], invariants of Lie algebras, [7], and much more. The method also provides an intriguing alternative to the Cartan exterior differential systems approach to solving general equivalence problems, [70], and has recently been used to produce the first rigorous proof of the termination of Cartan's equivalence method, [2, 3]. Extensions to the discrete realm can be found in [4, 41, 42, 69]. See [24, 25] for an algebraic reformulation of the method. Further developments can be found in the book [40] and survey article [53].

In the standard implementation of the equivariant method, the pseudo-group action is first prolonged to a sufficiently high order jet space so that the action becomes (locally) free. Once freeness is achieved, the specification of a local cross-section to the prolonged pseudo-group orbits enables the construction of a moving frame by solving the associated normalization equations for the pseudo-group parameters. With a moving frame in hand, the corresponding invariantization map is used to produce complete systems of differential invariants, invariant differential operators, and invariant differential forms. The most important new contribution of the equivariant method is the *recurrence formulae*, that relate the differentiated and normalized differential invariants and invariant differential forms, and thereby specify the entire structure of the algebra of differential invariants, as well as the invariant variational bicomplex, [32]. Remarkably, these formulae can be explicitly determined, through elementary linear algebra, ab initio using only the equations for the cross-section that will be used to construct the moving frame and the formulae for the infinitesimal generators of the pseudo-group action.

While the idea of prolonging the pseudo-group action to reach freeness proved to be of crucial importance in the initial development of the general theory and basic algorithms, the computation of the prolonged action, which relies on implicit differentiation, can lead to unwieldy expressions that limit the method's practical scope and implementation through symbolic computation. The aim of the recursive method is, in the spirit of the original Cartan approach, [10, 21, 27], to recursively normalize as many pseudo-group parameters as possible before prolonging to the next higher order jet space. The overall goal is to avoid, as much as possible, unnecessary large intermediate formulae and thereby control, or at least moderate, the associated expression swell that can overwhelm available computational resources.

For local Lie group actions, the foundations of the recursive method were introduced in [54], which generalized and extended Kogan's inductive method, [30]. The key insight was to base the computations on the lifted recurrence formulas and the normalized Maurer–Cartan forms, rather than on the implicit differentiation operators employed in the standard prolongation approach. This paper culminates with an example which reveals that the proposed algorithm is not entirely recursive in the case of infinite-dimensional Lie pseudo-groups because it relies on the explicit (potentially unwieldy) expressions for the higher order Maurer–Cartan forms, which need to be computed in advance. Thus, while this version of a recursive algorithm successfully avoids computing the higher order prolonged actions at the outset, it trades this complication for similarly intricate expressions for the higher order Maurer–Cartan forms. Our present contribution is a *fully recursive* algorithm that enables one to *simultaneously* construct the partially normalized moving frame, the invariants, and the Maurer–Cartan forms. In particular, the partially normalized prolonged pseudo-group transformations and higher order Maurer–Cartan forms are only computed when required, after lower order normalizations are already in place, thereby significantly reducing the overall complexity of the required expressions. Moreover, our approach can be viewed as a new, fully recursive algorithm for finite-dimensional Lie group actions that can be applied even in the absence of explicit formulae for the associated Maurer–Cartan forms.

In order to justify the recursive moving frame algorithm, certain mild assumptions on the (prolonged) pseudo-group action are imposed. As in [56, 57], the pseudo-group is assumed to act freely on a sufficiently high order jet space, so that the recursive method produces a moving frame. Extending the recursive algorithm to pseudo-group actions that do not eventually become free is an interesting open problem. Indeed, even in suitable non-free cases, our algorithm can still be implemented to produce differential invariants, even though a Fundamental Basis Theorem for differential invariants is not yet known for such actions. There are two additional technical requirements that we need to impose, which can be found in Propositions 8.6 and 8.19. They involve the coordinate dependencies of the associated Maurer–Cartan forms, and serve to guarantee the success of our recursive construction of the partially normalized Maurer–Cartan forms. It is straightforward to verify these assumptions in practice; indeed, they hold for the vast majority of examples arising in applications. In particular, they are satisfied by all "translational Lie pseudo-groups", meaning those that contain the transitive sub-pseudo-group of pure translations $z \longmapsto z + c$ in some coordinate system, cf. Definition 2.7. More generally, if $\mathcal{G}$ is the one-to-one prolongation of a translational Lie pseudo-group, as in Definition 2.10, then the recursive algorithm presented in this paper succeeds. Our algorithm can also be readily adapted to intransitive Lie pseudo-groups, again provided the required technical assumptions hold.

We remain unsatisfied that, while applicable to a wide range of pseudo-group actions arising in applications, our underlying assumptions are coordinate-dependent. However, we have so far been frustrated in our search for a fully coordinate-free formulation, or, even better, a recursive algorithm that succeeds without such technical restrictions. Thus, determining the weakest constraints on the pseudo-group action that allows the implementation of a fully recursive algorithm is still an open question. On the other hand, in the event that a pseudo-group does not meet the constraints imposed in the present paper, one can always employ the "slightly less" recursive algorithm introduced in [54].

2 Lie Pseudo-Groups

Let M be an m-dimensional manifold. We assume smoothness, so that M is of class C^∞, although many treatments of Lie pseudo-groups make the stronger assumption of analyticity due to its role in the Cartan–Kähler existence theorem, [22, 37, 48]. Let us begin by recalling the standard definition of a pseudo-group.

Definition 2.1. A collection $\mathcal{G}$ of local diffeomorphisms of a manifold M is called a *pseudo-group* if

- $U \subset M$ is an open set and $\varphi : U \to M$ belongs to $\mathcal{G}$, then its restriction $\varphi \mid V \in \mathcal{G}$ for all open set $V \subset U$.
- $U_\alpha \subset M$ are open subsets, $U = \bigcup_\alpha U_\alpha$, and $\varphi : U \to M$ is a local diffeomorphism with $\varphi \,|\, U_\alpha \in \mathcal{G}$ for all α, then $\varphi \in \mathcal{G}$.
- $\varphi : U \to M$ and $\psi : V \to M$ are two local diffeomorphisms belonging to $\mathcal{G}$ with $\varphi(U) \subset V$, then $\psi \circ \varphi \in \mathcal{G}$.
- $\varphi : U \to M$ is in $\mathcal{G}$, and $V = \varphi(U)$, then $\varphi^{-1} \colon V \to M$ is also in $\mathcal{G}$.

Note that these requirements imply that $\mathcal{G}$ necessarily contains the identity diffeomorphism, denoted $\mathbb{1}(z) \equiv z$ for all $z \in M$. The simplest example of a pseudo-group is the collection of all local diffeomorphisms of a manifold M, denoted $\mathcal{D} = \mathcal{D}(M)$.

Remark on notation: For brevity, we allow the domain of a function to be an open subset of its source set; in particular, we will write a local diffeomorphism as $\varphi\colon M \to M$ even when $\operatorname{dom} \varphi \subsetneq M$. As in [55, 56], we employ Cartan's notational convention and write the local coordinate formulas for a local diffeomorphism as $Z = \varphi(z)$, where we systematically use lower case letters — in this case z — to denote the *source coordinates* and the corresponding capital letters — in this case Z — for the *target coordinates.*

Given $0 \le n \le \infty$, let $\mathcal{D}^{(n)} \subset \mathrm{J}^n(M, M)$ denote the bundle consisting of the n–jets of local diffeomorphisms. Coordinates on $\mathcal{D}^{(n)}$ are denoted by $\mathrm{j}_n\varphi|_z = (z, Z^{(n)})$, where $Z^{(n)}$ indicates the collection of target jet coordinates Z^a_B, representing the derivative $\partial^k Z^a/\partial z^{b^1} \cdots \partial z^{b^k}$, where $1 \le a \le m$, and $B = (b^1, \ldots, b^k)$ is a symmetric multi-index with $1 \le b^\nu \le m$ and $0 \le k = \# B \le n$. When $0 \le n < k \le \infty$, we let $\pi^k_n \colon \mathcal{D}^{(k)} \to \mathcal{D}^{(n)}$ denote the standard projection that takes a k-jet to an n-jet. To keep the notation uncluttered, we consistently identify (differential) functions and differential forms on $\mathcal{D}^{(n)}$ with their pull-backs under π^k_n for all $k > n$.

The diffeomorphism pseudo-group jet bundle $\mathcal{D}^{(n)}$ is endowed with the structure of a groupoid, [39, 72], with source map $\boldsymbol{\sigma}^{(n)}(z, \varphi^{(n)}) = z$ and target map $\boldsymbol{\tau}^{(n)}(z, \varphi^{(n)}) = \varphi(z) = Z$:

$$\begin{array}{ccc} & \mathcal{D}^{(n)} & \\ {\scriptstyle \boldsymbol{\sigma}^{(n)}} \swarrow & & \searrow {\scriptstyle \boldsymbol{\tau}^{(n)}} \\ M & & M \end{array}$$

In the (projective) limit, we write $\boldsymbol{\sigma} = \boldsymbol{\sigma}^{(\infty)}$ and $\boldsymbol{\tau} = \boldsymbol{\tau}^{(\infty)}$. The diffeomorphism pseudo-group $\mathcal{D}$ acts on $\mathcal{D}^{(n)}$ either by right composition

$$R_\varphi(\mathrm{j}_n\psi|_z) = \mathrm{j}_n\psi|_z \cdot \mathrm{j}_n\varphi^{-1}|_{\varphi(z)} = \mathrm{j}_n(\psi \circ \varphi^{-1})|_{\varphi(z)}, \quad \text{where} \quad z \in \operatorname{dom} \varphi \cap \operatorname{dom} \psi, \tag{2.1a}$$

or left composition

$$L_\varphi(\mathrm{j}_n\psi|_z) = \mathrm{j}_n\varphi|_{\psi(z)} \cdot \mathrm{j}_n\psi|_z = \mathrm{j}_n(\varphi \circ \psi)|_z, \quad \text{where} \quad z \in \operatorname{dom} \psi \cap \psi^{-1}(\operatorname{dom} \varphi). \tag{2.1b}$$

Several versions, not all equivalent, of the technical definition of a Lie pseudo-group can be found in the literature: [11, 22, 28, 35, 37, 64]. In this paper, we rely on the following variant.

Definition 2.2. A sub-pseudo-group $\mathcal{G} \subset \mathcal{D}$ is called a *Lie pseudo-group* if there exists $n^* \ge 1$ such that, for all $n^* \le n < \infty$, the pseudo-group jets $\mathcal{G}^{(n)} \subset \mathcal{D}^{(n)}$ form a subbundle, the induced projection $\pi^{n+1}_n \colon \mathcal{G}^{(n+1)} \to \mathcal{G}^{(n)}$ is a fibration, and every local diffeomorphism $\varphi \in \mathcal{D}$ that satisfies $\mathrm{j}_n\varphi \subset \mathcal{G}^{(n)}$ belongs to $\mathcal{G}$. The minimal value of n^* is called the *order* of the Lie pseudo-group.

In local coordinates, the subbundle $\mathcal{G}^{(n)} \subset \mathcal{D}^{(n)}$ is described by a nonlinear system of partial differential equations

$$F^{(n)}(z, Z^{(n)}) = 0, \tag{2.2}$$

known as the n^{th} order *determining system* of the Lie pseudo-group.

In the sequel, we will use $g^{(n)}$ to denote a convenient system of *pseudo-group parameters* of order n, so that the algebraic solution to the determining system (2.2) is given (locally) by $Z^{(n)} = \Psi^{(n)}(z, g^{(n)})$. These parameters can be either obtained via elimination as the remaining independent components of the diffeomorphism jet coordinates $Z^{(n)}$ once the system (2.2) has been algebraic solved, or as some alternative convenient set of variables that is used to parametrize their solution.

As a consequence of the pseudo-group axioms, the determining system forms a system of *Lie equations*, [36], meaning that the composition of any two solutions is, where defined, also a solution, as is the inverse of any solution. This implies that the determining system (2.2) is invariant under right and left composition (2.1). Furthermore, Definition 2.2 implies the following formal integrability/involutivity property of the determining equations of a Lie pseudo-group: for any $n \geq n^*$, the n^{th} order determining system is obtained by differentiating the determining system of order n^*. More precisely, let

$$\mathbb{D}_{z^b} = \frac{\partial}{\partial z^b} + \sum_{a=1}^{m} \sum_{\#B\geq 0} Z^a_{B,b} \frac{\partial}{\partial Z^a_B}, \qquad b = 1, \ldots, m, \tag{2.3}$$

denote the standard *total derivative operators* on the diffeomorphism jet bundle $\mathcal{D}^{(\infty)}$. Then the n^{th} order determining system is given by

$$\mathbb{D}^B_z\, F^{(n^*)}(z, Z^{(n^*)}) = 0, \qquad \text{for all} \qquad 0 \leq \#B \leq n - n^*, \tag{2.4}$$

where

$$\mathbb{D}^B_z = \mathbb{D}_{z^{b^1}} \cdots \mathbb{D}_{z^{b^k}}, \qquad 1 \leq b^\nu \leq m, \qquad k = \#B. \tag{2.5}$$

See [26, 55] for an in depth discussion of how the formal integrability and involutivity of Lie pseudo-groups is related to the underlying definition.

Remark 2.3. It is essential that, when determining the structure equations of the Lie pseudo-group, [55], and, moreover, for the implementation of our algorithm to succeed, we consider the determining equations including *all* their differential consequences as in (2.4), at *all* orders n, and not just the order n^* of the pseudo-group.

Transitioning to the infinitesimal setting, let $\mathbf{v}$ denote a (locally defined) vector field on M, that is a local section of the tangent bundle $\mathbf{v}\colon M \to TM$. For $0 \leq n \leq \infty$, let $\mathrm{j}_n\mathbf{v}\colon M \to \mathrm{J}^n TM$ be its n^{th} order jet. In the local coordinates $z = (z^1, \ldots, z^m)$ on M, we write

$$\mathbf{v} = \sum_{a=1}^{m} \zeta^a(z) \frac{\partial}{\partial z^a}. \tag{2.6}$$

The induced local coordinates on $\mathrm{J}^n TM$ are denoted

$$\zeta^{(n)} = (\ \ldots\ \zeta^a_B\ \ldots\ ), \qquad 1 \leq a \leq m, \qquad 0 \leq \#B \leq n, \tag{2.7}$$

where ζ^a_B represents the derivative $\partial^k \zeta^a / \partial z^{b^1} \cdots \partial z^{b^k}$ with $1 \leq b^\nu \leq m$ and $k = \#B$.

Let $\mathfrak{g}$ denote the space of *infinitesimal generators* of the pseudo-group, which consists of all local vector fields $\mathbf{v}$ whose corresponding flow $\exp(t\,\mathbf{v})$ forms a (local) one-parameter subgroup of $\mathcal{G}$. A vector field (2.6) is in $\mathfrak{g}$ if and only if, for each $n \geq n^*$, its n^{th} order jet $\mathrm{j}_n\mathbf{v}$ satisfies the *infinitesimal determining system*

$$L^{(n)}(z, \zeta^{(n)}) = 0, \tag{2.8}$$

obtained by linearizing the (nonlinear) n^{th} order determining system (2.2) at the identity jet $\mathbb{1}^{(n)}$. The infinitesimal determining system (2.8) is a *linear Lie equation*, [36], meaning that the Lie bracket of any two solutions is again a solution on their common domain of definition, which thereby formalizes the Lie algebraic structure of $\mathfrak{g}$.

Remark 2.4. The local solvability of the infinitesimal determining system (2.8) does not seem to follow from the above definitions. It is thus customary to assume that Lie pseudo-groups are *tame*, meaning that the infinitesimal determining equations are locally solvable, [55, 56]. To the best of our knowledge, all known examples satisfy the tameness condition, and recently it was shown in [2] that a rather general class of Lie pseudo-groups are tame. On the other hand, the regularity assumptions in Definition 2.2 imply the formal integrability/involutivity of the equations (2.8).

Example 2.5. As one of our main illustrative examples of a Lie pseudo-group, we consider the *Lie–Tresse–Kumpera action*, [35, 56],

$$X = f(x), \qquad Y = y, \qquad U = \frac{u}{f'(x)}, \qquad \text{where} \qquad f \in \mathcal{D}(\mathbb{R}), \tag{2.9}$$

defined on $M = \mathbb{R}^3 \setminus \{u = 0\}$. (The condition $u \neq 0$ is to ensure that this example satisfies the underlying regularity assumptions of Definition 2.2.) The first order determining equations of this Lie pseudo-group are

$$\begin{gathered} X_y = X_u = 0, \qquad Y = y, \qquad Y_x = Y_u = 0, \qquad Y_y = 1, \\ U\,X_x = u, \qquad U_u\,X_x = 1, \qquad U_y = 0. \end{gathered} \tag{2.10}$$

It is not hard to show that this system of partial differential equations is involutive, [62], and hence the higher order determining equations are obtained by (total) differentiation of (2.10) with respect to x, y, u, so that (2.9) represents their general solution.

If

$$\mathbf{v} = \xi(x,y,u)\frac{\partial}{\partial x} + \eta(x,y,u)\frac{\partial}{\partial y} + \varphi(x,y,u)\frac{\partial}{\partial u} \tag{2.11}$$

denotes a (local) vector field in $\mathbb{R}^3 \setminus \{u = 0\}$, the linearization of the first order determining system (2.10) at the identity jet yields the first order infinitesimal determining equations

$$\xi_y = \xi_u = 0, \quad \eta = \eta_x = \eta_y = \eta_u = 0, \quad \varphi = -u\,\xi_x, \quad \varphi_y = 0, \quad \varphi_u = -\xi_x. \tag{2.12}$$

Again, involutivity of this linear system of partial differential equations implies that the higher order infinitesimal determining equations are obtained by (total) differentiation with respect to x, y, u. The general solution to (2.12) is obviously

$$\mathbf{v} = a(x)\frac{\partial}{\partial x} - a'(x)\,u\frac{\partial}{\partial u}, \tag{2.13}$$

where $a(x)$ is an arbitrary smooth function, which prescribes the general infinitesimal generator of the Lie pseudo-group action (2.9).

Example 2.6. Our second main example involves the following Lie pseudo-group introduced in [56]:

$$X = f(x), \quad Y = f'(x)y + g(x) = e(x,y), \quad U = u + \frac{f''(x)y + g'(x)}{f'(x)} = u + \frac{e_x}{f_x}, \tag{2.14}$$

where $f \in \mathcal{D}(\mathbb{R})$ and $g \in \mathrm{C}^\infty(\mathbb{R})$ is an arbitrary differentiable function. The first order determining equations of this Lie pseudo-group are

$$X_u = X_y = 0, \qquad Y_y = X_x, \qquad Y_u = 0, \qquad U = u + \frac{Y_x}{X_x}, \qquad U_u = 1, \tag{2.15}$$

which form an involutive system of partial differential equations. Linearizing (2.15) at the identity jet yields the infinitesimal determining system

$$\xi_y = \xi_u = 0, \qquad \eta_y = \xi_x, \qquad \eta_u = 0, \qquad \varphi = \eta_x, \qquad \varphi_u = 0, \tag{2.16}$$

which is also involutive. The general solution to (2.16) determines the space of infinitesimal generators, namely,

$$\mathbf{v} = a(x)\frac{\partial}{\partial x} + \big[\, a'(x)y + b(x) \,\big]\frac{\partial}{\partial y} + \big[\, a''(x)y + b'(x) \,\big]\frac{\partial}{\partial u},$$

where $a(x)$ and $b(x)$ are two arbitrary smooth functions.

The following fairly general classes of Lie pseudo-groups will play an important role in the justification of the recursive moving frame algorithm presented in Section 8.

Definition 2.7. We will call a Lie pseudo-group $\mathcal{G}$ *translational* if, in a neighborhood of each point $z \in U \subset M$, it contains a finite-dimensional abelian Lie sub-(pseudo-)group $\mathcal{A} \subset \mathcal{G}$ that acts transitively and locally freely at each point of U. Equivalently, on U, the pseudo-group's space of infinitesimal generators contains an abelian subalgebra $\mathfrak{a} \subset \mathfrak{g}$ of dimension $m = \dim M$, spanned by m pointwise linearly independent mutually commuting vector fields $\mathbf{v}_1, \ldots, \mathbf{v}_m$, i.e., an abelian frame or pointwise basis of the tangent bundle $TU \subset TM$.

Equivalently, $\mathcal{G}$ is translational if and only if there exist *local translational coordinates* $z = (z^1, \ldots, z^m)$ such that $\mathcal{G}$ contains the *translational group*

$$\mathcal{T} = \{ Z^a = z^a + c^a \mid c^a \in \mathbb{R},\, a = 1, \ldots, m \} \subset \mathcal{G}$$

as a sub-pseudo-group. Indeed, given an abelian frame $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ in the neighborhood of a point $z_0 \in M$, the translational coordinates can be constructed by exponentiation:

$$(z^1, \ldots, z^m) \;\longleftrightarrow\; \exp(z^1 \mathbf{v}_1 + \;\cdots\; + z^m \mathbf{v}_m)\, z_0.$$

In the sequel, when using the translational condition, we will always work in these adapted translational coordinates.

Proposition 2.8. A Lie pseudo-group is translational, if and only if there exists local coordinates (z, Z) on $M \times M$ such that its determining system is independent of the source and target coordinates z^a, Z^a, i.e., of the form

$$F^{(n)}(Z^{(1,n)}) = 0, \tag{2.17}$$

where

$$Z^{(1,n)} = (\ldots\, Z^a_B \,\ldots) \qquad 1 \leq a \leq m,, \qquad 1 \leq \#B \leq n,$$

denotes the pseudo-group jet coordinates of order between 1 and n.

Proof. We begin by proving sufficiency. Since $Z^{(n)} = (Z, Z^{(1,n)})$, let

$$F^{(n)}(z, Z, Z^{(1,n)}) = 0$$

be the n^{th} order determining system of $\mathcal{G}$, and consider an arbitrary translation

$$\psi \colon z \longmapsto Z = z + c, \tag{2.18}$$

so $\psi \in \mathcal{T} \subset \mathcal{G}$. Then, the invariance of the determining system under left and right composition implies

$$\begin{aligned} 0 &= L_\psi^* F^{(n)}(z, Z, Z^{(1,n)}) = F^{(n)}(z, Z + c, Z^{(1,n)}), \\ 0 &= R_\psi^* F^{(n)}(z, Z, Z^{(1,n)}) = F^{(n)}(z + c, Z, Z^{(1,n)}). \end{aligned}$$

which yields (2.17).

To show necessity, we observe that the translation (2.18) and the identity map $\mathbb{1}(z) = z$ share the same pseudo-group jets of order ≥ 1:

$$\psi^{(1,n)} = \mathbb{1}^{(1,n)}.$$

Since the identity map belongs to any pseudo-group, the latter is a solution of the determining system (2.17), and thus

$$F^{(n)}(\psi^{(1,n)}) = F^{(n)}(\mathbb{1}^{(1,n)}) = 0.$$

Definition 2.2 implies that $\psi \in \mathcal{G}$, and hence $\mathcal{T} \subset \mathcal{G}$. □

Corollary 2.9. A Lie pseudo-group $\mathcal{G}$ is translational if and only if, in translational coordinates, its infinitesimal determining system has the form

$$L^{(n)}(\zeta^{(1,n)}) = 0, \tag{2.19}$$

where $\zeta^{(1,n)} = (\ldots\ \zeta_B^a\ \ldots)$, $1 \leq \#B \leq n$, $1 \leq a \leq m$, denotes the vector field jets of order between 1 and n.

A substantial generalization of the translational condition is provided by pseudo-groups that are one-to-one prolongations of translational pseudo-groups, in accordance with the following construction.

Definition 2.10. A Lie pseudo-group $\mathcal{G}$ acting on M is said to be an *one-to-one prolongation* of a Lie pseudo-group $\mathcal{H}$ acting on N if there is a surjective submersion $\pi \colon M \to N$ such that there is a one-to-one correspondence between elements $h \in \mathcal{H}$ and $g \in \mathcal{G}$ satisfying $\pi \circ g = h \circ \pi$. We further assume that the correspondence $g \longleftrightarrow h$ is smooth, that is, for all $n \geq 0$, the map relating their respective pseudo-group parameters $g^{(n)} \longleftrightarrow h^{(n)}$ is smooth.

Remark 2.11. In [38, 59, 65] and elsewhere, the smoothness assumption on the correspondence $g \longleftrightarrow h$ is not explicitly stated, but, to the best of our understanding, is implicitly assumed.

Remark 2.12. In [59], the pseudo-group $\mathcal{G}$ satisfying the condition in Definition 2.10 was called an "isomorphic prolongation" of the pseudo-group $\mathcal{H}$. In his book [65], Stormark, in fact, introduces two inequivalent notions of "isomorphic prolongation" of pseudo-groups. Definition 2.10 is in conformity with that provided on page 170 for general pseudo-groups, but not with the one on page 381 for Lie pseudo-groups, which is formulated in terms of the size of the solution space to their determining equations. For the latter context, Stormark introduces the more restrictive term "one-to-one prolongation", which coincides with our Definition 2.10. Hence, in order to avoid any possible confusion, we will use the term "one-to-one prolongation" throughout.

Example 2.13. The Lie–Tresse–Kumpera pseudo-group (2.9) is a one-to-one prolongation of the diffeomorphism pseudo-group $X = f(x)$.

Example 2.14. The pseudo-group (2.14) is a one-to-one prolongation of the pseudo-group action

$$X = f(x), \qquad Y = f'(x)y + g(x), \tag{2.20}$$

on $N = \mathbb{R}^2$.

Many of the Lie pseudo-groups $\mathcal{G}$ of importance in applications, are *one-to-one prolongations* of Lie pseudo-groups $\mathcal{H}$ acting on lower dimensional manifolds. Examples include infinite-dimensional symmetry groups of integrable differential equations in more than one space dimension, such as the KP equation, [15] and the Davey–Stewartson equations, [13], as well as many important equations arising in fluid dynamics, including the Euler and Navier–Stokes equations, and systems arising in boundary layer theory, [9,47]. Equivalence pseudo-groups of coframes that appear in Cartan's equivalence method are also often of this form, [48,70]. Given a Lie pseudo-group $\mathcal{H}$, the n^{th} order prolonged pseudo-group $\mathcal{H}^{(n)}$ acting on the diffeomorphism jet bundle $\mathcal{D}^{(n)}(N)$ is also a Lie pseudo-group, and is naturally a one-to-one prolongation of $\mathcal{H}$. The pseudo-group $\mathcal{H}^{(n)}$ is called a *normal prolongation* of $\mathcal{H}$.

When $\mathcal{G}$ is a one-to-one prolongation of $\mathcal{H}$, the structural properties of the pseudo-group $\mathcal{G}$ are effectively encapsulated in its "projection" $\mathcal{H}$. Given a one-to-one prolongation $\mathcal{G}$ of $\mathcal{H}$, we introduce the adapted local coordinates

$$\begin{array}{c} z = (z_\sharp, z_\flat) = (\ldots z^{a_\sharp} \ldots z^{b_\flat} \ldots) = (z^1, \ldots, z^\ell, z^{\ell+1}, \ldots, z^m) \in M \\ \Big\downarrow \pi \\ z_\sharp = (\ldots z^{a_\sharp} \ldots) = (z^1, \ldots, z^\ell) \in N \end{array} \tag{2.21}$$

in accordance with Definition 2.10. This induces a splitting of the pseudo-group target variables and vector field coefficients

$$\begin{array}{ccc} Z = (Z_\sharp, Z_\flat) = (Z^1, \ldots, Z^m) \in \mathcal{G} & \qquad & \zeta = (\zeta_\sharp, \zeta_\flat) = (\zeta^1, \ldots, \zeta^m) \in \mathfrak{g} \\ \Big\downarrow \pi & & \Big\downarrow \pi \\ Z_\sharp = (Z^1, \ldots, Z^\ell) \in \mathcal{H} & & \zeta_\sharp = (\zeta^1, \ldots, \zeta^\ell) \in \mathfrak{h}. \end{array}$$

In terms of these adapted local coordinates, we can formulate the following characterization of a one-to-one prolongation of a Lie pseudo-group.

Proposition 2.15. If $\mathcal{G}$ is a one-to-one prolongation of $\mathcal{H}$, then its determining system is of the form

$$F^{(n)}_{\mathcal{H}}(z_\sharp, Z^{(n)}_\sharp) = 0, \qquad \frac{\partial Z_\sharp}{\partial z_\flat} = 0, \qquad Z_\flat = F_\flat(z, Z^{(n)}_\sharp), \tag{2.22a}$$

where the first set of equations represents the determining system of $\mathcal{H}$. Infinitesimally, the determining system of $\mathfrak{g}$ takes the form

$$L^{(n)}_{\mathfrak{h}}(z_\sharp, \zeta^{(n)}_\sharp) = 0, \qquad \frac{\partial \zeta_\sharp}{\partial z_\flat} = 0, \qquad \zeta_\flat = L_\flat(z, \zeta^{(n)}_\sharp). \tag{2.22b}$$

Proof. In the system of local coordinates (2.21), since $\mathcal{H}$ is a Lie pseudo-group acting on N, any diffeomorphism $\varphi \in \mathcal{H}$ is of the form $Z_\sharp = \varphi(z_\sharp)$. Thus, φ is independent of the fiber coordinates $z_\flat$, which is reflected in the second equation in (2.22a), and satisfies the determining system given by the first equation of (2.22a). Next, since $\mathcal{G}$ is a one-to-one prolongation of $\mathcal{H}$, the uniqueness assumption requires all jets of $\psi \in \mathcal{G}$ at the source point z to be uniquely specified by the jets of some diffeomorphism $\varphi \in \mathcal{H}$, which is encapsulated in the third equation in (2.22a). Finally, the infinitesimal determining system is obtained by linearizing the determining system (2.22a) at the identity jet, leading immediately to (2.22b). □

Corollary 2.16. If $\mathcal{G}$ is a one-to-one prolongation of a translational pseudo-group $\mathcal{H}$, then its determining system is of the form

$$F^{(n)}_{\mathcal{H}}(Z^{(1,n)}_\sharp) = 0, \qquad \frac{\partial Z_\sharp}{\partial z_\flat} = 0, \qquad Z_\flat = F_\flat(z_\flat, Z^{(n)}_\sharp),$$

and the infinitesimal determining system of $\mathfrak{g}$ takes the form

$$L^{(n)}_{\mathfrak{h}}(\zeta^{(1,n)}_\sharp) = 0, \qquad \frac{\partial \zeta_\sharp}{\partial z_\flat} = 0, \qquad \zeta_\flat = L_\flat(z_\flat, \zeta^{(n)}_\sharp). \tag{2.23}$$

Example 2.17. To illustrate Proposition 2.15, we revisit Examples 2.6 and 2.14. Following the notation of Proposition 2.15, we have

$$z_\sharp = (x, y) \qquad \text{and} \qquad z_\flat = u.$$

Recalling the determining equations (2.15) of the Lie pseudo-group (2.14), the first equation in (2.22a) corresponds to the equations

$$X_y = 0, \qquad Y_y = X_x.$$

On the other hand, the middle equation of (2.22a) corresponds to the equations

$$X_u = Y_u = 0. \tag{2.24a}$$

Finally, the third equation in (2.22a) is given by the determining equation

$$U = u + \frac{Y_x}{X_x}. \tag{2.24b}$$

The remaining determining equation in (2.15), namely $U_u = 1$, is a differential consequence of (2.24).

3 The Lifted Bundle

Let $\mathcal{G}$ be a Lie pseudo-group acting on an m-dimensional manifold M. We are particularly interested in the induced action on submanifolds $S \subset M$ of dimension $1 \leq p < m$. For $0 \leq n \leq \infty$, let $\mathrm{J}^n = \mathrm{J}^n(M,p)$ denote the n^{th} order extended jet bundle consisting of equivalence classes of p-dimensional submanifolds under the equivalence relation of n^{th} order contact, [48]. (*Note*: the submanifold jet bundle $\mathrm{J}^n = \mathrm{J}^n(M,p)$ is *not* the preceding Cartesian product jet bundle $\mathrm{J}^n(M,M)$ containing the diffeomorphism and pseudo-group jets.) For $n < k$, let $\widetilde{\pi}^k_n \colon \mathrm{J}^k \longrightarrow \mathrm{J}^n$ denote the standard projection. We introduce the local coordinates $z = (x,u) = (x^1,\ldots,x^p,u^1,\ldots,u^q) = (\ldots\, x^i \,\ldots\, u^\alpha \,\ldots)$ on M, where $q = m - p$, so that submanifolds that are transverse to the vertical fibers $\{x = x_0\}$ are (locally) given as graphs of smooth functions $u = f(x)$. (Non-transversal submanifolds can easily be handled by adopting alternative local coordinates.) The induced coordinates on J^n are given by

$$z^{(n)} = (x, u^{(n)}) = (\ \ldots\ x^i\ \ldots\ u^\alpha_K\ \ldots\ ),$$

where u^α_K represents the derivative $\partial^\ell u^\alpha / \partial x^{k^1} \cdots \partial x^{k^\ell}$ for $1 \leq \alpha \leq q$, and $K = (k^1,\ldots,k^\ell)$ is a symmetric multi-index with $1 \leq k^\nu \leq p$ and $0 \leq \ell = \#K \leq n$.

The action of the Lie pseudo-group $\mathcal{G}$ on M induces an action on its submanifolds, and hence on the submanifold jet bundle J^n, written

$$Z^{(n)} = g^{(n)} \cdot z^{(n)}, \qquad g \in \mathcal{G}, \tag{3.1}$$

and called the n^{th} order *prolonged action* of $\mathcal{G}$. Its local coordinate expressions are obtained by successively applying the *lifted total derivative operators*, also known as the operators of *implicit differentiation*, to the target coordinates U^α:

$$U^\alpha_K = D^K_X U^\alpha = D_{X^{k^1}} \cdots D_{X^{k^\ell}} U^\alpha, \qquad 1 \leq \alpha \leq q, \qquad \ell = \#K, \qquad 1 \leq k^\nu \leq p. \tag{3.2}$$

Here

$$D_{X^i} = \sum_{j=1}^{p} W^j_i \, D_{x^j}, \quad 1 \leq i \leq p, \qquad \text{where} \qquad (W^j_i) = (D_{x^i} X^j)^{-1} \tag{3.3}$$

is the inverse total Jacobian matrix, while

$$D_{x^i} = \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \sum_{\#K \geq 0} u^\alpha_{K,i} \frac{\partial}{\partial u^\alpha_K} + \sum_{a=1}^{m} \sum_{\#B \geq 0} \left(Z^a_{B,i} + \sum_{\alpha=1}^{q} Z^a_{B,p+\alpha} \, u^\alpha_i \right) \frac{\partial}{\partial Z^a_B} \tag{3.4}$$

can be viewed as (extended) total derivative operators on the lifted bundle, noting that the expression in parentheses in the final summation is the "total derivative" of the pseudo-group jet coordinate Z^a_B with respect to x^i. We also note that the lifted total derivative operators (3.3) mutually commute:

$$[\, D_{X^i}, D_{X^j} \,] = 0, \qquad\qquad i,j = 1,\ldots,p.$$

The prolonged action (3.1) involves both the pseudo-group parameters $g^{(n)}$ and the submanifold jets $z^{(n)}$. Together they coordinatize the n^{th} order *lifted bundle* $\mathcal{B}^{(n)} \to \mathrm{J}^n$ obtained by pulling back the pseudo-group bundle $\mathcal{G}^{(n)} \to M$ by the standard projection $\widetilde{\pi}^n_0 \colon \mathrm{J}^n \to M$. Further, for $n < k$, we let $\widehat{\pi}^k_n \colon \mathcal{B}^{(k)} \to \mathcal{B}^{(n)}$ denote the induced projection maps of bundles. As with $\mathcal{G}^{(n)}$, the lifted bundle $\mathcal{B}^{(n)}$ also admits a natural groupoid structure, [39, 72]:

$$
\begin{array}{ccc}
 & \mathcal{B}^{(n)} & \\
 {\scriptstyle \widetilde{\boldsymbol{\sigma}}^{(n)}} \swarrow & & \searrow {\scriptstyle \widetilde{\boldsymbol{\tau}}^{(n)}} \\
 \mathrm{J}^n & & \mathrm{J}^n
\end{array}
$$

with source map $\widetilde{\boldsymbol{\sigma}}^{(n)}(z^{(n)}, g^{(n)}) = z^{(n)}$ and target map $\widetilde{\boldsymbol{\tau}}^{(n)}(z^{(n)}, g^{(n)}) = Z^{(n)} = g^{(n)} \cdot z^{(n)}$ prescribed by the prolonged action. As before, we abbreviate $\widetilde{\boldsymbol{\sigma}} = \widetilde{\boldsymbol{\sigma}}^{(\infty)}$ and $\widetilde{\boldsymbol{\tau}} = \widetilde{\boldsymbol{\tau}}^{(\infty)}$. The pseudo-group $\mathcal{G}$ acts by *right multiplication* on $\mathcal{B}^{(n)}$:

$$
\widetilde{R}_h(z^{(n)}, g^{(n)}) = (h^{(n)} \cdot z_n, R_h(g^{(n)})) = (h^{(n)} \cdot z^{(n)}, g^{(n)} \cdot (h^{(n)})^{-1}), \quad h \in \mathcal{G},
$$

where defined.

4 Differential Invariants and Invariant Differential Forms

Given the lifted bundle $\mathcal{B}^{(n)} \to \mathrm{J}^n$, a (locally defined) function $I \colon \mathcal{B}^{(n)} \to \mathbb{R}$ is called a *lifted differential invariant*, of order n, if it remains unchanged under *right multiplication*: $I(\widetilde{R}_h(z^{(n)}, g^{(n)})) = I(z^{(n)}, g^{(n)})$ where defined. It is not hard to see that the entries of the prolonged target map $Z^{(n)} = (\ \ldots\ X^i\ \ldots\ U^\alpha_K\ \ldots\ )$ form a complete system of lifted differential invariants of order n, [56], meaning that any other lifted differential invariant of order n can be locally written as a function thereof: $I = F(Z^{(n)})$. More generally, we will make repeated use of the corresponding right-invariant differential forms, for which we now introduce a convenient basis.

At infinite order, the bundle structure $\widetilde{\boldsymbol{\sigma}} \colon \mathcal{B}^{(\infty)} \to \mathrm{J}^\infty$ induces a splitting of the cotangent bundle $T^*\mathcal{B}^{(\infty)}$ into jet and group components, [56]. The *jet forms* are spanned by the *horizontal forms*

$$
dx^i, \qquad i = 1, \ldots, p, \tag{4.1a}
$$

and the basic *submanifold contact forms*

$$
\theta^\alpha_K = du^\alpha_K - \sum_{k=1}^{p} u^\alpha_{K,k}\, dx^k, \qquad \alpha = 1, \ldots, q, \qquad \#K \geq 0. \tag{4.1b}
$$

The group component is spanned by the *group forms*

$$
\Upsilon^a_B = dZ^a_B - \sum_{b=1}^{m} Z^a_{B,b}\, dz^b, \qquad a = 1, \ldots, m, \qquad \#B \geq 0, \tag{4.2}
$$

which are obtained by restricting the standard contact forms on $\mathcal{D}^{(\infty)}$ to the pseudo-group jet bundle $\mathcal{G}^{(\infty)}$ and then pulling back to $\mathcal{B}^{(\infty)}$. As before, we do not distinguish between functions and forms on $\mathcal{G}^{(\infty)}$ and their counterparts on $\mathcal{B}^{(\infty)}$. Because they are obtained by restriction to a subbundle, the group forms are not linearly independent. Fortunately, the linear relations among them are easily expressed by "lifting" the infinitesimal determining equations; see Theorem 5.1 below for the precise result.

Accordingly, the space of differential forms $\boldsymbol{\Omega}^* = \boldsymbol{\Omega}^*(\mathcal{B}^{(\infty)})$ decomposes into

$$\boldsymbol{\Omega}^* = \bigoplus_{k,l} \boldsymbol{\Omega}^{k,l}, \tag{4.3}$$

where $\boldsymbol{\Omega}^{k,l}$ is the subspace spanned by wedge products of k jet forms (4.1) and l group forms (4.2). We denote by $\boldsymbol{\Omega}^*_J = \bigoplus_k \boldsymbol{\Omega}^{k,0}$ the subspace of pure jet forms[1], and define the projection map $\pi_J \colon \boldsymbol{\Omega}^* \to \boldsymbol{\Omega}^*_J$ which takes a differential form on $\mathcal{B}^{(\infty)}$ to its jet component by annihilating all the group forms therein. As with the variational bicomplex, [1, 55], the splitting (4.3) decomposes the differential into jet and group components

$$d = d_J + d_G,$$

thereby forming a bicomplex. The jet component d_J can, in turn, be split into horizontal and vertical components, as prescribed by the horizontal forms (4.1a) and contact forms (4.1b), thereby forming the *lifted variational tricomplex*, [32].

Definition 4.1. The *lift* of a differential form ω on J^∞ to $\mathcal{B}^{(\infty)}$ is the invariant jet form

$$\boldsymbol{\lambda}(\omega) = \pi_J \big[\, \widetilde{\boldsymbol{\tau}}^* \omega \,\big]. \tag{4.4}$$

In particular, the lift of the differential of a form is obtained by taking the jet differential of its lift:

$$\boldsymbol{\lambda}(d\omega) = \pi_J \big[\, \widetilde{\boldsymbol{\tau}}^* d\omega \,\big] = \pi_J \big[\, d\, \widetilde{\boldsymbol{\tau}}^* \omega \,\big] = d_J \boldsymbol{\lambda}(\omega).$$

For the submanifold jet coordinates x^i, u^α_K, viewed as degree zero differential forms, the lift map (4.4) reduces to the prolonged action (3.1):

$$X^i = \boldsymbol{\lambda}(x^i), \qquad U^\alpha_K = \boldsymbol{\lambda}(u^\alpha_K). \tag{4.5}$$

The lift of the basic jet forms (4.1) are denoted by

$$\sigma^i = \sigma^{X^i} = \boldsymbol{\lambda}(dx^i) = d_J X^i, \qquad \vartheta^\alpha_K = \boldsymbol{\lambda}(\theta^\alpha_K), \tag{4.6}$$

and called, respectively, the *lifted horizontal forms* and the *lifted contact forms*. We also introduce the *lifted jet forms*

$$\sigma^\alpha_K = \sigma^{U^\alpha_K} = \boldsymbol{\lambda}(du^\alpha_K) = d_J U^\alpha_K = \sum_{i=1}^p U^\alpha_{K,i}\, \sigma^i + \vartheta^\alpha_K, \tag{4.7}$$

so that

$$\vartheta^\alpha_K = \sigma^\alpha_K - \sum_{i=1}^p U^\alpha_{K,i}\, \sigma^i = d_J U^\alpha_K - \sum_{i=1}^p U^\alpha_{K,i}\, d_J X^i, \tag{4.8}$$

which can also be obtained by directly lifting the contact forms (4.1b).

Switching our attention to vector fields, the *lifts*

$$\mathcal{D}_i = \mathcal{D}_{X^i} = \boldsymbol{\lambda}\big(D_{x^i}\big), \qquad \mathbb{D}^K_\alpha = \mathbb{D}_{U^\alpha_K} = \boldsymbol{\lambda}\left(\frac{\partial}{\partial u^\alpha_K}\right), \tag{4.9}$$

[1]Keep in mind that, while a pure jet form does not involve any of the group forms (4.2), its coefficients may still depend upon the pseudo-group parameters.

of the differential operators $\{ D_{x^i}, \partial_{u^\alpha_K} \}$ dual to the jet coframe $\{ dx^i, \theta^\alpha_J \}$, cf., (4.1), are defined by the identity

$$d_J F = \sum_{i=1}^{p} (\mathcal{D}_i F)\, \sigma^i \; + \sum_{\alpha=1}^{q} \sum_{\#K \geq 0} (\mathbb{D}^K_\alpha F)\, \vartheta^\alpha_K, \tag{4.10}$$

for any function $F\colon \mathcal{B}^{(\infty)} \longrightarrow \mathbb{R}$. Alternatively, the operators (4.9) are defined by the linear algebraic interior product equations (pairings)

$$\begin{array}{lll} \mathcal{D}_i \,\lrcorner\, \sigma^j = \delta^j_i, & \mathcal{D}_i \,\lrcorner\, \vartheta^\beta_I = 0, & \mathcal{D}_i \,\lrcorner\, \Upsilon^a_B = 0, \\ \mathbb{D}^K_\alpha \,\lrcorner\, \sigma^j = 0, & \mathbb{D}^K_\alpha \,\lrcorner\, \vartheta^\beta_I = \delta^\beta_\alpha \delta^K_I, & \mathbb{D}^K_\alpha \,\lrcorner\, \Upsilon^a_B = 0. \end{array}$$

Similarly, dual to the lifted coframe $\{ \sigma^i, \sigma^\alpha_K \} = \boldsymbol{\lambda} \{ dx^i, du^\alpha_K \}$ cf., (4.6), (4.7), are the *lifted vector fields*

$$\mathbb{D}_i = \mathbb{D}_{X^i} = \boldsymbol{\lambda}\left(\frac{\partial}{\partial x^i} \right), \qquad \mathbb{D}^K_\alpha = \mathbb{D}_{U^\alpha_K} = \boldsymbol{\lambda}\left(\frac{\partial}{\partial u^\alpha_K} \right), \tag{4.11}$$

defined by the identity

$$d_J F = \sum_{i=1}^{p} (\mathbb{D}_i F)\, \sigma^i \; + \sum_{\alpha=1}^{q} \sum_{\#K \geq 0} (\mathbb{D}^K_\alpha F)\, \sigma^\alpha_K, \tag{4.12}$$

or, equivalently, by the linear algebraic system

$$\begin{array}{lll} \mathbb{D}_i \,\lrcorner\, \sigma^j = \delta^j_i, & \mathbb{D}_i \,\lrcorner\, \sigma^\alpha_K = 0, & \mathbb{D}_i \,\lrcorner\, \Upsilon^a_B = 0, \\ \mathbb{D}^K_\alpha \,\lrcorner\, \sigma^j = 0, & \mathbb{D}^K_\alpha \,\lrcorner\, \sigma^\beta_I = \delta^\beta_\alpha \delta^K_I, & \mathbb{D}^K_\alpha \,\lrcorner\, \Upsilon^a_B = 0. \end{array} \tag{4.13}$$

Observe that, in view of (4.7), the vertical operators $\mathbb{D}^K_\alpha$ in (4.9) and (4.11) are the same; only the invariant horizontal operators differ: $\mathcal{D}_i \neq \mathbb{D}_i$. Their interrelationship is obtained by lifting the standard formula for the total derivative operators

$$D_{x^i} = \frac{\partial}{\partial x^i} \; + \sum_{\alpha=1}^{q} \sum_{\#K \geq 0} u^\alpha_{K,i} \frac{\partial}{\partial u^\alpha_K}, \qquad i = 1, \ldots, p, \tag{4.14}$$

defined on J^∞, i.e. the non-group part of (3.4), thus producing the following interesting formula:

$$\mathcal{D}_i = \mathbb{D}_i \; + \sum_{\alpha=1}^{q} \sum_{\#K \geq 0} U^\alpha_{K,i}\, \mathbb{D}^K_\alpha, \qquad i = 1, \ldots, p. \tag{4.15}$$

Example 4.2. We illustrate the above considerations using the Lie–Tresse–Kumpera pseudo-group (2.9) acting on surfaces $S \subset M \subset \mathbb{R}^3$. For simplicity, we assume that a surface is parametrized as the graph of a function, $S = \{ (x, y, u(x, y)) \}$, with $u \neq 0$. (Surfaces possessing vertical tangents are handled by introducing an alternative system of local coordinates, e.g., by switching the independent and dependent variables.) First, note that the lifted horizontal forms (4.6) are

$$\sigma^X = d_J X = f_x\, dx, \qquad \sigma^Y = d_J Y = dy, \tag{4.16}$$

and hence the corresponding lifted differential operators (3.3) are

$$D_X = \frac{1}{f_x} D_x, \qquad D_Y = D_y,$$

where, according to (3.4),

$$\begin{aligned} D_x &= \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + u_{xy} \frac{\partial}{\partial u_y} + \cdots + f_x \frac{\partial}{\partial f} + f_{xx} \frac{\partial}{\partial f_x} + \cdots , \\ D_y &= \frac{\partial}{\partial y} + u_y \frac{\partial}{\partial u} + u_{xy} \frac{\partial}{\partial u_x} + u_{yy} \frac{\partial}{\partial u_y} + \cdots . \end{aligned} \tag{4.17}$$

(Since $\partial f/\partial y \equiv 0$, the formula for D_y contains no group terms.) Thus, from (3.2), the prolonged action, up to order 2, is[2]

$$\begin{aligned} X &= f, \qquad Y = y, \qquad U = \frac{u}{f_x}, \\ U_X &= D_X U = \frac{f_x u_x - f_{xx} u}{f_x^3}, \qquad U_Y = D_Y U = \frac{u_y}{f_x}, \\ U_{XX} &= D_X^2 U = \frac{f_x^2 u_{xx} - 3\, f_x f_{xx}\, u_x - (f_x f_{xxx} + 3\, f_{xx}^2)\, u}{f_x^5}, \\ U_{XY} &= D_X D_Y U = \frac{f_x u_{xy} - f_{xx} u_y}{f_x^3}, \qquad U_{YY} = D_Y^2 U = \frac{u_{yy}}{f_x}. \end{aligned} \tag{4.18}$$

The lifts (4.7) of the vertical one-forms $du, du_x, du_y, \ldots$, yields, up to order 1, the lifted jet forms

$$\begin{aligned} \sigma^U &= d_J U = \frac{1}{f_x}\, du - \frac{u\, f_{xx}}{f_x^2}\, dx = \frac{1}{f_x}\, \theta + \frac{f_x u_x - f_{xx} u}{f_x^2}\, dx + \frac{u_y}{f_x}\, dy, \\ \sigma^{U_X} &= d_J U_X = \frac{1}{f_x^2}\, du_x - \frac{f_{xx}}{f_x^3}\, du - \frac{2\, f_x f_{xx}\, u_x - (f_x f_{xxx} + 3\, f_{xx}^2)\, u}{f_x^4}\, dx \\ &= \frac{1}{f_x^2}\, \theta_x - \frac{f_{xx}}{f_x^3}\, \theta + \frac{f_x^2 u_{xx} - 3\, f_x f_{xx}\, u_x - (f_x f_{xxx} + 3\, f_{xx}^2)\, u}{f_x^4}\, dx + \frac{f_x u_{xy} - f_{xx} u_y}{f_x^3}\, dy, \\ \sigma^{U_Y} &= d_J U_Y = \frac{1}{f_x}\, du_y - \frac{f_{xx}\, u_y}{f_x^2}\, dx = \frac{1}{f_x}\, \theta_y + \frac{f_x u_{xy} - f_{xx}\, u_y}{f_x^2}\, dx + \frac{u_{yy}}{f_x}\, dy, \end{aligned} \tag{4.19}$$

where

$$\theta = du - u_x\, dx - u_y\, dy, \qquad \theta_x = du_x - u_{xx}\, dx - u_{xy}\, dy, \qquad \theta_y = du_y - u_{xy}\, dx - u_{yy}\, dy, \tag{4.20}$$

are the basic contact forms of order ≤ 1. Alternatively, the lifts (4.8) of the contact forms (4.20) are, up to order 1,

$$\begin{aligned} \vartheta &= \boldsymbol{\lambda}(\theta) = \frac{\theta}{f_x} = \sigma^U - U_X \sigma^X - U_Y \sigma^Y, \\ \vartheta_x &= \boldsymbol{\lambda}(\theta_x) = \frac{f_x\, \theta_x - f_{xx}\, \theta}{f_x^3} = \sigma^{U_X} - U_{XX} \sigma^X - U_{XY} \sigma^Y, \\ \vartheta_y &= \boldsymbol{\lambda}(\theta_y) = \frac{\theta_y}{f_x} = \sigma^{U_Y} - U_{XY} \sigma^X - U_{YY} \sigma^Y, \end{aligned} \tag{4.21}$$

[2] Keep in mind that, in the calculations, $f, f_x, f_{xx}, \ldots$, are pseudo-group parameters prescribed by the jets of the defining function $f(x)$.

in accord with (4.8). One completes the coframes on the lifted bundles by appending the group forms of the appropriate orders:

$$\Upsilon_k = df_k - f_{k+1}\, dx, \qquad k \geq 0. \tag{4.22}$$

As for the lifted vector fields defined by (4.13) we have, up to order 1,

$$\begin{aligned}
\mathbb{D}_X = \boldsymbol{\lambda}\left(\frac{\partial}{\partial x}\right) &= \frac{1}{f_x}\frac{\partial}{\partial x} + \frac{f_{xx}\, u}{f_x^2}\frac{\partial}{\partial u} + \frac{(f_x f_{xxx} - 2 f_{xx}^2)\, u + 2 f_x f_{xx}\, u_x}{f_x^3}\frac{\partial}{\partial u_x} \\
&\qquad + \frac{f_{xx}\, u_y}{f_x^2}\frac{\partial}{\partial u_y} + \cdots + \frac{\partial}{\partial f} + \frac{f_{xx}}{f_x}\frac{\partial}{\partial f_x} + \frac{f_{xxx}}{f_x}\frac{\partial}{\partial f_{xx}} + \cdots, \\
\mathbb{D}_Y = \boldsymbol{\lambda}\left(\frac{\partial}{\partial y}\right) &= \frac{\partial}{\partial y}, \\
\mathbb{D}_U = \boldsymbol{\lambda}\left(\frac{\partial}{\partial u}\right) &= f_x \frac{\partial}{\partial u} + f_{xx}\frac{\partial}{\partial u_x} + f_{xxx}\frac{\partial}{\partial u_{xx}} + \cdots, \\
\mathbb{D}_{U_X} = \boldsymbol{\lambda}\left(\frac{\partial}{\partial u_x}\right) &= f_x^2 \frac{\partial}{\partial u_x} + 3 f_x f_{xx}\frac{\partial}{\partial u_{xx}} + \cdots, \\
\mathbb{D}_{U_Y} = \boldsymbol{\lambda}\left(\frac{\partial}{\partial u_y}\right) &= f_x \frac{\partial}{\partial u_y} + f_{xx}\frac{\partial}{\partial u_{xy}} + \cdots.
\end{aligned} \tag{4.23}$$

One can verify that, in accordance with (4.15) and (4.17),

$$\begin{aligned}
\mathcal{D}_X = \boldsymbol{\lambda}(D_x) &= \frac{1}{f_x} D_x = \mathbb{D}_X + U_X\, \mathbb{D}_U + U_{XX}\, \mathbb{D}_{U_X} + U_{XY}\, \mathbb{D}_{U_Y} + \cdots, \\
\mathcal{D}_Y = \boldsymbol{\lambda}(D_y) &= \quad D_y \quad = \mathbb{D}_Y + U_Y\, \mathbb{D}_U + U_{XY}\, \mathbb{D}_{U_X} + U_{YY}\, \mathbb{D}_{U_Y} + \cdots.
\end{aligned}$$

5 Maurer–Cartan Forms and Structure Equations

Let us next review the direct construction of the Maurer–Cartan forms and associated structure equations of a Lie pseudo-group that was developed in [55]. We begin with the full diffeomorphism pseudo-group $\mathcal{D} = \mathcal{D}(M)$. A *right-invariant Maurer–Cartan form* for $\mathcal{D}$ is, by definition, a right-invariant contact form on $\mathcal{D}^{(\infty)}$. In local coordinates, a basis of right-invariant Maurer–Cartan forms for the diffeomorphism pseudo-group is provided by the order zero group forms

$$\mu^a = \Upsilon^a = dZ^a - \sum_{b=1}^{m} Z_b^a\, dz^b, \qquad a = 1, \ldots, m,$$

and their successive (Lie) derivatives

$$\mu_B^a = \mathbb{D}_Z^B \mu^a = \mathbb{D}_{Z^{b^1}} \cdots \mathbb{D}_{Z^{b^k}} \mu^a, \qquad 1 \leq a \leq m, \qquad k = \#B \geq 0, \tag{5.1}$$

where $\mathbb{D}_{Z^1}, \ldots, \mathbb{D}_{Z^m}$ coincide with the lifted vector fields $\mathbb{D}_{X^1}, \ldots \mathbb{D}_{X^p}, \mathbb{D}_{U^1}, \ldots, \mathbb{D}_{U^q}$ given in (4.11), the latter corresponding to the case $\#K = 0$. We let

$$\mu^{(n)} = (\ \ldots\ \mu_B^a\ \ldots), \qquad 0 \leq \#B \leq n,$$

denote the collection of Maurer–Cartan forms of order $\leq n$.

Given a Lie pseudo-group $\mathcal{G}$ acting on M, its *right-invariant Maurer–Cartan forms* are obtained by restricting the right-invariant Maurer–Cartan forms of the diffeomorphism pseudo-group to the subbundle $\mathcal{G}^{(\infty)} \subset \mathcal{D}^{(\infty)}$. We will continue to use the same notation μ^a_B for the restricted forms, as well as their counterparts obtained by pulling back to the lifted bundle $\mathcal{B}^{(\infty)}$. Of course, the restricted Maurer–Cartan forms are no longer linearly independent. However, one can easily write down the complete system of linear dependencies they satisfy. To explain this, and for later purposes, as in [55], we will extend the lift map (4.4) to the vector field jets (2.7) by defining the lift of ζ^a_B to be the corresponding basic *right-invariant Maurer–Cartan form* μ^a_B:

$$\boldsymbol{\lambda}(\zeta^a_B) = \mu^a_B, \qquad a = 1, \ldots, m, \qquad \#B \geq 0. \tag{5.2}$$

With this convention, as first discovered in [55], the complete system of linear relations among the Maurer–Cartan forms for the pseudo-group $\mathcal{G}$ is obtained by lifting its infinitesimal determining equations!

Theorem 5.1. For any $n \geq n^*$, the Maurer–Cartan forms $\mu^{(n)}$ of a Lie pseudo-group $\mathcal{G}$ are constrained by the linear relations obtained by lifting the infinitesimal determining system (2.8) of $\mathcal{G}$:

$$\boldsymbol{\lambda}\big[L^{(n)}(z, \zeta^{(n)})\big] = L^{(n)}(Z, \mu^{(n)}) = 0. \tag{5.3}$$

Example 5.2. For the Lie–Tresse–Kumpera pseudo-group of Example 2.5, let

$$\boldsymbol{\lambda}(\xi_{x^a y^b u^c}) = \mu^x_{X^a Y^b U^c}, \qquad \boldsymbol{\lambda}(\eta_{x^a y^b u^c}) = \mu^y_{X^a Y^b U^c}, \qquad \boldsymbol{\lambda}(\varphi_{x^a y^b u^c}) = \mu^u_{X^a Y^b U^c},$$

denote the restrictions of the diffeomorphism Maurer–Cartan forms obtained by lifting the vector field jet coordinates corresponding to (2.11). Lifting the first order infinitesimal determining equations (2.12), we obtain the linear algebraic relations

$$\mu^x_Y = \mu^x_U = 0, \quad \mu^y = \mu^y_X = \mu^y_Y = \mu^y_U = 0, \quad \mu^u = -U\,\mu^x_X, \quad \mu^u_Y = 0, \quad \mu^u_U = -\mu^x_X, \tag{5.4}$$

among the Maurer–Cartan forms of order ≤ 1. (The skeptical reader can verify these by writing out the explicit formulae, [55].) Higher order relations are obtained by "formal differentiation", that is, by lifting the differentiated first order infinitesimal determining equations. It is then not hard to see that a basis of Maurer–Cartan forms is given by

$$\mu_k = \mu^x_{X^k} = \mathbb{D}^k_X \mu^x, \qquad k \geq 0, \tag{5.5}$$

where $\mu^x = \Upsilon$ is the order zero group form in (4.22), and $\mathbb{D}_X$ is the invariant differential operator appearing in (4.23).

Example 5.3. For the pseudo-group studied in Example 2.6, the lift of the first order infinitesimal determining equations (2.16) yields the linear relations

$$\mu^x_Y = \mu^x_U = 0, \qquad \mu^y_Y = \mu^x_X, \qquad \mu^y_U = 0, \qquad \mu^u = \mu^y_X, \qquad \mu^u_U = 0$$

among the Maurer–Cartan forms of order ≤ 1. Again, since the first order system is involutive, all the higher order relations are obtained by "formal differentiation". We deduce that a basis of Maurer–Cartan forms is thus given by

$$\mu_k = \mu^x_{X^k}, \qquad \nu_k = \mu^y_{X^k}, \qquad k \geq 0.$$

Their coordinate expressions may be found in [55], although we emphasize that these are not needed for the implementation of the recursive moving frame algorithm introduced in Section 8.

For later purposes, we will also need the *left-invariant Maurer–Cartan forms.* For the diffeomorphism pseudo-group, these are the contact forms on the diffeomorphism jet bundles $\mathcal{D}^{(\infty)}$ that remain invariant under *left multiplication* (2.1b). The order zero left-invariant Maurer–Cartan forms are given by

$$\lambda^a = \sum_{b=1}^m w_b^a \, \Upsilon^b, \qquad a = 1, \ldots, m, \qquad \text{where} \qquad (w_b^a) = (Z_b^a)^{-1}$$

is the inverse of the $m \times m$ Jacobian matrix $\nabla Z = (Z_b^a)$, written in terms of first order jets. Their higher order counterparts are obtained by repeated Lie differentiation with respect to the total derivative operators (2.3) on the jet bundle $\mathcal{D}^{(\infty)}$, as in (2.5):

$$\lambda_B^a = \mathbb{D}_z^B \left(\sum_{b=1}^m w_b^a \, \Upsilon^b \right). \tag{5.6}$$

As before, the *left-invariant Maurer–Cartan forms* for the pseudo-group $\mathcal{G}$ are merely obtained by restricting the left-invariant diffeomorphism Maurer–Cartan forms (5.6) to the pseudo-group jet subbundle $\mathcal{G}^{(\infty)}$. We will again adopt the same notation for the restricted forms and their lifts to $\mathcal{B}^{(\infty)}$. The linear relations satisfied by the left-invariant Maurer–Cartan forms are analogous to the relations satisfied by the right-invariant Maurer–Cartan forms, [55].

As with finite-dimensional Lie groups, the pseudo-group inversion map[3]

$$i(g^{(\infty)}) = (g^{-1})^{(\infty)} \tag{5.7}$$

switches right- and left-invariant objects. Note that inversion switches source and target: $\boldsymbol{\sigma}(i(g^{(\infty)})) = \boldsymbol{\tau}(g^{(\infty)})$ and conversely. As with finite-dimensional Lie groups, the right and left invariant Maurer–Cartan forms are related by pseudo-group inversion, modulo a sign.

Proposition 5.4. Pseudo-group inversion (5.7) maps the right-invariant Maurer–Cartan forms to the negatives of their left-invariant counterparts:

$$i^*(\mu_B^a) = -\lambda_B^a. \tag{5.8}$$

Proof. Using the definitions (5.1) and (5.6) for the right and left Maurer–Cartan forms, respectively, we have that

$$\begin{aligned} i^*(\mu_B^a) = i^*\big(\mathbb{D}_Z^B \, \Upsilon^a\big) &= i^*\left(\mathbb{D}_Z^B\left(dZ^a - \sum_{b=1}^m Z_{z^b}^a \, dz^b\right)\right) = \mathbb{D}_z^B\left(dz^a - \sum_{b=1}^m z_{Z^b}^a \, dZ^b\right) \\ &= \mathbb{D}_z^B\left(\sum_{b=1}^m -z_{Z^b}^a\left(dZ^b - \sum_{c=1}^m Z_{z^c}^b \, dz^c\right)\right) = -\,\mathbb{D}_z^B\left(\sum_{b=1}^m z_{Z^b}^a \, \Upsilon^b\right) \\ &= -\,\mathbb{D}_z^B\left(\sum_{b=1}^m w_b^a \, \Upsilon^b\right) = -\lambda_B^a. \end{aligned}$$

□

[3]To streamline the exposition, we will at times write $g^{(\infty)}$ in formulas involving pseudo-group jets even though we may only require some associated finite order jet: $g^{(n)} = \pi_n^\infty(g^{(\infty)})$ for some $n < \infty$. This notational convention allows us to avoid repeatedly specifying exactly which order n is required at each step in the construction.

Example 5.5. Consider the case of the one-dimensional diffeomorphism pseudo-group $\mathcal{D}(\mathbb{R})$, with jet coordinates

$$g^{(\infty)} = (x, f, f_x, f_{xx}, f_{xxx}, \ldots)$$

representing the derivatives of the diffeomorphism $f(x) \in \mathcal{D}(\mathbb{R})$ with source x and target $X = f(x)$. The inversion map is explicitly given by

$$i(g^{(\infty)}) = (g^{-1})^{(\infty)} = \left(f, x, \frac{1}{f_x}, -\frac{f_{xx}}{f_x^3}, \frac{3f_{xx}^2 - f_x f_{xxx}}{f_x^5}, \ldots \right),$$

the entries being the standard calculus formulas for the successive derivatives of the inverse map $x = f^{-1}(X)$ in terms of derivatives of $X = f(x)$. Up to order 2, the right Maurer–Cartan forms are

$$\mu = \Upsilon, \qquad \mu_X = \frac{\Upsilon_x}{f_x}, \qquad \mu_{XX} = \frac{f_x\, \Upsilon_{xx} - f_{xx}\, \Upsilon_x}{f_x^3}, \tag{5.9}$$

where

$$\Upsilon = df - f_x\, dx, \qquad \Upsilon_x = df_x - f_{xx}\, dx, \qquad \Upsilon_{xx} = df_{xx} - f_{xxx}\, dx,$$

are the basic group forms, as in (4.22). Similarly, the left Maurer–Cartan forms are

$$\lambda = \frac{\Upsilon}{f_x}, \qquad \lambda_x = \frac{f_x\, \Upsilon_x - f_{xx}\, \Upsilon}{f_x^2}, \qquad \lambda_{xx} = \frac{f_x^2\, \Upsilon_{xx} - 2 f_x\, f_{xx}\, \Upsilon_x - (f_x\, f_{xxx} - 2 f_{xx}^2)\, \Upsilon}{f_x^3}. \tag{5.10}$$

One can then readily verify formula (5.8) for this example:

$$i^*(\mu) = -\lambda, \qquad i^*(\mu_X) = -\lambda_x, \qquad i^*(\mu_{XX}) = -\lambda_{xx}, \qquad \ldots .$$

Finally, we recall the new construction of the structure equations of a Lie pseudo-group found in [55]. This approach possesses several advantages over the traditional Cartan method. First, it is immediate and completely constructive, requiring only simple linear algebra to implement. Moreover, in contrast to Cartan's method, it does not rely on the introduction of special coordinates, nor does it involve any recondite prolongation procedure. In addition, it applies equally well to transitive and intransitive pseudo-group actions, and, as argued in [59], gives the "correct" structure equations in the latter situation, unlike Cartan's construction which exhibits several problematic features.

The starting point is the structure equations for the diffeomorphism pseudo-group, [55]. To state this result, we introduce the standard *multinomial coefficients*

$$\binom{B}{A} = \binom{B}{C} = \frac{B\,!}{A\,!\; C\,!}, \qquad \text{when} \qquad B = (A, C).$$

Here, the *factorial* of a symmetric multi-index $B = (b^1, \ldots, b^k)$ with $1 \leq b^\nu \leq m$, is given by the usual formula

$$B\,! = \widetilde{b}^1\,!\;\widetilde{b}^2\,!\;\cdots\;\widetilde{b}^m\,!,$$

where $\widetilde{b}^\ell$ indicates the number of occurrences of the integer ℓ in B.

Proposition 5.6. The structure equations of the diffeomorphism pseudo-group $\mathcal{D}(M)$ are

$$d\sigma^a = \sum_{b=1}^{m} \mu_b^a \wedge \sigma^b, \qquad d\mu_B^a = \sum_{c=1}^{m} \left[\sigma^c \wedge \mu_{B,c}^a + \sum_{\substack{B=(A,C)\\ \#C \geq 1}} \binom{B}{A} \mu_{A,c}^a \wedge \mu_C^c \right], \tag{5.11}$$

where $\sigma^a = \boldsymbol{\lambda}(dz^a)$, $a = 1, \ldots, m$, are the order 0 right-invariant lifted jet forms introduced in (4.6) and (4.7) when $\#K = 0$, also denoted

$$\sigma^1 = \sigma^{X^1}, \ \ldots \ , \sigma^p = \sigma^{X^p}, \qquad \sigma^{p+1} = \sigma^{U^1}, \ \ldots \ , \sigma^m = \sigma^{p+q} = \sigma^{U^q}.$$

Example 5.7. Specializing (5.11) to the one-dimensional case $M = \mathbb{R}$, we deduce the well known structure equations of the diffeomorphism pseudo-group $\mathcal{D}(\mathbb{R})$, [11, 55]:

$$d\sigma = \mu_1 \wedge \sigma^X, \qquad d\mu_k = \sigma^X \wedge \mu_{k+1} - \sum_{j=1}^{[(k+1)/2]} \frac{k-2j+1}{k+1} \binom{n+1}{j} \mu_j \wedge \mu_{k+1-j}, \tag{5.12}$$

where $\sigma^X = d_J X = f_x\, dx$ and the right invariant Maurer–Cartan forms are $\mu_k = \mathbb{D}_X^k \mu_0$, the first few of which can be found in (5.9).

Since the Maurer–Cartan forms of the Lie pseudo-group $\mathcal{G}$ are obtained by restriction to the subbundle $\mathcal{G}^{(\infty)}$, the structure equations for $\mathcal{G}$ are merely obtained by restriction of the diffeomorphism structure equations. Thus, in view of Theorem 5.1:

Theorem 5.8. The structure equations of a Lie pseudo-group $\mathcal{G}$ are obtained by restricting the diffeomorphism structure equations (5.11) to the kernel of the infinite order lifted infinitesimal determining system, that is, (5.3) for $n = \infty$.

Examples can be found below and in [55, 59]. Finally, the following result is used in most of our examples.

Proposition 5.9. Let $\mathcal{G}$ be a one-to-one prolongation of $\mathcal{H}$. Then the pull-backs of a basis of Maurer–Cartan forms for $\mathcal{H}$ under the projection $\pi\colon M \to N$ forms a basis for the Maurer–Cartan forms of $\mathcal{G}$.

Proof. Using the notation introduced at the end of Section 2, the lift of the prolonged infinitesimal determining equations (2.22b) yields the linear relations

$$L_{\mathfrak{h}}^{(\infty)}(Z_\sharp, \mu_\sharp^{(\infty)}) = 0, \qquad \mu_{\sharp, Z_\flat} = 0, \qquad \mu_\flat = L_\flat(Z, \mu_\sharp^{(n)}),$$

among the Maurer–Cartan forms of $\mathcal{G}$. It follows that the subset of the invariant group forms $\mu_\sharp^{(\infty)}$ corresponding to the coordinates $z_\sharp$ on $\mathcal{H}$ provides a basis of Maurer–Cartan forms for $\mathcal{G}$, and hence also constitutes a basis of Maurer–Cartan forms of $\mathcal{H}$. □

6 Moving Frames and Freeness

We refer to the foundational paper [56] for a detailed account of how the equivariant moving frame method, first introduced in [17] to treat finite-dimensional Lie group actions, can be adapted to Lie pseudo-groups. For finite-dimensional Lie groups, the existence of a moving frame requires the action to be free and regular on an open subset of the jet space, [17]. Similar conditions must hold in the pseudo-group framework, although infinite-dimensionality of a Lie pseudo-group precludes freeness of its action on any finite dimensional jet space, in the ordinary sense of having trivial isotropy at a point. Instead, we work with the following refined notion of freeness, [56], which is based on the observation that the action of the pseudo-group on the n^{th} order submanifold jets depends only on the n^{th} order jets of the pseudo-group diffeomorphisms, and hence the freeness condition should not involve any higher order pseudo-group parameters. For finite-dimensional Lie group actions, this leads to a slight generalization of the usual notion of freeness.

Definition 6.1. The n^{th} order *isotropy subgroup* of the submanifold jet $z^{(n)} \in \mathrm{J}^n$ is

$$\mathcal{G}^{(n)}_{z^{(n)}} = \left\{ g^{(n)} \in \mathcal{G}^{(n)}|_z \;\middle|\; g^{(n)} \cdot z^{(n)} = z^{(n)} \right\}.$$

Definition 6.2. A Lie pseudo-group $\mathcal{G}$ acts *freely* at $z^{(n)} \in \mathrm{J}^n$ if $\mathcal{G}^{(n)}_{z^{(n)}} = \{\mathbb{1}^{(n)}_z\}$, and *locally freely* if $\mathcal{G}^{(n)}_{z^{(n)}}$ is a discrete subgroup of $\mathcal{G}^{(n)}|_z$. The pseudo-group $\mathcal{G}$ is said to act (locally) *freely at order* n if it acts (locally) freely on an open subset $\mathcal{V}^n \subset \mathrm{J}^n$, called the set of *regular* n*–jets.*

In the standard equivariant moving frame construction, [56], once the action becomes free at some order n, a moving frame section $\varrho^{(n)} \colon \mathcal{V}^n \to \mathcal{B}^{(n)}$ is constructed by choosing a *cross-section* $\mathcal{K}^n \subset \mathcal{V}^n$, that is, a submanifold intersecting the (regular) prolonged pseudo-group orbits uniquely and transversally. The corresponding right-equivariant moving frame associates to each $z^{(n)} \in \mathcal{V}^n$ the unique (as guaranteed by freeness) pseudo-group jet $g^{(n)} = \rho^{(n)}(z^{(n)}) \in \mathcal{G}^{(n)}$ that maps $z^{(n)}$ to the chosen cross-section: $g^{(n)} \cdot z^{(n)} \in \mathcal{K}^n$. The resulting *moving frame section*

$$\varrho^{(n)}(z^{(n)}) = (z^{(n)}, \rho^{(n)}(z^{(n)})) \tag{6.1}$$

is thus obtained by solving the *normalization equations*

$$Z^{(n)} = \boldsymbol{\lambda}(z^{(n)}) = \widetilde{\boldsymbol{\tau}}^{(n)}(z^{(n)}, g^{(n)}) = g^{(n)} \cdot z^{(n)} \in \mathcal{K}^n$$

for the pseudo-group parameters $g^{(n)} = \rho^{(n)}(z^{(n)})$. By construction, $\varrho^{(n)} \colon \mathcal{V}^n \to \mathcal{B}^{(n)}$ is a (locally defined) right-equivariant section of the lifted bundle $\mathcal{B}^{(n)}$, meaning that

$$\widetilde{R}_g(\varrho^{(n)}(z^{(n)})) = \varrho^{(n)}(g^{(n)} \cdot z^{(n)}), \qquad g \in \mathcal{G}|_z,$$

whenever defined.

Once the prolonged pseudo-group action becomes free at order n, a result known as the *persistence of freeness*, [57, 58], guarantees that the action remains free under prolongation.

Theorem 6.3. Let $\mathcal{G}$ be a Lie pseudo-group of order n^*. Suppose $\mathcal{G}$ acts (locally) freely at $z^{(n)} \in \mathrm{J}^n$, where $n \geq n^*$. Then, for all $k \geq 0$, it also acts (locally) freely at $z^{(n+k)} \in \mathrm{J}^{(n+k)}$ with $\widetilde{\pi}^{n+k}_n(z^{(n+k)}) = z^{(n)}$.

Persistence of freeness then guarantees the existence of an infinite sequence of *compatible moving frames* on higher order jet spaces $\mathcal{V}^n \leftarrow \mathcal{V}^{n+1} \leftarrow \mathcal{V}^{n+2} \leftarrow \cdots$ that are based on *compatible cross-sections* $\mathcal{K}^n \leftarrow \mathcal{K}^{n+1} \leftarrow \mathcal{K}^{n+2} \leftarrow \cdots$, satisfying

$$\widetilde{\pi}^{k+1}_k(\mathcal{K}^{k+1}) = \mathcal{K}^k \qquad \text{for all} \qquad k \geq n. \tag{6.2}$$

This implies the desired compatibility conditions

$$\widetilde{\pi}^{k+1}_k(\varrho^{(k+1)}(z^{(k+1)})) = \varrho^{(k)}(z^{(k)}) \qquad \text{for all} \qquad k \geq n,$$

on the resulting moving frame sections. To avoid technical problems with shrinking domains of definition of the moving frames $\rho^{(k)}$ as $k \to \infty$, we impose the technical assumptions used in [56], that hold in all explicit examples treated to date. Namely, we assume that the lowest order moving frame $\rho^{(n)}$ is defined on a domain $\mathcal{W}^n \subset \mathcal{V}^n \subset \mathrm{J}^n$, while each higher order compatible moving frame $\rho^{(k)}$ for $k \geq n$ is defined on $\mathcal{W}^k = (\widetilde{\pi}^k_n)^{-1}(\mathcal{W}^n) \subset \mathcal{V}^k$. We let $\rho = \rho^{(\infty)}$ denote the induced infinite order moving frame, and $\varrho = \varrho^{(\infty)} \colon \mathcal{W}^\infty = (\widetilde{\pi}^\infty_n)^{-1}(\mathcal{W}^n) \longrightarrow \mathcal{B}^{(\infty)}$ its associated (local) section.

Once a moving frame section has been constructed, the *invariantization map*

$$\iota = \varrho^* \circ \boldsymbol{\lambda} \colon \boldsymbol{\Omega}^*(\mathcal{W}^\infty) \;\longrightarrow\; \boldsymbol{\Omega}^*(\mathcal{W}^\infty),$$

sends jet forms to their invariant counterparts. When applied to the basic jet forms (4.1), the invariantization map yields an invariant coframe on $\mathcal{W}^\infty \subset \mathcal{V}^\infty$ that plays a fundamental role in the analysis of the *invariant variational bicomplex*[4], [32, 67].

In the following we add hats over quantities that have been pulled-back (normalized) by a moving frame map[5]. For example,

$$\widehat{X}^i = \varrho^*(X^i) = \iota(x^i), \qquad \widehat{U}^\alpha_K = \varrho^*(U^\alpha_K) = \iota(u^\alpha_K),$$
$$\widehat{\sigma}^i = \varrho^*(\sigma^i), \qquad \widehat{\sigma}^\alpha_K = \varrho^*(\sigma^\alpha_K), \qquad \widehat{\mu}^a_B = \varrho^*(\mu^a_B).$$

The hat decoration is also used to denote the normalization of the lifted invariant operators (4.9), (4.11):

$$\mathcal{D}_i \;\longrightarrow\; \widehat{\mathcal{D}}_i, \qquad \mathbb{D}_i \;\longrightarrow\; \widehat{\mathbb{D}}_i, \qquad \mathbb{D}^\alpha_K \;\longrightarrow\; \widehat{\mathbb{D}}^\alpha_K.$$

These operators are defined by identities corresponding to the normalization of (4.10) and (4.12):

$$dF = \sum_{i=1}^p (\widehat{\mathcal{D}}_i F)\, \widehat{\sigma}^i + \sum_{\alpha=1}^q \sum_{\#K \geq 0} (\widehat{\mathbb{D}}^K_\alpha F)\, \widehat{\vartheta}^\alpha_K = \sum_{i=1}^p (\widehat{\mathbb{D}}_i F)\, \widehat{\sigma}^i + \sum_{\alpha=1}^q \sum_{\#K \geq 0} (\widehat{\mathbb{D}}^K_\alpha F)\, \widehat{\sigma}^\alpha_K, \tag{6.3}$$

where $F \colon \mathrm{J}^{(\infty)} \to \mathbb{R}$ is any differential function, or, equivalently, by the linear algebraic system resulting from the pairings

$$\begin{aligned} &\widehat{\mathcal{D}}_i \,\lrcorner\, \widehat{\sigma}^j = \delta^j_i, && \widehat{\mathcal{D}}_i \,\lrcorner\, \widehat{\vartheta}^\alpha_K = 0, && \widehat{\mathbb{D}}^K_\alpha \,\lrcorner\, \widehat{\sigma}^j = 0, && \widehat{\mathbb{D}}^K_\alpha \,\lrcorner\, \widehat{\vartheta}^\beta_I = \delta^\beta_\alpha \delta^K_I, \\ &\widehat{\mathbb{D}}_i \,\lrcorner\, \widehat{\sigma}^j = \delta^j_i, && \widehat{\mathbb{D}}_i \,\lrcorner\, \widehat{\sigma}^\alpha_K = 0, && \widehat{\mathbb{D}}^K_\alpha \,\lrcorner\, \widehat{\sigma}^j = 0, && \widehat{\mathbb{D}}^K_\alpha \,\lrcorner\, \widehat{\sigma}^\beta_I = \delta^\beta_\alpha \delta^K_I. \end{aligned} \tag{6.4}$$

As above, the horizontal operators $\widehat{\mathcal{D}}_i$, $\widehat{\mathbb{D}}_i$, are, in general, not the same, whereas the vertical operators $\widehat{\mathbb{D}}^K_\alpha$ coincide. Pulling back (4.8) by a right moving frame section, we obtain the identity

$$\widehat{\vartheta}^\alpha_K = \widehat{\sigma}^\alpha_K - \sum_{i=1}^p \widehat{U}^\alpha_{K,i}\, \widehat{\sigma}^i.$$

[4] Although, when the action is non-projectable, it would be more accurate to talk about the invariant variational "quasi-tricomplex" as the exterior derivative splits into three components over the space of invariant jet forms, [32]. Interestingly, this quasi-tricomplex structure also arises in the study of the topology of foliations, [66].

[5] In section 8, we will extend the hat notation to pull-backs by partial moving frames; see Definition 8.2.

Combined with the pairings (6.4), it follows that

$$\widehat{\mathcal{D}}_i = \widehat{\mathbb{D}}_i + \sum_{\alpha=1}^{q} \sum_{\#K \geq 0} \widehat{U}^\alpha_{K,i}\, \widehat{\mathbb{D}}^K_\alpha, \qquad i = 1, \ldots, p, \tag{6.5}$$

which corresponds to the invariantization of the differential operators (4.14).

The invariantization of the submanifold jet coordinates plays an essential role in the construction of a complete system of differential invariants, [57].

Proposition 6.4. The *normalized differential invariants*

$$\widehat{X}^i = \iota(x^i), \qquad \widehat{U}^\alpha_K = \iota(u^\alpha_K), \tag{6.6}$$

contain a complete set of functionally independent differential invariants.

Suppose that a cross-section is defined by setting a suitable number of differential functions to constants. Then the normalized differential invariants corresponding to these differential functions are constant, and are known as the *phantom differential invariants*. The remaining, non-phantom normalized differential invariants, provide the complete set referred to in Proposition 6.4. Indeed, those corresponding to the invariantized local cross-section coordinates are functionally independent. Completeness is easily established via the Replacement Theorem, [17, 56], which follows immediately from the fact that invariantization does not affect invariant quantities and hence the local coordinate formula for any differential invariant does not change when one replaces each jet coordinate by its invariantization (6.6).

7 Recurrence Relations

We now discuss the all-important recurrence relations that collectively govern the algebra of differential invariants and, more generally, the invariant differential forms, as prescribed by the choice of a moving frame. Let

$$\mathbf{v} = \sum_{i=1}^{p} \xi^i(x,u) \frac{\partial}{\partial x^i} \; + \; \sum_{\alpha=1}^{q} \varphi_\alpha(x,u) \frac{\partial}{\partial u^\alpha} \; \in \; \mathfrak{g} \tag{7.1}$$

be an infinitesimal generator of the pseudo-group $\mathcal{G}$, written in terms of the selected local coordinates $z = (x, u)$. Thus, the notation used in (2.6) has become $(\ldots \zeta^a \ldots) = (\ldots \xi^i \ldots \varphi_\alpha \ldots)$, in accordance with standard notational conventions, [47, 48]. For $0 \leq n \leq \infty$, its prolongation to the submanifold jet space $\mathrm{J}^n = \mathrm{J}^n(M,p)$ is

$$\mathbf{v}^{(n)} = \sum_{i=1}^{p} \xi^i(x,u) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \sum_{0 \leq \#K \leq n} \varphi^K_\alpha(x, u^{(k)}) \frac{\partial}{\partial u^\alpha_K} \; \in \; \mathfrak{g}^{(n)}, \tag{7.2a}$$

whose coefficients are provided by the standard prolongation formula, [48],

$$\varphi^K_\alpha = D_K Q^\alpha + \sum_{i=1}^{p} \xi^i u^\alpha_{K,i}, \quad \text{where} \quad Q^\alpha = \varphi_\alpha - \sum_{i=1}^{p} \xi^i u^\alpha_i, \quad 1 \leq \alpha \leq q, \tag{7.2b}$$

are the components of the *characteristic* of the vector field (7.1). We note that each prolonged infinitesimal generator coefficient φ^K_α is a certain universal linear combination of the vector field jet coordinates (2.7), whose coefficients depend polynomially on the submanifold jet coordinates u^β_L of orders $1 \leq \# L \leq \# K$.

Owing to the appearance of the jet projection π_J in the definition of the lift map (4.4), the lift and exterior differential operators do not commute. Their lack of commutativity is encapsulated in the *universal recurrence relation*, [56].

Proposition 7.1. Let ω be a differential form on J^∞. Then

$$d\big[\,\boldsymbol{\lambda}(\omega)\,\big] = \boldsymbol{\lambda}\big[\,d\omega + \mathbf{v}^{(\infty)}(\omega)\,\big]. \tag{7.3}$$

Here $\mathbf{v}^{(\infty)}(\omega)$ denotes the Lie derivative of the differential form ω with respect to the prolonged infinitesimal generator (7.2), while the lift map $\boldsymbol{\lambda}$ acts on the resulting expressions in accordance with (4.5), (4.6), (5.2).

In particular, by setting ω to be each of the submanifold jet coordinates x^i, u^α_K, we obtain the *lifted recurrence relations*

$$dX^i = \sigma^i + \mu^i, \qquad dU^\alpha_K = \sigma^\alpha_K + \psi^K_\alpha = \sum_{k=1}^{p} U^\alpha_{K,k}\,\sigma^k + \vartheta^\alpha_K + \psi^K_\alpha, \tag{7.4}$$

where the *lifted infinitesimal generator coefficient*

$$\psi^K_\alpha = \boldsymbol{\lambda}(\varphi^K_\alpha) \tag{7.5}$$

is obtained by lifting the corresponding prolonged infinitesimal generator coefficient (7.2b). As a result, each ψ^K_α is a certain universal linear combination of Maurer–Cartan forms μ^a_B, whose coefficients depend polynomially on the lifted differential invariants U^β_L of orders $1 \leq \# L \leq \# K$.

Finally, pulling-back the lifted recurrence relations (7.4) by the moving frame section ϱ produces the *fundamental recurrence relations*

$$d\widehat{X}^i = \widehat{\sigma}^i + \widehat{\mu}^i, \qquad d\widehat{U}^\alpha_K = \sum_{k=1}^{p} \widehat{U}^\alpha_{K,k}\,\widehat{\sigma}^k + \widehat{\vartheta}^\alpha_K + \widehat{\psi}^K_\alpha, \tag{7.6}$$

among the normalized differential invariants. Those corresponding to the phantom differential invariants can be uniquely solved for the basis of pulled-back Maurer–Cartan forms $\widehat{\mu}^a_B = \varrho^*(\mu^a_B)$. Substituting the resulting expressions into the remaining non-phantom recurrence relations produces a complete system of differential identities that completely characterizes the algebra of differential invariants, [57]. It is a remarkable fact that this structure can be completely determined without recourse to the explicit expressions for either the differential invariants or the invariant differential operators, or even the moving frame, requiring only the universal prolongation formula (7.2) for the infinitesimal generators along with some elementary linear algebra. As such, the procedure can be completely and directly automated in any standard computer algebra system e.g. MATHEMATICA or MAPLE. Examples of the implementation of the equivariant moving frame construction may be found in [14, 56]. Extensions to invariant differential forms and the invariant variational (quasi-)tricomplex, and further applications to invariant variational problems and invariant geometric flows appear in [32, 52, 68].

8 Recursive Moving Frames

Since the implicit derivative operators (3.3) involve the entries of the inverse total Jacobian matrix, the explicit formulas for the prolonged action (3.2) rapidly lead to unwieldy expressions that can easily overwhelm any attempted automated computation of substantial examples, even though ultimately many of these expressions will end up being considerably simplified once the moving frame computation is carried through to completion. Thus, the aim of the recursive moving frame algorithm is to suppress all unnecessary intermediate expression swell, that may well prevent the symbolic calculation from being carried out on existing computer platforms. We remind the reader that, in order for our algorithm to succeed, we will need to impose some mild restrictions on the pseudo-group action, as described in detail in Propositions 8.6 and 8.19. In particular, translational pseudo-groups and those that are their one-to-one prolongations will satisfy these restrictions.

As in the recursive algorithm developed in [54], we employ a succession of pseudo-group parameter normalizations at each order, based on a sequence of *compatible cross-sections* $\mathcal{K}^k \subset \mathrm{J}^k$*, now for all* $k \geq 0$*,* to the regular prolonged pseudo-group orbits. As in (6.2), compatibility requires that

$$\mathcal{K}^k = \widetilde{\pi}_k^{k+1}(\mathcal{K}^{k+1}) \qquad \text{for all} \qquad k \geq 0. \tag{8.1}$$

When $k < n$, the order of freeness, the prolonged action is not free and the normalization equations

$$Z^{(k)} = g^{(k)} \cdot z^{(k)} \in \mathcal{K}^k \tag{8.2}$$

will not uniquely determine the pseudo-group jet $g^{(k)}$. However, they can be solved for *some* of the pseudo-group parameters in terms of the k^{th} order submanifold jet coordinates and the remaining pseudo-group parameters of order $\leq k$. The result will be a right-equivariant *partial moving frame* of order k, as formally described in Definition 8.2. Compatibility of the cross-sections (8.1) implies that one can retain the already established lower order normalizations as the order is increased. As a consequence of the partially normalized lifted recurrence relations, expressions for the moving frame and the differential invariants are eventually obtained without any need to compute the full prolonged action (3.2). Before going through the details of the general algorithm, it will help to first work through a simple example.

Example 8.1. As in Example 4.2, we consider the Lie–Tresse–Kumpera pseudo-group (2.9) acting on surfaces that are locally represented as graphs of functions $(x, y, u(x, y))$ with $u \neq 0$. The first step is to compute the order 0 jet forms, which was already done in (4.16) and (4.19):

$$\sigma^X = \boldsymbol{\lambda}(dx) = f_x\, dx, \qquad \sigma^Y = \boldsymbol{\lambda}(dy) = dy, \tag{8.3a}$$

along with

$$\sigma^U = \boldsymbol{\lambda}(du) = \frac{du}{f_x} - \frac{u\, f_{xx}}{f_x^2} dx = U_X \sigma^X + U_Y \sigma^Y + \vartheta, \tag{8.3b}$$

the latter expression following from the general formula (4.8) as in (4.21).

Initiating the normalization procedure, we choose the cross-section

$$\mathcal{K}^0 = \{x = 0,\ u = 1\} \subset \mathrm{J}^0 = M,$$

i.e., the line parallel to the y-axis passing through $(0,0,1)$. Solving the corresponding normalization equations

$$X = 0, \qquad U = 1,$$

produces the order 0 pseudo-group normalizations

$$f = 0, \qquad f_x = u. \tag{8.4a}$$

In the standard moving frame procedure, implementation of the higher order pseudo-group normalizations requires computing the prolonged action in full detail. However, one can avoid such a computation by noticing that, on a pseudo-group orbit, most of group forms (4.2) will pull back to contact forms, and thereby vanish on a pseudo-group section $z \mapsto (z, g^{(\infty)}(z)) \in \mathcal{G}^{(\infty)}$. (A precise statement of this observation is given in Proposition 8.6.) In the following we introduce the equivalence relation $\equiv$ to denote equality modulo group contact forms on $\mathcal{B}^{(\infty)}$. Substituting (8.4a) into the basis group forms $\Upsilon_k = df_k - f_{k+1}\,dx$, cf. (4.22), yields the partially normalized group forms

$$\widehat{\Upsilon} = -u\,dx, \qquad \widehat{\Upsilon}_x = du - f_{xx}\,dx, \qquad \widehat{\Upsilon}_k = df_k - f_{k+1}\,dx, \qquad k \geq 2.$$

On a pseudo-group orbit, $\widehat{\Upsilon}$ clearly cannot pull-back to a contact form, but the higher order group forms can. Setting[6] $\widehat{\Upsilon}_k \equiv 0$ for $k \geq 1$, it follows that

$$f_{k+1} = D_x^k(u) = u_{k,0}, \qquad k \geq 0. \tag{8.4b}$$

Thus, (8.4) immediately reproduce the pseudo-group normalizations that were obtained in [56].

At this stage, we have already constructed normalizations for all the pseudo-group parameters, and hence this example may not seem especially recursive. In more complicated situations, additional iterations of the normalization procedure will be required, making the recursive nature of the algorithm more evident.

Behind the scene there is a cross-section $\mathcal{K}^\infty$ inducing the moving frame normalizations. We can easily recover $\mathcal{K}^\infty$ since, by construction, the moving frame ρ restricts to the identity jet on its defining cross-section:

$$\rho(z_0^{(\infty)}) = \mathbb{1}^{(\infty)}, \qquad \text{whenever} \qquad z_0^{(\infty)} \in \mathcal{K}^\infty. \tag{8.5}$$

On the other hand, at the base point $z = (x, y, u)$, the identity jet for the Lie–Tresse–Kumpera pseudo-group (2.9) is recovered from the pseudo-group parameter values

$$f = x, \qquad f_x = 1, \qquad f_{k+1} = 0, \qquad k \geq 1. \tag{8.6}$$

Substituting (8.6) into (8.4) yields the cross-section

$$\mathcal{K}^\infty = \{x = f = 0, \ \ u = f_x = 1, \ \ u_{k,0} = f_{k+1} = 0, \ \ k \geq 1\} \tag{8.7}$$

that produces our moving frame map. Therefore, the corresponding normalization equations are

$$X = 0, \qquad U = 1, \qquad U_{X^k} = 0, \qquad k \geq 1. \tag{8.8}$$

[6] This important fact will be justified in Example 8.5.

To find coordinate expressions for the differential invariants we substitute the normalizations (8.4) into the order 0 jet forms (8.3):

$$\widehat{\sigma}^X = u\,dx, \qquad \widehat{\sigma}^Y = dy, \qquad \widehat{\sigma}^U = \frac{du}{u} - \frac{u_x}{u}dx, \tag{8.9}$$

where the hat decoration will be our notation for the pull-back via the moving frame ρ defined by (8.4). In terms of the standard jet coframe (4.1),

$$\widehat{\sigma}^U = \frac{\theta}{u} + \frac{u_y}{u}\,dy = \frac{\theta}{u} + \frac{u_y}{u}\,\widehat{\sigma}^Y, \tag{8.10}$$

where $\theta = du - u_x\,dx - u_y\,dy$ is the ordinary order 0 contact form for surfaces. On the other hand, the pull-back of (8.3b) by ρ yields

$$\widehat{\sigma}^U = \widehat{U}_X\,\widehat{\sigma}^X + \widehat{U}_Y\,\widehat{\sigma}^Y + \widehat{\vartheta}. \tag{8.11}$$

Comparing the non-contact components of (8.10) and (8.11), we deduce that $\widehat{U}_X = 0$ is a phantom invariant, as we already knew from (8.8), while

$$\widehat{U}_Y = \frac{u_y}{u}$$

yields the coordinate expression for the basic differential invariant.

The coordinate expressions of the remaining normalized invariants $\widehat{U}_{Y^\ell}$, $\widehat{U}_{X^kY^\ell}$, $k \geq 0$, $\ell \geq 1$, can now be found through the use of the fundamental recurrence relations. To determine these, first note that the prolonged infinitesimal generator for this action is obtained by substituting (2.13) into the prolongation formula:

$$\mathbf{v}^{(\infty)} = a\frac{\partial}{\partial x} - a_x\,u\frac{\partial}{\partial u} - (a_{xx}u + 2\,a_x u_x)\frac{\partial}{\partial u_x} - (a_x u_y)\frac{\partial}{\partial u_y} \;-\; \cdots\,.$$

In accordance with (5.2), we identify the lifted jet coordinates with the corresponding Maurer–Cartan forms (5.5):

$$\boldsymbol{\lambda}(a) = \mu, \qquad \boldsymbol{\lambda}(a_x) = \mu_X, \qquad \boldsymbol{\lambda}(a_{xx}) = \mu_{XX}, \qquad \ldots\,.$$

Substitution into (7.4) produces the lifted recurrence relations which, up to order 1, are

$$\begin{aligned} dX = \sigma^X + \mu, \qquad dY = \sigma^Y, \qquad dU = \sigma^U - U\mu_X,\\ dU_X = \sigma^{U_X} - U\mu_{XX} - 2U_X\mu_X, \qquad dU_Y = \sigma^{U_Y} - U_Y\mu_X. \end{aligned} \tag{8.12}$$

Pulling-back (8.12) by the moving frame, which is equivalent to making the substitutions (8.4), we obtain the *phantom recurrence relations* associated with the cross-section coordinates $X, U, U_X, \ldots$, each of which is set equal to a constant, and hence their differentials all vanish: $0 = dX = dU = dU_X = \cdots$. Using these in (8.12) yields

$$\widehat{\mu} = -\widehat{\sigma}^X, \qquad \widehat{\mu}_X = \widehat{\sigma}^U, \qquad \widehat{\mu}_{XX} = \widehat{\sigma}^{U_X}. \tag{8.13}$$

Substituting (8.13) into the remaining non-phantom recurrence relations, and then invoking (4.7), one obtains, modulo contact forms,

$$d\widehat{U}_Y = \widehat{\sigma}^{U_Y} - \widehat{U}_Y\widehat{\sigma}^U \equiv \widehat{U}_{XY}\widehat{\sigma}^X + (\widehat{U}_{YY} - \widehat{U}_Y^2)\widehat{\sigma}^Y. \tag{8.14}$$

On the other hand, from (6.3),

$$d\widehat{U}_Y \equiv (\widehat{\mathcal{D}}_X \widehat{U}_Y)\, \widehat{\sigma}^X + (\widehat{\mathcal{D}}_Y \widehat{U}_Y)\, \widehat{\sigma}^Y, \tag{8.15}$$

where

$$\widehat{\mathcal{D}}_X = \frac{1}{u} D_x, \qquad \widehat{\mathcal{D}}_Y = D_y, \tag{8.16}$$

are the invariant dual total derivative operators to $\widehat{\sigma}^X$ and $\widehat{\sigma}^Y$, given in (8.9). Equating (8.14) and (8.15) we find that

$$\widehat{U}_{XY} = \widehat{\mathcal{D}}_X \widehat{U}_Y, \qquad \widehat{U}_{YY} = \widehat{\mathcal{D}}_Y \widehat{U}_Y + \widehat{U}_Y^2.$$

Further analysis of the higher order recurrence relations leads us to deduce that every differential invariant can be written as a function of the fundamental invariant $\widehat{U}_Y$ and its invariant derivatives obtained by successively applying the invariant differential operators (8.16).

Finally, the invariant operators dual to $\{\widehat{\sigma}^X, \widehat{\sigma}^Y, \widehat{\sigma}^{U_{i,j}}\}$, obtained by solving (6.4), are, up to order 1,

$$\begin{aligned}
\widehat{\mathbb{D}}_X &= \frac{1}{u}\,\frac{\partial}{\partial x} + \frac{u_x}{u}\,\frac{\partial}{\partial u} + \frac{u_{xx}}{u}\,\frac{\partial}{\partial u_x} + \frac{u_x\, u_y}{u^2}\,\frac{\partial}{\partial u_y} + \frac{u_{xxx}}{u}\,\frac{\partial}{\partial u_{xx}} \\
&\qquad\qquad + \frac{u\, u_y\, u_{xx} + 2u\, u_x\, u_{xy} - 2u_x^2 u_y}{u^3}\,\frac{\partial}{\partial u_{xy}} + \frac{u_x\, u_{yy}}{u^2}\,\frac{\partial}{\partial u_{yy}} + \cdots\,, \\
\widehat{\mathbb{D}}_Y &= \frac{\partial}{\partial y}, \qquad\qquad \widehat{\mathbb{D}}_U = u\frac{\partial}{\partial u} + u_x\frac{\partial}{\partial u_x} + u_{xx}\frac{\partial}{\partial u_{xx}} + \cdots\,, \\
\widehat{\mathbb{D}}_{U_X} &= u^2\frac{\partial}{\partial u_x} + 3uu_x\frac{\partial}{\partial u_{xx}} + \cdots\,, \qquad \widehat{\mathbb{D}}_{U_Y} = u\frac{\partial}{\partial u_y} + u_x\frac{\partial}{\partial u_{xy}} + \cdots\,.
\end{aligned}$$

By direct computation, one can verify (at least for the first few terms), that the formulae (6.5) hold; namely,

$$\widehat{\mathcal{D}}_X = \widehat{\mathbb{D}}_X + \sum_{i,j=0}^{\infty} \widehat{U}_{i+1,j+1}\, \widehat{\mathbb{D}}_{U_{i,j+1}}, \qquad \widehat{\mathcal{D}}_Y = \widehat{\mathbb{D}}_Y + \sum_{i,j=0}^{\infty} \widehat{U}_{i,j+1}\, \widehat{\mathbb{D}}_{U_{i,j}}.$$

We further note the identities

$$\widehat{\mu}_k = \widehat{\mathbb{D}}_X^k \widehat{\mu} = -\widehat{\mathbb{D}}_X^k \widehat{\sigma}^X, \qquad k \geq 0,$$

which are the normalized counterparts of (5.5), and will be justified in Example 8.13.

Using this example as a reference point, we can now describe the geometric framework underlying the recursive algorithm. As in the non-recursive moving frame algorithm, [56, 57], we assume that the prolonged Lie pseudo-group action becomes eventually free in order to guarantee termination of the recursive algorithm and the existence of a moving frame. Nevertheless, even in cases where termination is not guaranteed, the recursive algorithm can yield important information, including any differential invariants found during the course of its implementation, even in the absence of a formal Basis Theorem.

The following definition of a partial moving frame was proposed in [54], and formalizes the geometric object obtained by normalizing some of the pseudo-group parameters.

Definition 8.2. A *partial moving frame* of *order* k is a right-invariant (local) subbundle $\widehat{\mathcal{B}}^{(k)} \subset \mathcal{B}^{(k)}$ of the lifted bundle $\mathcal{B}^{(k)} \to \mathrm{J}^k$. *Right-invariance* means that $\widetilde{R}_g(\widehat{\mathcal{B}}^{(k)}) \subset \widehat{\mathcal{B}}^{(k)}$ for all $g \in \mathcal{G}$, where defined.

We note that, in the case of a moving frame, the right-invariant subbundle is simply the image of the moving frame section (6.1), $\widehat{\mathcal{B}}^{(k)} = \varrho^{(k)}(\mathcal{W}^k)$, and so its fiber over each point $z^{(k)} \in \mathcal{W}^k$ is the single point $\varrho^{(k)}(z^{(k)})$, that is

$$\widehat{\mathcal{B}}^{(k)}|_{z^{(k)}} = \left\{ \varrho^{(k)}(z^{(k)}) = (z^{(k)}, \rho^{(k)}(z^{(k)})) \right\} \subset \mathcal{B}^{(k)}|_{z^{(k)}}. \tag{8.17}$$

In general, if $\widehat{\mathcal{B}}^{(k)} \subset \mathcal{B}^{(k)}$ is right-invariant, then its inverse image under the standard projection, $(\widehat{\pi}^l_k)^{-1}(\widehat{\mathcal{B}}^{(k)}) \subset \mathcal{B}^{(l)}$ for any $l > k$, is itself right-invariant.

Given a right-invariant subbundle $\widehat{\mathcal{B}}^{(k)} \subset \mathcal{B}^{(k)}$ we now consider the restricted target map $\widehat{\boldsymbol{\tau}}^{(k)} \colon \widehat{\mathcal{B}}^{(k)} \longrightarrow \mathrm{J}^k$. Observe that, if $\mathcal{K}^k \subset \mathrm{J}^k$ is *any* subset, the evident identity

$$\widehat{\boldsymbol{\tau}}^{(k)}[\widetilde{R}_h(z^{(k)}, g^{(k)})] = \widehat{\boldsymbol{\tau}}^{(k)}(z^{(k)}, g^{(k)}) = g^{(k)} \cdot z^{(k)}$$

implies that $(\widehat{\boldsymbol{\tau}}^{(k)})^{-1}\mathcal{K}^k \subset \widehat{\mathcal{B}}^{(k)}$ is a right-invariant subset. In order to ensure that the resulting subset be a (local) subbundle, we impose the following transversality condition; see also [54].

Proposition 8.3. If $\mathcal{K}^k \subset \mathrm{J}^k$ is a cross-section to the prolonged pseudo-group orbits, or, more generally, satisfies $T\mathcal{K}^k|_{z^{(k)}} + \mathfrak{g}^{(k)}|_{z^{(k)}} = T\mathrm{J}^k|_{z^{(k)}}$ for all $z^{(k)} \in \mathcal{K}^k$, then $\widehat{\mathcal{B}}^{(k)} = (\widetilde{\boldsymbol{\tau}}^{(k)})^{-1}\mathcal{K}^k$ defines a partial moving frame of order k.

In particular, when the action becomes free (and regular) on an open subset of J^n, the partial moving frame $\widehat{\mathcal{B}}^{(n)} = (\widetilde{\boldsymbol{\tau}}^{(n)})^{-1}\mathcal{K}^n \subset \mathcal{B}^{(n)}$ associated with a local cross-section $\mathcal{K}^n \subset \mathrm{J}^n$ coincides with the image of an equivariant moving frame section (8.17), reproducing the standard construction, [56].

Given a point $z_0^{(\infty)} \in \mathrm{J}^\infty|_{z_0}$, let

$$\mathcal{B}_0 = \mathcal{B}^{(\infty)}\Big|_{z_0^{(\infty)}} = \widetilde{\boldsymbol{\sigma}}^{-1}(z_0^{(\infty)}) \simeq \boldsymbol{\sigma}^{-1}(z_0) = \mathcal{G}^{(\infty)}|_{z_0}$$

denote the corresponding source fiber of the lifted bundle sitting over $z_0^{(\infty)}$. Its image under the target map is the *orbit* containing $z_0^{(\infty)}$, denoted

$$\mathcal{O}_0 = \mathcal{O}_{z_0^{(\infty)}} = \widetilde{\boldsymbol{\tau}}\big(\mathcal{B}_0\big),$$

which, by freeness, is diffeomorphic to the fiber: $\mathcal{O}_0 \simeq \mathcal{B}_0 \simeq \mathcal{G}^{(\infty)}|_{z_0}$. Further, if $\varrho = \varrho^{(\infty)} \colon \mathrm{J}^\infty \to \mathcal{B}^{(\infty)}$ is a right moving frame section, let

$$\mathcal{R}_0 = \varrho(\mathcal{O}_0) \subset \mathcal{B}^{(\infty)}$$

denote the image of the orbit $\mathcal{O}_0$ in the lifted bundle.

Proposition 8.4. Let $\mathcal{G}$ be a Lie pseudo-group with right and left Maurer–Cartan forms μ^a_B and λ^a_B, respectively, that acts locally freely on an open subset of J^∞. Let $\mathcal{K}^\infty \subset \mathrm{J}^\infty$ be a cross-section, and $\varrho = \varrho^{(\infty)} \colon \mathrm{J}^\infty \to \mathcal{B}^{(\infty)}$ the corresponding right moving frame section. Then, given $z_0^{(\infty)} \in \mathcal{K}^\infty$, using the above notations,

$$\widetilde{\boldsymbol{\tau}}^*(\widehat{\mu}^a_B \mid \mathcal{O}_0) = (\varrho \circ \widetilde{\boldsymbol{\tau}})^*(\mu^a_B \mid \mathcal{R}_0) = -\,\lambda^a_B \mid \mathcal{B}_0. \tag{8.18}$$

Proof. Since $z_0^{(\infty)} \in \mathcal{K}^\infty$, we have $\varrho(z_0^{(\infty)}) = (z_0^{(\infty)}, \mathbb{1}^{(\infty)}) \in \mathcal{B}_0$, cf., (8.5). Thus, by right equivariance of the moving frame, given any $(z_0^{(\infty)}, g^{(\infty)}) \in \mathcal{B}_0$, we have

$$\begin{aligned}\varrho \circ \widetilde{\boldsymbol{\tau}}(z_0^{(\infty)}, g^{(\infty)}) = \varrho(g^{(\infty)} \cdot z_0^{(\infty)}) = \widetilde{R}_{g^{(\infty)}}(\varrho(z_0^{(\infty)})) = \widetilde{R}_{g^{(\infty)}}(z_0^{(\infty)}, \mathbb{1}^{(\infty)}) \\ = (g^{(\infty)} \cdot z_0^{(\infty)}, (g^{(\infty)})^{-1}) = i(z_0^{(\infty)}, g^{(\infty)}),\end{aligned}$$

where i denotes the pseudo-group inversion map (5.7). In other words, $\varrho \circ \widetilde{\boldsymbol{\tau}} \mid \mathcal{B}_0 = i \mid \mathcal{B}_0$, which implies

$$\mathcal{R}_0 = \varrho\big(\mathcal{O}_0\big) = \varrho \circ \widetilde{\boldsymbol{\tau}}\big(\mathcal{B}_0\big) = i\big(\mathcal{B}_0\big) = \left\{ (g^{(\infty)} \cdot z_0^{(\infty)}, (g^{(\infty)})^{-1}) \;\middle|\; g^{(\infty)} \in \mathcal{B}_0 \right\}.$$

Thus, in view of Proposition 5.4

$$\widetilde{\boldsymbol{\tau}}^*(\widehat{\mu}_B^a \mid \mathcal{O}_0) = (\varrho \circ \widetilde{\boldsymbol{\tau}})^*(\mu_B^a \mid \mathcal{R}_0) = i^*(\mu_B^a \mid \mathcal{R}_0) = -\,\lambda_B^a \mid \mathcal{B}_0,$$

which establishes the result. □

Example 8.5. To illustrate Proposition 8.4, we consider the Lie–Tresse–Kumpera pseudo-group (2.9). A cross-section to the prolonged action is given by (8.7). Let $z_0^{(\infty)} \in \mathcal{K}^\infty$ be the submanifold jet on the cross-section given by

$$x = 0, \qquad y = y^0, \qquad u = 1, \qquad u_{k,0} = 0, \qquad u_{i,j} = u_{i,j}^0, \qquad j, k \geq 1, \qquad i \geq 0,$$

where y^0, $u_{i,j}^0$ are constants. Then, using (4.18), the transformed submanifold jet $z^{(n)} = g^{(n)} \cdot z_0^{(n)} = \widetilde{\boldsymbol{\tau}}^{(n)}(z_0^{(n)}, g^{(n)})$ is given, up to order 2, by

$$\begin{gathered} x = f, \qquad y = y^0, \qquad u = \frac{1}{f_x}, \qquad u_x = -\frac{f_{xx}}{f_x^3}, \qquad u_y = \frac{u_y^0}{f_x}, \\ u_{xx} = -\frac{f_{xxx}}{f_x^4} + \frac{3f_{xx}^2}{f_x^5}, \qquad u_{xy} = \frac{u_{xy}^0}{f_x^2} - \frac{f_{xx}\, u_y^0}{f_x^3}, \qquad u_{yy} = \frac{u_{yy}^0}{f_x}, \end{gathered} \tag{8.19}$$

where the source of the local diffeomorphism jet is $\boldsymbol{\sigma}^{(n)}(g^{(n)}) = (0, y^0, 1)$.

To verify formula (8.18), we substitute (8.19) into the normalized Maurer–Cartan forms (8.13). Recalling the formulae (5.10) for the left invariant Maurer–Cartan forms $\lambda, \lambda_x, \lambda_{xx}, \ldots$, of the diffeomorphism pseudo-group $\mathcal{D}(\mathbb{R})$, the result, up to order 2, is

$$\begin{aligned} \widetilde{\boldsymbol{\tau}}^*(\widehat{\mu} \mid \mathcal{O}_0) &= -\,\widetilde{\boldsymbol{\tau}}^*(\widehat{\sigma}^X \mid \mathcal{O}_0) = -u\,dx\Big|_{(8.19)} = -\frac{1}{f_x}df = -\,\lambda \mid \mathcal{B}_0, \\ \widetilde{\boldsymbol{\tau}}^*(\widehat{\mu}_X \mid \mathcal{O}_0) &= \widetilde{\boldsymbol{\tau}}^*(\widehat{\sigma}^U \mid \mathcal{O}_0) = \frac{du - u_x\,dx}{u}\bigg|_{(8.19)} = -\frac{1}{f_x}df_x + \frac{f_{xx}}{f_x^2}df \\ &\qquad = -\frac{\Upsilon_x}{f_x} + \frac{f_{xx}\,\Upsilon}{f_x^2} = -\lambda_x, \\ \widetilde{\boldsymbol{\tau}}^*(\widehat{\mu}_{XX} \mid \mathcal{O}_0) &= \widetilde{\boldsymbol{\tau}}^*(\widehat{\sigma}^{U_X} \mid \mathcal{O}_0) = \frac{1}{u^2}du_x - \frac{u_x}{u^3}du + \left(\frac{u_x^2}{u^3} - \frac{u_{xx}}{u^2}\right)dx\bigg|_{(8.19)} \\ &= -\frac{df_{xx}}{f_x} + 2\frac{f_{xx}}{f_x^2}df_x + \left(\frac{f_{xxx}}{f_x^2} - 2\frac{f_{xx}^2}{f_x^3}\right)df \\ &= -\frac{\Upsilon_{xx}}{f_x} + 2\frac{f_{xx}\,\Upsilon_x}{f_x^2} + \left(\frac{f_{xxx}}{f_x^2} - 2\frac{f_{xx}^2}{f_x^3}\right)\Upsilon = -\lambda_{xx}, \end{aligned}$$

noting that the restriction to $\mathcal{B}_0$ does not affect $\lambda_x, \lambda_{xx}, \ldots$. In general,

$$\widetilde{\boldsymbol{\tau}}^*(\widehat{\mu} \mid \mathcal{O}_0) = -\lambda \mid \mathcal{B}_0, \qquad \widetilde{\boldsymbol{\tau}}^*(\widehat{\mu}_k \mid \mathcal{O}_0) = -\lambda_k, \qquad k \geq 1. \tag{8.20}$$

We observe that all normalized Maurer–Cartan forms of orders ≥ 1 restrict along an orbit to genuine contact forms. Consequently, the second equality in (8.20) implies that the normalized Maurer–Cartan forms of order ≥ 1 vanish, modulo contact forms, on the orbit:

$$\widetilde{\boldsymbol{\tau}}^*(\widehat{\mu}_k \mid \mathcal{O}_0) \ \equiv \ 0, \qquad k \geq 1. \tag{8.21}$$

By the definition (5.5) of the Maurer–Cartan forms μ_k, equation (8.21) is equivalent to

$$\widetilde{\boldsymbol{\tau}}^*(\widehat{\Upsilon}_k \mid \mathcal{O}_0) \ \equiv \ 0, \qquad k \geq 1,$$

at the level of the basic group forms. This justifies the claims that led to the prolonged pseudo-group normalizations (8.4b) in Example 8.1. Note that, in contrast, the normalized order zero Maurer–Cartan form does not vanish, modulo contact forms:

$$\widetilde{\boldsymbol{\tau}}^*(\widehat{\mu} \mid \mathcal{O}_0) = \widetilde{\boldsymbol{\tau}}^*(\widehat{\Upsilon} \mid \mathcal{O}_0) = -u\, dx \ \not\equiv \ 0.$$

In the language of exterior differential systems, this non-vanishing requirement corresponds to an *independence condition* on the integral solutions, [48].

From the previous example we observed that when

$$\widetilde{\boldsymbol{\tau}}^*(\widehat{\mu}^a_B \mid \mathcal{O}_0) = -\, \lambda^a_B \mid \mathcal{B}_0 \tag{8.22}$$

is a genuine contact form, then

$$\widetilde{\boldsymbol{\tau}}^*(\widehat{\mu}^a_B \mid \mathcal{O}_0) \ \equiv \ 0. \tag{8.23}$$

The observed identity (8.23) holds whenever λ^a_B is independent of the coordinates $z^1, \ldots, z^m$ and their differentials $dz^1, \ldots, dz^m$, which leads to the following result.

Proposition 8.6. If the left Maurer–Cartan forms λ^a_B of order ≥ 1 are independent of the coordinates $z^1, \ldots, z^m$ and their differentials $dz^1, \ldots, dz^m$, then the normalized group forms

$$\widetilde{\boldsymbol{\tau}}^*(\widehat{\Upsilon}^a_B \mid \mathcal{O}_0) \ \equiv \ 0, \qquad \#B \geq 1, \tag{8.24}$$

vanish, modulo contact forms, along the orbit $\mathcal{O}_0$.

Proof. By assumption, whenever $\#B \geq 1$, equality (8.22) holds, which then implies (8.23). Further, according to their defining formula (5.1), each right Maurer–Cartan form μ^a_B of order ≥ 1 is a linear combination of the group forms Υ^a_B of order ≥ 1 and conversely, which immediately establishes (8.24). □

Corollary 8.7. If $\mathcal{G}$ is a translational Lie pseudo-group, then (8.24) holds in translational coordinates.

Proof. According to (5.6), the left Maurer–Cartan forms λ^a_B of order ≥ 1 are obtained by Lie differentiating the order zero Maurer–Cartan forms

$$\lambda^a = \sum_{b=1}^{m} w^a_b\, \Upsilon^b = \sum_{b=1}^{m} w^a_b \left(dZ^b - \sum_{c=1}^{m} Z^b_c\, dz^c \right) = -dz^a + \sum_{b=1}^{m} w^a_b\, dZ^b, \qquad a = 1, \ldots, m,$$

where $(w^a_b) = (Z^a_b)^{-1}$ denotes the inverse of the Jacobian matrix whose entries are given by the total derivative of Z^a with respect to the operators (2.3). Since $\mathbb{D}_{z^b}(dz^a) = 0$,

$$\lambda^a_B = \mathbb{D}^B_z \left(\sum_{b=1}^{m} w^a_b \, dZ^b \right) \quad \text{for} \quad \#B \geq 1. \tag{8.25}$$

The latter expression involves the pseudo-group jets Z^a_B and their differentials dZ^a_B for $\#B \geq 1$. Since, in translational coordinates, the source variables z^a do not appear in the determining equations (2.17), the right hand-side of (8.25) is independent of z^a and dz^a once restricted to the lifted determining system (2.17). □

In the context of the recursive algorithm, it suffices that (8.24) holds on a basis of the group forms. For example, if $\mathcal{G}$ is a one-to-one prolongation of $\mathcal{H}$, then according to (5.9), the pull-back of the Maurer–Cartan forms of $\mathcal{H}$ by the projection map forms a basis of Maurer–Cartan forms for $\mathcal{G}$.

Corollary 8.8. If $\mathcal{G}$ is the one-to-one prolongation of a translational Lie pseudo-group $\mathcal{H}$, then, in translational coordinates, the (pull-back to $\mathcal{G}$ of the) group forms of $\mathcal{H}$ satisfy

$$\widetilde{\boldsymbol{\tau}}^*(\widehat{\Upsilon}^{a_\sharp}_{B_\sharp} \mid \mathcal{O}_0) \;\equiv\; 0, \qquad \#B_\sharp \geq 1. \tag{8.26}$$

Remark 8.9. We note that in Proposition 8.6 and Corollaries 8.7, 8.8, nothing is mentioned about the order zero normalized group forms $\widehat{\Upsilon}^a$. Some of them can be nonzero, as occurs in Examples 8.5, 10.1, 10.2, and 10.3, and thereby specify independence conditions.

Once the low order pseudo-group normalizations have been prescribed, if the hypothesis of Proposition 8.6 holds, equations (8.24) or (8.26) yield constraints on the higher order pseudo-group normalizations. The induced *prolonged pseudo-group normalizations* will correspond to a certain choice of (partial) cross-section. By definition, a *partial cross-section* $\mathcal{K}^k$ is a submanifold intersecting the prolonged pseudo-group orbits transversally, as cross-sections do, but whose intersection with each orbit is a submanifold of dimension ≥ 1 so that $T\mathcal{K}^k|_{z^{(k)}} \cap \mathfrak{g}^{(k)}|_{z^{(k)}} \neq \{0\}$ for $z^{(k)} \in \mathcal{K}^k$. To obtain the defining equations of the (partial) cross-section we use the fact that on a (partial) cross-section the (partial) moving frame includes the identity jet, (8.5). Thus, substituting the identity jet's pseudo-group parameters, $g^{(k)} = \mathbb{1}^{(k)}$, into the prolonged pseudo-group normalizations produces the corresponding (partial) cross-section.

After solving the normalization equations (8.2) for the partial moving frame, some of the pseudo-group parameters $g^{(k)}_1$ will be expressed in terms of the submanifold jets coordinates and the remaining pseudo-group parameters $g^{(k)}_2$. We call the parameters $g^{(k)}_1$ the *partially normalized pseudo-group parameters*, while $g^{(k)}_2$ will be called (*partially*) *unnormalized pseudo-group parameters*.

Employing the preceding hat notation to also indicate partial normalization via a partial moving frame, let

$$\widehat{\sigma}^i, \qquad \widehat{\sigma}^\alpha_K, \qquad \widehat{\mu}^a_B, \qquad \widehat{\Upsilon}^a_B, \tag{8.27}$$

denote the restrictions (pull-backs) of the jet, Maurer–Cartan, and group forms to the k^{th} order partial moving frame bundle $\widehat{\mathcal{B}}^{(k)}$. We will refer to them as *partially normalized* *forms.* As with the pseudo-group parameters, some of the *partially normalized Maurer–Cartan forms* $\widehat{\mu}_1^{(k)}$ will be expressed in terms of the partially normalized jet forms $\widehat{\sigma}^i$, $\widehat{\sigma}_K^\alpha$, $1 \leq i \leq p$, $1 \leq \alpha \leq q$, $0 \leq \#K \leq k$, and the remaining (*partially*) *unnormalized Maurer–Cartan forms* $\widehat{\mu}_2^{(k)}$. This can all be done symbolically through the use of the partially normalized recurrence relations (7.4). The same holds for the partially normalized group forms.

Given a partial moving frame $\widehat{\mathcal{B}}^{(k)}$ of order k, if the pseudo-group action is not free at this order, the next step in the recursive algorithm is to compute the expressions for the $(k+1)^{\text{th}}$ order partially normalized prolonged action. This is accomplished by using the (partially normalized) recurrence relations (7.4), as we now explain. We begin by stating a key identity that enables us to proceed.

Lemma 8.10. Let $\widehat{\mu}_B^a$ denote the partially normalized Maurer–Cartan forms corresponding to the partial moving frame $\widehat{\mathcal{B}}^{(k-1)}$. Then

$$\widehat{\mu}_{B,b}^a = \widehat{\mathbb{D}}_b \widehat{\mu}_B^a + L_{B,b}^a, \qquad a, b = 1, \ldots, m, \qquad \#B \leq k-1, \tag{8.28}$$

where $L_{B,b}^a$ is a *correction term* of order $\leq k-1$ that can be explicitly determined.

Here $\widehat{\mathbb{D}}_1, \ldots, \widehat{\mathbb{D}}_m$ are the partially normalized differential operators dual to the partially normalized jet forms $\widehat{\sigma}^1, \ldots, \widehat{\sigma}^m$, cf. (8.27), and whose (truncated) coordinate expressions are obtained by solving the system of linear equations

$$\begin{aligned} \widehat{\mathbb{D}}_b \,\lrcorner\, \widehat{\sigma}^a &= \delta_b^a, \\ \widehat{\mathbb{D}}_b \,\lrcorner\, \widehat{\sigma}_K^\alpha &= 0, \end{aligned} \qquad 1 \leq a, b \leq m, \quad 1 \leq \alpha \leq q, \quad 1 \leq \#K \leq k-1, \tag{8.29a}$$

along with the conditions

$$\widehat{\mathbb{D}}_b \,\lrcorner\, \widehat{\Upsilon}_B^a = 0 \tag{8.29b}$$

imposed on a basis of (partially) unnormalized group forms of order $\#B \leq k$. Note that the higher order terms in $\widehat{\mathbb{D}}_b$ can be omitted since they do not appear in (8.28).

In favorable cases, the correction terms in (8.28) all vanish, that is $L_{B,b}^a = 0$ for all a, b, B, and then the normalized Maurer–Cartan forms are given by the same formula as their unnormalized counterparts:

$$\widehat{\mu}_{B,b}^a = \widehat{\mathbb{D}}_b \widehat{\mu}_B^a. \tag{8.30}$$

More generally, under somewhat milder conditions on the pseudo-group, (8.30) only holds for some a, b, B, but those cases are the ones of importance for implementing the recursive algorithm. Either results in a significant simplification in the recursive algorithm, and hence an a priori characterization of such cases is of importance. Unfortunately, we do not yet know the most general conditions on the pseudo-group that will guarantee that (8.30) holds in the cases of importance, and indeed we will see in Example 10.3 that it does not hold in complete generality. We defer the proof of Lemma 8.10 until we finish our presentation of the recursive algorithm. The proof will contain the explicit formulae for the correction terms in (8.28); see (8.39) and the subsequent discussion for details.

Now, in order to implement the recursive step, we must require that the linear system (8.29b) be equivalent to the linear system

$$\widehat{\mathbb{D}}_b \,\lrcorner\, \widehat{\mu}_B^a = 0 \tag{8.31}$$

restricted to a basis of (partially) unnormalized Maurer–Cartan forms. The latter conditions will guarantee that $\widehat{\mathbb{D}}_b$ is an invariant differential operator. The equivalence of (8.29b) and (8.31) imposes some further restrictions on the pseudo-group action, which will be analyzed at the end of this section.

Once the expressions for the partially normalized Maurer–Cartan forms of order $\leq k$ have been obtained using (8.28), and restricted (pulled-back) to the partial moving frame $\widehat{\mathcal{B}}^{(k)}$, the order k partially normalized recurrence relations

$$\sum_{k=1}^{p} \widehat{U}^\alpha_{K,k}\, \widehat{\sigma}^k \equiv \widehat{\sigma}^\alpha_K = d\widehat{U}^\alpha_K - \widehat{\psi}^K_\alpha, \tag{8.32}$$

cf. (4.7), (7.6), yield the expressions for the $(k+1)^{\text{th}}$ partially normalized prolonged action. Here $\widehat{\psi}^K_\alpha = \widehat{\psi}^K_\alpha(\widehat{X}, \widehat{U}^{(k)}, \widehat{\mu}^{(k)})$ denotes the k^{th} order partially normalized lifted infinitesimal generator coefficient $\psi^K_\alpha = \boldsymbol{\lambda}(\varphi^K_\alpha)$ defined in (7.5). Essential to the recursive algorithm is the observation that their coordinate expressions can be computed recursively. Indeed, at order k, the expressions for the partially normalized invariants $(\widehat{X}, \widehat{U}^{(k)})$ are known, while the expressions for the partially normalized Maurer–Cartan forms $\widehat{\mu}^{(k)}$ are obtained by pulling-back (8.28) or, in favorable cases, (8.30) to the partial moving frame $\widehat{\mathcal{B}}^{(k)}$. The resulting coordinate expressions for the partially normalized lifted invariants allow one to make further pseudo-group normalizations. The procedure is then repeated until a moving frame is constructed at the order of freeness of the pseudo-group. Details can be found in Section 9.

Proof of Lemma 8.10. Using Cartan's formula for Lie differentiation, [48],

$$L^a_{B,b} = \widehat{\mu}^a_{B,b} - \widehat{\mathbb{D}}_b \widehat{\mu}^a_B = \widehat{\mu}^a_{B,b} - \widehat{\mathbb{D}}_b \lrcorner\, d\widehat{\mu}^a_B - d(\widehat{\mathbb{D}}_b \lrcorner\, \widehat{\mu}^a_B). \tag{8.33}$$

To compute the second term we use the structure equations of the Lie pseudo-group $\mathcal{G}$. To simplify the notation, let

$$\Sigma^a_B = \sum_{c=1}^{m} \sum_{\substack{B=(A,C)\\ \#C\geq 1}} \binom{B}{A} \mu^a_{A,c} \wedge \mu^c_C \Big|_{L^{(\infty)}(Z,\mu^{(\infty)})=0} \tag{8.34}$$

denote the second term in the structure equations (5.11) restricted to the lifted infinitesimal determining equations (5.3) that prescribe the linear dependencies among the Maurer–Cartan forms. As a result, the partially normalized structure equations assume the form

$$d\widehat{\mu}^a_B = \sum_{c=1}^{m} \widehat{\sigma}^c \wedge \widehat{\mu}^a_{B,c} + \widehat{\Sigma}^a_B. \tag{8.35}$$

For a fixed $1 \leq b \leq m$, we single out the linear dependency of $\widehat{\mu}^a_B$ on $\widehat{\sigma}^b$, and write[7]

$$\widehat{\mu}^a_B = I^a_{B;b}\, \widehat{\sigma}^b + \widehat{\nu}^a_{B;b}, \qquad \text{where} \qquad I^a_{B;b} = \widehat{\mathbb{D}}_b \lrcorner\, \widehat{\mu}^a_B, \tag{8.36}$$

so that $\widehat{\nu}^a_{B;b}$ is independent of $\widehat{\sigma}^b$. Note, if $\widehat{\mu}^a_B$ is already normalized, then the right-hand side of (8.36) can be obtained symbolically through use of the recurrence relations

[7]There is no summation over repeated indices here.

for the phantom invariants. On the other hand, for the unnormalized Maurer–Cartan forms, $I^a_{B;b} = 0$ as a consequence of (8.31). Then, from (5.11), (8.35),

$$\widehat{\mathbb{D}}_b \mathbin{\lrcorner} d\widehat{\mu}^a_B = \widehat{\nu}^a_{B,b;b} - \sum_{\substack{c=1 \\ c\neq b}}^{m} I^a_{B,c;b}\,\widehat{\sigma}^c - \widehat{\Sigma}^a_{B;b}, \qquad \text{where} \qquad \widehat{\Sigma}^a_{B;b} = \widehat{\mathbb{D}}_b \mathbin{\lrcorner} \widehat{\Sigma}^a_B. \tag{8.37}$$

Using (8.36), the third term in (8.33) becomes

$$d(\widehat{\mathbb{D}}_b \mathbin{\lrcorner} \widehat{\mu}^a_B) = dI^a_{B;b}. \tag{8.38}$$

Substituting (8.37) and (8.38) into (8.33), yields

$$\begin{aligned} L^a_{B,b} &= \widehat{\mu}^a_{B,b} - \widehat{\nu}^a_{B,b;b} + \sum_{\substack{c=1 \\ c\neq b}}^{m} I^a_{B,c;b}\,\widehat{\sigma}^c + \widehat{\Sigma}^a_{B;b} - dI^a_{B;b} \\ &= I^a_{B,b;b}\,\widehat{\sigma}^b + \sum_{\substack{c=1 \\ c\neq b}}^{m} I^a_{B,c;b}\,\widehat{\sigma}^c + \widehat{\Sigma}^a_{B;b} - dI^a_{B;b} = \sum_{c=1}^{m} I^a_{B,c;b}\,\widehat{\sigma}^c - dI^a_{B;b} + \widehat{\Sigma}^a_{B;b}, \end{aligned} \tag{8.39}$$

which is the promised formula for the correction terms.

The correction term (8.39) is of order $\leq k-1$ and can be computed recursively, as we now explain. Assume $\#B \leq k-1$. Then once the $(k-1)^{\text{st}}$ iteration of the recursive algorithm is completed, expressions for the (partially) normalized jet and Maurer–Cartan forms $\widehat{\sigma}^i, \widehat{\sigma}^\alpha_K, \widehat{\mu}^a_C$, for $\#K, \#C \leq k-1$, are known. Moreover, in view of the general formulae for the correction terms $\widehat{\psi}^K_\alpha$ in the partially normalized recurrence relations — see the discussion following (8.32) — when the partially normalized Maurer–Cartan forms $\widehat{\mu}^a_C$ are expressed in terms of the basis of partially normalized jet forms $\widehat{\sigma}^i$, $\widehat{\sigma}^\alpha_K$, and the remaining unnormalized Maurer–Cartan forms, their resulting coefficients are explicit functions of the partially normalized invariants $\widehat{X}$, $\widehat{U}^{(k-1)}$ of order $\leq k-1$. According to (8.34), since $\widehat{\Sigma}^a_B$ only involves wedge products of (partially) normalized Maurer–Cartan forms of order $\leq k-1$, the term $\widehat{\Sigma}^a_{B;b}$ in (8.39) can be computed via (8.37), and is of order $\leq k-1$. Similarly, the invariant $I^a_{B;b}$ can be determined from its definition (8.36) and is of order $\leq k-1$.

Further, when $\#B \leq k-2$, each term in the initial summand of (8.39) can be determined from (8.36) and is of order $\leq k-1$. To explain how to proceed when $\#B = k-1$, let $\mu^{\circ k} = \{\mu^{\circ k}_1, \ldots, \mu^{\circ k}_{d_k}\}$ be a basis of k^{th} order Maurer–Cartan forms. Then the restriction of the k^{th} order diffeomorphism Maurer–Cartan forms to $\mathcal{G}^{(k)}$ are certain linear combinations of the Maurer–Cartan forms $\mu^{(k-1)}$ and $\mu^{\circ k}$

$$\mu^a_{B,c} = \mathcal{M}^a_{B,c}(Z, \mu^{(k-1)}, \mu^{\circ k})$$

so that, once pulled-back to $\widehat{\mathcal{B}}^{(k-1)}$,

$$\widehat{\mu}^a_{B,c} = \mathcal{M}^a_{B,c}(\widehat{Z}, \widehat{\mu}^{(k-1)}, \widehat{\mu}^{\circ k}).$$

Since the order k Maurer–Cartan forms have yet to be normalized once the $(k-1)^{\text{th}}$ iteration of the recursive algorithm is completed,

$$\widehat{\mathbb{D}}_b \mathbin{\lrcorner} \widehat{\mu}^{\circ k}_\ell = 0, \qquad b = 1, \ldots, m \quad \text{and} \quad \ell = 1, \ldots, d_k,$$

by virtue of (8.31). It therefore follows that each term

$$I^a_{B,c;b} = \widehat{\mathbb{D}}_b \,\lrcorner\, \widehat{\mu}^a_{B,c} = \mathcal{M}^a_{B,c}(\widehat{Z}, \widehat{\mathbb{D}}_b \,\lrcorner\, \widehat{\mu}^{(k-1)})$$

in the initial sum of (8.39) can be recursively computed and is of order $\leq k-1$. □

The preceding manipulations will be considerably simplified if we know in advance that the relevant correction terms $L^a_{B,b}$ in (8.28) vanish, and hence (8.30) holds. Let us now discuss conditions on the pseudo-group that guarantee this.

Theorem 8.11. If $\mathcal{G}$ is a translational pseudo-group, then, in the translational coordinates, (8.30) holds for all a, b, B.

Proof. Since the action is transitive, at the initial iteration of the recursive moving frame algorithm we can set $Z^a = c^a$ for any convenient constants $c^1, \ldots, c^m$. From the recurrence relations (7.4), we have that

$$\widehat{\mu}^a = -\widehat{\sigma}^a, \qquad a = 1, \ldots, m.$$

In light of Corollary 2.9, the fact that the infinitesimal determining equations (2.19) are independent of the order zero vector field jets ζ^a implies, after lifting the equations, that the Maurer–Cartan forms of order ≥ 1 are linearly independent of the order zero Maurer–Cartan forms μ^a. From the formula for the prolonged vector field coefficients (7.2b) and the definition of the correction terms $\psi^K_\alpha = \boldsymbol{\lambda}(\varphi^K_\alpha)$ in the recurrence relations (7.4), it follows that ψ^K_α is independent of μ^a when $\#K \geq 1$:

$$\psi^K_\alpha = \psi^K_\alpha(X, U^{(k)}, \mu^{(1,k)}), \qquad \#K = k \geq 1,$$

where, as above, $\mu^{(1,k)}$ denotes the collection of Maurer–Cartan forms μ^a_B of orders $1 \leq \#B \leq k$. Thus, once (partially) normalized, the Maurer–Cartan forms $\widehat{\mu}^{(1,\infty)}$ are themselves independent of $\widehat{\mu}^a = -\widehat{\sigma}^a$. This implies that

$$\widehat{\mathbb{D}}_b \,\lrcorner\, \widehat{\mu}^a_B = 0 \qquad \text{whenever} \qquad \#B \geq 1, \qquad a, b = 1, \ldots, m. \tag{8.40}$$

Now, to compute the correction term (8.33), we first observe that, by (8.35),

$$\widehat{\mathbb{D}}_b \,\lrcorner\, d\widehat{\mu}^a_B = \widehat{\mathbb{D}}_b \,\lrcorner\, \left[\sum_{c=1}^m \widehat{\sigma}^c \wedge \widehat{\mu}^a_{B,c} + \widehat{\Sigma}^a_B\right] = \widehat{\mu}^a_{B,b} + \widehat{\mathbb{D}}_b \,\lrcorner\, \widehat{\Sigma}^a_B = \widehat{\mu}^a_{B,b}, \tag{8.41}$$

the final equality following from (8.40) using the fact that $\widehat{\Sigma}^a_B$ only involves (partially) normalized Maurer–Cartan forms of order ≥ 1. On the other hand, again by (8.35), (8.40),

$$d(\widehat{\mathbb{D}}_b \,\lrcorner\, \widehat{\mu}^a_B) = d \begin{Bmatrix} -1, & \text{if } B = b, \\ 0, & \text{otherwise}, \end{Bmatrix} = 0. \tag{8.42}$$

Substituting (8.41), (8.42) into the right-hand side of (8.33) produces $L^a_{B,b} = 0$, as claimed. □

From the perspective of the recursive algorithm, it would suffice to know that the correction term in (8.28) is zero when the equation is restricted to a basis of Maurer–Cartan forms.

Theorem 8.12. Let $\mathcal{G}$ be a one-to-one prolongation of a translational pseudo-group $\mathcal{H}$. Then, in the adapted system of coordinates (2.21), the correction terms associated with the Maurer–Cartan forms of $\mathcal{H}$ vanish:

$$\widehat{\mu}^{a_\sharp}_{B_\sharp,b_\sharp} = \widehat{\mathbb{D}}_{b_\sharp}\widehat{\mu}^{a_\sharp}_{B_\sharp} + L^{a_\sharp}_{B_\sharp,b_\sharp} = \widehat{\mathbb{D}}_{b_\sharp}\widehat{\mu}^{a_\sharp}_{B_\sharp}.$$

Proof. Recalling Corollary 2.16, the lift of the infinitesimal determining system (2.23) yields the following linear relations

$$L^{(n)}_{\mathfrak{h}}(\mu^{(1,n)}_\sharp) = 0, \qquad \mu_{\sharp,Z_\flat} = 0, \qquad \mu_\flat = L_\flat(Z_\flat, \mu^{(1,n)}_\sharp)$$

among the Maurer–Cartan forms of order $\leq n$. It follows that $\mu^{(1,n)}_\sharp, \mu^{(n)}_\flat$ are linearly independent of the order zero Maurer–Cartan forms $\mu_\sharp = (\mu^1, \ldots, \mu^\ell)$. Adapting the proof of Theorem 8.11, one can, at order zero, at least make the normalizations

$$Z^1 = c^1, \quad \ldots \quad Z^\ell = c^\ell.$$

Then, the order zero recurrence relations

$$\begin{aligned} dZ^{a_\sharp} &= \sigma^{a_\sharp} + \mu^{a_\sharp}, & 1 \leq a_\sharp \leq \ell, \\ dZ^{a_\flat} &= \sigma^{a_\flat} + \mu^{a_\flat}, & \ell + 1 \leq a_\flat \leq m, \end{aligned}$$

imply that

$$\widehat{\mu}^{a_\sharp} = -\widehat{\sigma}^{a_\sharp}, \qquad 1 \leq a_\sharp \leq \ell.$$

Since $\mu^{a_\flat}$ and ψ^α_K are independent of $\mu^{a_\sharp}$, the normalized Maurer–Cartan forms $\widehat{\mu}^{a_\sharp}_{B_\sharp}$ of order $\# B_\sharp \geq 1$ will be linearly independent of $\widehat{\sigma}^{a_\sharp}$. In view of the structure equations for $\mathcal{H}$,

$$d\sigma^{a_\sharp} = \sum_{b_\sharp=1}^{\ell} \mu^{a_\sharp}_{b_\sharp} \wedge \sigma^{b_\sharp}, \qquad d\mu^{a_\sharp}_{B_\sharp} = \sum_{c_\sharp=1}^{\ell} \sigma^{c_\sharp} \wedge \mu^{a_\sharp}_{B_\sharp,c_\sharp} + \Sigma^{a_\sharp}_{B_\sharp},$$

the same computations as in the proof of Theorem 8.11 yields $L^{a_\sharp}_{B_\sharp,b_\sharp} = 0$. □

Example 8.13. As mentioned in Example 2.13, the Lie–Tresse–Kumpera pseudo-group (2.9) is a one-to-one prolongation of the diffeomorphism pseudo-group $X = f(x)$. Since the (infinitesimal) determining system of the diffeomorphism pseudo-group is vacuous, the hypotheses of Theorem 8.12 hold, and

$$\widehat{\sigma}^{U_{k-1,0}} = \widehat{\mu}_k = \widehat{\mathbb{D}}^k_X \widehat{\mu} = -\widehat{\mathbb{D}}^k_X \widehat{\sigma}^X, \qquad k \geq 1.$$

Theorem 8.12 does not mention what happens when $\widehat{\mu}_k$ is differentiated with respect to $\widehat{\mathbb{D}}_Y$ or $\widehat{\mathbb{D}}_U$. To complete this example, let us compute the corresponding correction terms. First, we show that $L^x_{X^k,Y} = 0$ vanishes using (8.33). From the lifted infinitesimal determining equations (5.4), $\mu_{k,1} = \mu^x_{X^k Y} = 0$ for all $k \geq 0$ and therefore $\widehat{\mu}_{k,1} = \widehat{\mu}^x_{X^k Y} = 0$. Recalling the structure equations of the diffeomorphism pseudo-group $\mathcal{D}(\mathbb{R})$ given in (5.12), and using the fact that the normalized Maurer–Cartan forms

$$\widehat{\mu} = -\widehat{\sigma}^X, \qquad \widehat{\mu}_{k+1} = \widehat{\sigma}^{U_{k,0}}, \qquad k \geq 1, \tag{8.43}$$

are independent of $\widehat{\sigma}^Y$, it follows that

$$\widehat{\mathbb{D}}_Y \lrcorner\, d\widehat{\mu}_k - d(\widehat{\mathbb{D}}_Y \lrcorner\, \widehat{\mu}_k) = 0.$$

Therefore, the right-hand side of (8.33) is zero and $L^x_{X^k,Y} = 0$. On the other hand, the correction term is not zero when $\widehat{\mu}_k$ is differentiated with respect to $\widehat{\mathbb{D}}_U$. Indeed, since $\widehat{\mu}_X = \widehat{\sigma}^U$, we have

$$L^x_{,U} = -\widehat{\mathbb{D}}_U \lrcorner \, (\widehat{\sigma}^X \wedge \widehat{\mu}_X) = \widehat{\sigma}^X$$

so that

$$0 = \widehat{\mu}_U = \widehat{\mathbb{D}}_U \widehat{\mu} + \widehat{\sigma}^X = -\widehat{\mathbb{D}}_U \widehat{\sigma}^X + \widehat{\sigma}^X.$$

More generally, for $k \geq 1$,

$$\begin{aligned} L^x_{X^k,U} &= \widehat{\mathbb{D}}_U \lrcorner \sum_{j=1}^{[(k+1)/2]} \frac{k-2j+1}{k+1} \binom{n+1}{j} \widehat{\mu}_j \wedge \widehat{\mu}_{k+1-j} \\ &= \frac{k-1}{k+1} \binom{k+1}{1} \widehat{\mu}_k = (k-1)\, \widehat{\mu}_k = (k-1)\, \widehat{\sigma}^{U_{k-1,0}}, \end{aligned}$$

where in the last equality we used the normalizations (8.43).

Example 8.14. As noted in Example 2.14, the pseudo-group (2.14) is a one-to-one prolongation of the pseudo-group (2.20). The infinitesimal determining system of the latter is

$$\xi_y = 0, \qquad \eta_y = \xi_x,$$

which satisfies the hypotheses of Theorem 8.12. Thus, the correction term in (8.28) is zero when the equation is restricted to the basis of (partially) normalized Maurer–Cartan forms $\widehat{\mu}_k$, $\widehat{\nu}_k$, $k \geq 0$, and the operators $\widehat{\mathbb{D}}_X$, $\widehat{\mathbb{D}}_Y$. Hence, we conclude that

$$\widehat{\mu}_k = \widehat{\mathbb{D}}^k_X \widehat{\mu}, \qquad \widehat{\nu}_k = \widehat{\mathbb{D}}^k_X \widehat{\nu},$$

and

$$0 = \widehat{\mu}_{X^k,Y} = \widehat{\mathbb{D}}_Y \widehat{\mu}_k, \qquad \widehat{\mu}_{k+1} = \widehat{\nu}_{X^k,Y} = \widehat{\mathbb{D}}_Y \widehat{\nu}_k.$$

The recursive construction of a moving frame for the pseudo-group (2.14) will be given below in Example 10.1.

The next theorem highlights another situation in which some of the correction terms in (8.28) vanish.

Theorem 8.15. Let $\mathcal{G}$ be an intransitive Lie pseudo-group action. Choose an adapted system of local coordinates[8] $z = (z_\sharp, z_\flat)$, such that the first ℓ coordinates $z_\sharp = (z^1, \ldots, z^\ell)$ are invariants, i.e.,

$$Z^{a_\sharp} = z^{a_\sharp}, \qquad 1 \leq a_\sharp \leq \ell.$$

Then

$$\widehat{\mu}^a_{B,b_\sharp} = \widehat{\mathbb{D}}_{b_\sharp} \widehat{\mu}^a_B, \qquad \text{for} \qquad B = (b^1, \ldots, b^n), \quad 1 \leq a, b^\nu \leq m, \quad 1 \leq b_\sharp \leq \ell. \tag{8.44}$$

Proof. The argument is similar to the proof of Theorem 8.12. According to the coordinate splitting $z = (z_\sharp, z_\flat)$, the infinitesimal determining system of the Lie pseudo-group takes the form

$$\zeta_\sharp = 0, \qquad L^{(n)}(z, \zeta^{(n)}_\flat) = 0,$$

[8]This system of coordinates is not necessarily the same as that introduced in (2.21). We note that Cartan, [11, 12], employs such coordinates in his approach to the structure theory of Lie pseudo-groups.

producing the following constraints on the Maurer–Cartan forms:

$$\mu_\sharp = 0, \qquad L^{(n)}(Z, \mu_\flat^{(n)}) = 0.$$

Then, the recurrence relations (7.4) reduce to

$$\begin{aligned} dZ^{a_\sharp} &= \sigma^{a_\sharp}, & 1 \le a_\sharp \le \ell, \\ dZ^{a_\flat} &= \sigma^{a_\flat} + \mu^{a_\flat}, & \ell + 1 \le a_\flat \le m, \\ dU^\alpha_K &= \sigma^\alpha_K + \psi^\alpha_K, & \#K \ge 1, \end{aligned}$$

with

$$\psi^\alpha_K = \psi^\alpha_K(X, U^{(k)}, \mu_\flat^{(k)}), \qquad k = \#K.$$

As one implements the recursive moving frame algorithm, the (partially) normalized Maurer–Cartan forms $\widehat{\mu}_\flat^{(k)}$ will be independent of $\widehat{\sigma}^{a_\sharp} = \sigma^{a_\sharp}$. Thus, the corresponding correction terms

$$L^{a_\flat}_{B_\flat, b_\sharp} = \widehat{\mu}^{a_\flat}_{B_\flat, b_\sharp} - \widehat{\mathbb{D}}_{b_\sharp} \lrcorner\, d\widehat{\mu}^{a_\flat}_{B_\flat} - d(\widehat{\mathbb{D}}_{b_\sharp} \lrcorner\, \widehat{\mu}^{a_\flat}_{B_\flat}) = \widehat{\mu}^{a_\flat}_{B_\flat, b_\sharp} - \widehat{\mu}^{a_\flat}_{B_\flat, b_\sharp} = 0$$

vanish. Since $\widehat{\mu}^{a_\sharp}_{B_\sharp} = 0$ is identically zero, we can extend the last equality to

$$L^a_{B, b_\sharp} = 0, \qquad \text{where} \qquad B = (b^1, \ldots, b^n), \quad n = \#B \ge 0, \quad 1 \le a, b^\nu \le m,$$

thereby establishing (8.44). □

Example 8.16. An illustration of Theorem 8.15 is provided by the pseudo-group (2.9) of Example 2.5 where $Y = y$ is an invariant. In Example 8.13 we explicitly found that $L^x_{X^k, Y} = 0$ for any $k \ge 0$.

Remark 8.17. In the context of Theorem 8.15, if we further assume that the action includes all pure translations in the $z_\flat$ coordinates, then all the correction terms $L^a_{B,b}$ vanish.

Remark 8.18. At first, one might be tempted to conclude that the variables $z_\sharp = (z^1, \ldots, z^\ell)$ in Theorem 8.15 are superfluous, and one may as well restrict the Lie pseudo-group action to the variables $z_\flat = (z^{\ell+1}, \ldots, z^m)$. However, this is not necessarily the case, due to the possible occurrence of essential invariants, [59]. A simple example is given by the abelian Lie pseudo-group

$$X = x, \qquad U = u + g(x),$$

where $g(x)$ is an arbitrary smooth function, which has x as an essential invariant. Assuming $u = u(x)$ is a function of x, and implementing the recursive moving frame algorithm, it is not hard to see that a basis of Maurer–Cartan forms is given by

$$\mu_k = \mu^u_{X^k}, \qquad k \ge 0, \qquad \text{and that once normalized} \qquad \widehat{\mu}_k = \widehat{\mathbb{D}}^k_X \widehat{\mu},$$

where

$$\widehat{\mathbb{D}}_X = D_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + \cdots, \qquad \widehat{\mu} = -\widehat{\sigma}^U = -(du - u_x\, dx).$$

Theorems 8.11, 8.12, and 8.15 cover a wide range of Lie pseudo-group actions encountered in applications. Determining the weakest constraints on the pseudo-group action guaranteeing the vanishing of the relevant correction terms $L^a_{B,b}$ remains an open problem.

We conclude this section by discussing the equivalence of the linear systems (8.29b) and (8.31), which is required for the success of our recursive algorithm. We begin with a fairly general result.

Proposition 8.19. If the basis of (partially) unnormalized Maurer–Cartan forms is linearly independent of the order zero jet forms $\widehat{\sigma}^1, \ldots, \widehat{\sigma}^m$, then (8.29b) and (8.31) are equivalent.

Proof. By the hypothesis, we can write the basis of (partially) unnormalized Maurer–Cartan forms $\widehat{\mu}^a_B$ of order[9] $1 \leq \#B \leq k$ in terms of the invariant jet forms $\widehat{\sigma}^\alpha_K$, $1 \leq \alpha \leq q$, $1 \leq \#K \leq k$, and a basis of (partially) unnormalized group forms $\widehat{\Upsilon}^a_B$, $1 \leq \#B \leq k$:

$$\widehat{\mu}^a_B = \sum_{\alpha=1}^{q} \sum_{\#K=1}^{k} F^{a,K}_{B,\alpha}(z^{(n)}, \widehat{Z}^{(n)})\, \widehat{\sigma}^\alpha_K + \sum_{\substack{\text{basis of}\\ \text{unnormalized}\\ \text{group forms}}} G^{a,A}_{B,c}(z^{(n)}, \widehat{Z}^{(n)})\, \widehat{\Upsilon}^c_A. \tag{8.45}$$

Taking the interior product of (8.45) with $\widehat{\mathbb{D}}_b$, and using (8.29b), we deduce (8.31). Conversely, since $\widehat{\mu}^a_B$ and $\widehat{\Upsilon}^a_B$ both form bases of (partially) unnormalized group forms, if (8.31) holds, then the interior product of (8.45) with $\widehat{\mathbb{D}}_b$ over the full basis of $\widehat{\mu}^a_B$ yields (8.29b). □

Next we demonstrate that the hypothesis of Proposition 8.19 holds when $\mathcal{G}$ is a translational pseudo-group.

Theorem 8.20. If $\mathcal{G}$ is a translational pseudo-group, then, in translational coordinates, the system of equations (8.29b) and (8.31) are equivalent.

Proof. The argument is similar to the proof of Theorem 8.11. Since the determining equations do not involve the order zero pseudo-group jets Z^a, taking the group differential of (2.17) implies that the pull-back to $\mathcal{G}^{(\infty)}$ of the group forms Υ^a_B of order $\#B \geq 1$ are linearly independent of the order zero group forms Υ^a. From the definition of the right invariant Maurer–Cartan forms (5.1), it follows that the pull-back to $\mathcal{G}^{(\infty)}$ of the Maurer–Cartan forms of order ≥ 1 are linearly independent of the order zero group forms. Since the (partially) normalized jet forms $\widehat{\sigma}^a$ will only occur via the normalization of the order zero group forms

$$\widehat{\mu}^a = \widehat{\Upsilon}^a = -\widehat{\sigma}^a,$$

the Maurer–Cartan forms of order ≥ 1 are linearly independent of $\widehat{\sigma}^a$, and the proof follows from that of Theorem 8.19. □

[9] At the initial iteration of the recursive algorithm, all the order zero Maurer–Cartan forms μ^a can be normalized. This explains why the (partially) unnormalized Maurer–Cartan forms $\widehat{\mu}^b_B$ are of order ≥ 1.

More generally, when $\mathcal{G}$ is the one-to-one prolongation of a Lie pseudo-group $\mathcal{H}$, as in Theorem 8.12, it is enough, for the purpose of the recursive moving frame algorithm, to restrict (8.29b) and (8.31) to $\mathcal{H}$.

Theorem 8.21. If $\mathcal{G}$ is the one-to-one prolongation of a translational Lie pseudo-group $\mathcal{H}$, then the equations

$$\widehat{\mathbb{D}}_{a_\sharp} \lrcorner\, \widehat{\Upsilon}^{b_\sharp}_{B_\sharp} = 0, \qquad B_\sharp = (b^1_\sharp, \ldots, b^k_\sharp), \quad 1 \leq a_\sharp, b_\sharp, b^\nu_\sharp \leq \ell,$$

restricted to a basis of (partially) unnormalized group forms of $\mathcal{H}$, are equivalent to the equations

$$\widehat{\mathbb{D}}_{a_\sharp} \lrcorner\, \widehat{\mu}^{b_\sharp}_{B_\sharp} = 0, \qquad B_\sharp = (b^1_\sharp, \ldots, b^k_\sharp), \quad 1 \leq a_\sharp, b_\sharp, b^\nu_\sharp \leq \ell,$$

restricted to a basis of (partially) unnormalized right-invariant Maurer–Cartan forms of $\mathcal{H}$.

This concludes our presentation of the currently known requirements that permit our recursive moving frame algorithm to succeed. Weakening, or even eliminating, these restrictions is a significant open problem.

9 The Recursive Algorithm

Let us summarize the recursive algorithm. Assume we are given a Lie pseudo-group $\mathcal{G}$ acting on M that eventually acts freely on an open subset of a sufficiently high order submanifold jet bundle. As elsewhere, [14, 53, 56–58], we do not have a means of a priori determining whether the prolonged action becomes eventually free. Although once this happens in practice, one detects it by being able to solve for all the normalized Maurer–Cartan forms in the recurrence relations (7.6), which indicates the existence of a moving frame. Alternatively, local freeness can be checked in advance by calculating the dimension of the span of the prolonged infinitesimal generators and checking whether it equals the dimension of the pseudo-group jet bundle of the same order. In most cases, lack of freeness is a consequence of the rate of growth of the dimensions of the pseudo-group jet spaces. It would be of great interest to find a condition that guarantees eventual freeness at the outset. Furthermore, even in the absence of freeness, our recursive algorithm will produce the differential invariants, if any, along the way. It may even produce one or more invariant differential operators that can then be used to construct additional higher order differential invariants. Indeed, if p independent differential invariants are found, their differentials constitute an invariant coframe, and hence produce p invariant differential operators. On the other hand, the fundamental Basis Theorem, as it currently exists, does not guarantee that these can be used to produce a complete system of differential invariants when the pseudo-group does not act freely. See [34] for possible examples and further developments related to globally defined differential invariants in the algebraic setting.

As we have seen, for our recursive algorithm to succeed, we must impose additional mild assumptions on the pseudo-group action. These hold if $\mathcal{G}$ is a translational pseudo-group, meaning that it contains an abelian subgroup that, in some coordinate system, coincides with the group of all translations. More generally, our method applies to one-to-one prolongations of translational pseudo-groups. Most known examples are of these types. It should be possible to prescribe intrinsic properties of the structure equations of $\mathcal{G}$ that guarantee the existence of the required abelian sub-pseudo-group of translations. Developing algorithms to achieve this would be particularly useful for the application of our recursive algorithm to more complex scenarios, but this will require further investigation of the structure theory of infinite-dimensional Lie pseudo-groups. Other pseudo-group actions can be treated provided the hypotheses in Propositions 8.6 and 8.19 are valid. The conditions in Proposition 8.6 can be checked before beginning the recursive algorithm; one would only need to know the determining equations of $\mathcal{G}$ to decide if the left Maurer–Cartan forms are independent of the coordinates $z^1, \ldots, z^m$ and their differentials $dz^1, \ldots, dz^m$. As for Proposition 8.19, verifying its conditions in advance is a little more subtle as it involves the (partially) normalized Maurer–Cartan forms.

Let us now summarize the steps involved in the recursive algorithm in a form that can be applied in practice. To begin the algorithm, the following data is required:

- The explicit formulas for the pseudo-group transformations of $\mathcal{G}$ on M.
- The linear determining system for its infinitesimal generators $\mathfrak{g}$.
- The choice of a basis for the group forms of order $\leq n$ corresponding to the order of freeness. Because the order of freeness cannot be determined from the outset, we can, instead, consider a basis of suitably high order k. If $k < n$, one can always include higher order group forms as the recursive algorithm proceeds. However, in practice, it is often easy to deduce a basis of order ∞.

Next, the initialization step is:

1. Write out the order 0 jet forms $\sigma^{X^i} = d_J X^i$, $i = 1, \ldots, p$, and $\sigma^{U^\alpha} = d_J U^\alpha$, $\alpha = 1, \ldots, q$.
2. Fix the order 0 cross-section and solve the corresponding normalization equations for $n_0 \leq m = \dim M$ pseudo-group parameters, thus producing the partial moving frame of order 0.
3. Substitute the existing normalizations into the order 0 Maurer–Cartan forms. Use (8.28) and (8.39), or, when justified, (8.30), e.g., for a translational Lie pseudo-group, to differentiate the partially normalized Maurer–Cartan forms of order zero to obtain the partially normalized Maurer–Cartan forms of order 1.
4. Substitute all existing normalizations into the order 0 jet forms to determine the partially normalized action on J^1. Use the latter expressions to further normalize pseudo-group parameters if possible.
5. Substitute the resulting formulae into the known group forms. If these can be set to zero modulo contact forms (along an orbit), e.g., if the pseudo-group is translational, or, more generally, by applying Proposition 8.6, solve the resulting equations for as many pseudo-group parameters as possible. This step is not necessary, but, when possible, it does simplify the ensuing calculations as it provides expressions for the pseudo-group normalizations without having to compute the prolonged action. The (partial) cross-section leading to these pseudo-group normalizations is obtain by substituting the identity jet into the normalizations for the pseudo-group parameters.

After initialization, one loops, starting with $k = 1$, through the following steps. The algorithm is guaranteed to terminate when k reaches the order of freeness, or, possibly, earlier, when the pseudo-group normalization is complete and a system of generating differential invariants and invariant differential operators has been found.

6. Substitute the current normalizations into the lifted recurrence relations at order k, the previously partially unnormalized Maurer–Cartan forms of order $\leq k$, and the partially normalized jet forms of order $\leq k - 1$.
7. Use (8.32) to write out the corresponding differentiated jet forms of order k. Comparison of the resulting expressions yields the required formulae for the partially normalized action on J^{k+1}.
8. Use (8.28) and (8.39), or, when justified, (8.30), e.g., for a translational Lie pseudo-group, to differentiate the partially normalized Maurer–Cartan forms of order k to obtain the partially normalized Maurer–Cartan forms of order $k + 1$.
9. If the action is not yet free, specify the order $k + 1$ cross-section and solve the corresponding normalization equations for n_{k+1} pseudo-group parameters.
10. While not necessary for the termination of the recursive algorithm, one can then repeat step 5 if desired.
11. Replace k by $k + 1$, and continue until termination.

Once k attains the order of freeness — which can be detected by the fact that one has solved for all the pseudo-group parameters of order $\leq k$ — the algorithm produces a moving frame of that order.

To then obtain a generating set of invariants, one might have to implement further iterations of the recursive algorithm. By the Fundamental Basis Theorem, the algebra of differential invariants is generated by a finite set of invariants of a certain order $k^* \geq k$, which can be determined algorithmically via Gröbner basis techniques, [57]. At order k^*, the recursive algorithm can be terminated as all the higher order differential invariants and invariant differential forms can be determined, using invariant differentiation, from the basic differential invariants and invariant differential forms of order $\leq k^*$ via the known recurrence formulas. Alternatively, if one desires explicit formulae for the higher order pseudo-group normalizations and resulting higher order compatible moving frames, one can continue the algorithm as described to any desired (finite) order. In this context, it would be of interest to understand how generally applicable are the formal power series methods for normalizing all pseudo-group parameters that were introduced, in the context of examples, in [55].

We remark that it is possible that, even if the pseudo-group does not act eventually freely, the recursive algorithm terminates, in the sense that none of the remaining pseudo-group parameters can be normalized, but one is still able to extract differential invariants and invariant differential operators even in the absence of a suitable version of the Basis Theorem.

10 Further Examples

In this section we work through three more examples of the recursive moving frame algorithm. More substantial examples and applications will be deferred to subsequent publications.

Example 10.1. Consider the Lie pseudo-group (2.14) that we introduced in Examples 2.6, 2.14, 8.14. A basis of group forms is given by

$$\Upsilon_k = df_k - f_{k+1}\,dx, \qquad \begin{aligned}\Psi_k &= de_{k,0} - e_{k+1,0}\,dx - f_{k+1}\,dy,\\ &= y\,df_{k+1} + dg_k - (y f_{k+2} + g_{k+1})dx,\end{aligned} \qquad k \geq 0. \tag{10.1}$$

The infinitesimal generator has the form

$$\mathbf{v} = a\frac{\partial}{\partial x} + (a_x\,y + b)\frac{\partial}{\partial y} + (a_{xx}\,y + b_x)\frac{\partial}{\partial u}\,. \tag{10.2}$$

The lift map will take the infinitesimal generator jet coordinates $a, a_x, a_{xx}, \ldots$, to the corresponding right Maurer–Cartan forms $\mu, \mu_X, \mu_{XX} \ldots$. We also identify the lift of the ∂_y coefficient, $a_x\,y+b$, with the independent Maurer–Cartan form ν; its x derivatives $a_{xx}\,y+b_x, a_{xxx}\,y+b_{xx}, \ldots$ are thereby lifted to $\nu_X, \nu_{XX}, \ldots$, while its y derivatives implies the identities $\nu_Y = \mu_X$, $\nu_{YY} = 0$, along with their formal differential consequences.

Applying the prolongation formula to (10.2), the lifted recurrence relations (7.4) are found to be, up to order 2,

$$\begin{aligned}
dX &= \sigma^X + \mu, \qquad dY = \sigma^Y + \nu, \qquad dU = \sigma^U + \nu_X,\\
dU_X &= \sigma^{U_X} + \nu_{XX} - U_X\,\mu_X - U_Y\,\nu_X, \qquad dU_Y = \sigma^{U_Y} + \mu_{XX} - U_Y\,\mu_X,\\
dU_{XX} &= \sigma^{U_{XX}} + \nu_{XXX} - U_X\,\mu_{XX} - U_Y\,\nu_{XX} - 2U_{XX}\,\mu_X - 2U_{XY}\,\nu_X,\\
dU_{XY} &= \sigma^{U_{XY}} + \mu_{XXX} - U_Y\,\mu_{XX} - 2U_{XY}\,\mu_X - U_{YY}\,\nu_X,\\
dU_{YY} &= \sigma^{U_{YY}} - 2U_{YY}\,\mu_X.
\end{aligned} \tag{10.3}$$

The first step in the implementation of the recursive algorithm is to compute the order zero jet forms

$$\sigma^X = f_x\,dx, \qquad \sigma^Y = e_x\,dx + f_x\,dy, \qquad \sigma^U = du + \left(\frac{e_{xx}}{f_x} - \frac{e_x\,f_{xx}}{f_x^2}\right)dx + \frac{f_{xx}}{f_x}dy. \tag{10.4}$$

Choosing the order zero cross-section $\mathcal{K}^0 = \{x = y = u = 0\}$, and solving the corresponding normalization equations $X = Y = U = 0$ yields the order zero partial moving frame

$$f = 0, \qquad e = 0, \qquad e_x = -u\,f_x. \tag{10.5}$$

Substituting (10.5) into the group forms (10.1) produces

$$\begin{aligned}\widehat{\Upsilon} &= -f_x\,dx,\\ \widehat{\Upsilon}_k &= df_k - f_{k+1}\,dx,\end{aligned} \qquad \begin{aligned}\widehat{\Psi} &= f_x(u\,dx - dy),\\ \widehat{\Psi}_x &= -d(u\,f_x) - e_{xx}\,dx - f_{xx}\,dy,\\ \widehat{\Psi}_{k+1} &= de_{k+1,0} - e_{k+2,0}\,dx - f_{k+2}\,dy,\end{aligned} \qquad k \geq 1.$$

In Example 2.14, we observed that the pseudo-group (2.14) is a one-to-one prolongation of the translational pseudo-group (2.20). Applying Corollary 8.8, the group forms

$$\widetilde{\boldsymbol{\tau}}^*(\widehat{\Upsilon}_k \mid \mathcal{O}_0) \;\equiv\; \widetilde{\boldsymbol{\tau}}^*(\widehat{\Psi}_k \mid \mathcal{O}_0) \;\equiv\; 0, \qquad k \geq 1$$

must vanish along an orbit, modulo contact forms. Corollary 8.8 does not mention what happens to the order zero group forms. Since the order zero left Maurer–Cartan form

$$\lambda^y = \frac{1}{f_x}de - \frac{e_x}{f_x^2}df$$

is independent of the order zero coordinates x, y, u and their differentials, along an orbit the corresponding right Maurer–Cartan form

$$\widetilde{\boldsymbol{\tau}}^*(\widehat{\mu}^y \mid \mathcal{O}_0) = \widetilde{\boldsymbol{\tau}}^*(\widehat{\Psi} \mid \mathcal{O}_0) \ \equiv\ 0$$

must vanish modulo contact forms, which yields the differential constraint

$$dy = u\,dx \qquad \text{on} \ \ \mathcal{O}_0. \tag{10.6}$$

Equation (10.6) implies that, along an orbit of the pseudo-group action, $y = y(x)$ and, moreover, satisfies the ordinary differential equation

$$\frac{dy}{dx} = u(x, y(x)).$$

Accordingly, we introduce the derivative operator

$$\widehat{D}_x = f_x\,\widehat{\mathcal{D}}_X = D_x + u\,D_y, \tag{10.7}$$

where

$$D_x = \frac{\partial}{\partial x} + \sum_{j,k=0}^{\infty} u_{k+1,j}\frac{\partial}{\partial u_{k,j}} + \sum_{k=0}^{\infty} f_{k+1}\frac{\partial}{\partial f_k} + e_{k+1,0}\frac{\partial}{\partial e_{k,0}},$$
$$D_y = \frac{\partial}{\partial y} + \sum_{j,k=0}^{\infty} u_{k,j+1}\frac{\partial}{\partial u_{k,j}} + \sum_{j=0}^{\infty} f_{k+1}\frac{\partial}{\partial e_{k,0}},$$

are the total derivative operators on the lifted bundle $\mathcal{B}^{(\infty)}$. The vanishing of the pulled-back group forms $\widetilde{\boldsymbol{\tau}}^*(\widehat{\Upsilon}_k \mid \mathcal{O}_0)$, modulo contact forms, implies

$$f_{k+1} = \widehat{D}_x^k f_x, \qquad k \geq 1. \tag{10.8}$$

The equalities (10.8) are trivial when the pseudo-group parameters have yet to be normalized. The key observation is that once a pseudo-group parameter f_k is normalized, the equalities (10.8) imply that all higher order pseudo-group parameters $f_{k+\ell}$ are obtained by differentiating f_k with respect to $\widehat{D}_x$.

Next, using (10.8) and (10.6), the requirement that $\widetilde{\boldsymbol{\tau}}^*(\widehat{\Psi}_x \mid \mathcal{O}_0) \ \equiv\ 0$ implies

$$e_{xx} = -(u_x + u\,u_y)f_x - 2\,u\,f_{xx}. \tag{10.9}$$

Finally, the constraints $\widetilde{\boldsymbol{\tau}}^*(\widehat{\Psi}_k \mid \mathcal{O}_0) \ \equiv\ 0$ for $k \geq 2$ yields

$$e_{k+2,0} = \widehat{D}_x e_{k+1,0} - u\,f_{k+2}, \qquad k \geq 1. \tag{10.10}$$

Therefore, substituting (10.9) in (10.10) with $k = 1$ and iterating we obtain the normalizations for the pseudo-group parameters $e_{k,0}$ for $k \geq 2$.

To normalize the remaining pseudo-group parameters, namely $f_x, f_{xx}, \ldots$, we consider the order 1 partially normalized prolonged action. Substituting (10.5) and (10.9) into the order 0 jet forms (10.4) we obtain

$$\widehat{\sigma}^X = f_x\,dx, \qquad \widehat{\sigma}^Y = f_x(dy - u\,dx),$$
$$\widehat{\sigma}^U = du - \left(u_x + u\,u_y + \frac{u\,f_{xx}}{f_x}\right)dx + \frac{f_{xx}}{f_x}dy. \tag{10.11}$$

Modulo contact forms,

$$\widehat{U}_X\,\widehat{\sigma}^X + \widehat{U}_Y\,\widehat{\sigma}^Y \equiv \widehat{\sigma}^U \equiv \left(\frac{u_y}{f_x} + \frac{f_{xx}}{f_x^2}\right)\widehat{\sigma}^Y,$$

and thus

$$\widehat{U}_X = 0, \qquad \widehat{U}_Y = \frac{u_y}{f_x} + \frac{f_{xx}}{f_x^2},$$

the former being a consequence of the pseudo-group normalization (10.9). Setting $\widehat{U}_Y = 0$, we obtain the further normalizations

$$f_{xx} = -u_y\,f_x \qquad \text{whence} \qquad e_{xx} = (u\,u_y - u_x)f_x. \tag{10.12}$$

Recalling (10.8) and (10.10), all higher order pseudo-group normalizations are obtained by differentiating (10.12) with respect to $\widehat{D}_x$, as given in (10.7). Thus, for example,

$$f_{xxx} = \widehat{D}_x(f_{xx}) = \widehat{D}_x(-u_y\,f_x) = (u_y^2 - u_{xy} - u\,u_{yy})f_x,$$

where we used the fact that $f_{xx} = -u_y\,f_x$. Similarly,

$$e_{xxx} = \widehat{D}_x(e_{xx}) - u f_{xxx} = (u\,u_{xy} + 2u^2 u_{yy} + 2u_y u_x - u_{xx} - u\,u_y^2)f_x.$$

At this stage, we have a partial moving frame, where the only unnormalized pseudo-group parameter is f_x. The corresponding partial cross-section, that produces the above moving frame normalizations is obtained by substituting the identity jet

$$f = x, \qquad f_x = 1, \qquad e = y, \qquad f_{k+2} = e_{k,0} = 0, \qquad k \geq 0,$$

into (10.5) and (10.12), yielding

$$\mathcal{K}^\infty = \{x = y = u_{x^k} = u_{x^k y} = 0, \quad k \geq 0\},$$

with corresponding normalization equations

$$X = Y = U_{X^k} = U_{X^k Y} = 0, \qquad k \geq 0. \tag{10.13}$$

To proceed further, we invoke the recurrence relations (10.3), now partially normalized through the preceding specifications of all of the pseudo-group parameters except f_x. Substituting (10.13) in (10.3), and then solving for the partially normalized Maurer–Cartan forms, we obtain

$$\widehat{\mu} = -\widehat{\sigma}^X, \qquad \widehat{\nu} = -\widehat{\sigma}^Y, \qquad \widehat{\nu}_X = -\widehat{\sigma}^U, \qquad \widehat{\mu}_{XX} = -\widehat{\sigma}^{U_Y}. \tag{10.14}$$

Recalling Example 8.14, the higher order partially normalized Maurer–Cartan forms $\widehat{\mu}_k$ are obtained by differentiating the equality $\widehat{\mu} = -\widehat{\sigma}^X$ with respect to

$$\widehat{\mathbb{D}}_X = \frac{1}{f_x}\frac{\partial}{\partial x} + \frac{u}{f_x}\frac{\partial}{\partial y} + \frac{u_x + u\,u_y}{f_x}\frac{\partial}{\partial u} + \frac{u_{xy} + u\,u_{yy}}{f_x}\frac{\partial}{\partial u_y} + \frac{u_{xx} + u\,u_{xy}}{f_x}\frac{\partial}{\partial u_x} + \cdots - u_y\frac{\partial}{\partial f_x},$$

which is defined by the linear algebraic system

$$\widehat{\mathbb{D}}_X \lrcorner\, \widehat{\sigma}^X = \widehat{\mathbb{D}}_X \lrcorner\, \widehat{\sigma}^Y = \widehat{\mathbb{D}}_X \lrcorner\, \widehat{\sigma}^{U_{k,j}} = 0, \qquad \widehat{\mathbb{D}}_X \lrcorner\, \widehat{\Upsilon}_x = 0.$$

Modulo contact forms, we obtain, up to order 2,

$$\widehat{\mu}_X = \widehat{\mathbb{D}}_X \widehat{\mu} = -\widehat{\mathbb{D}}_X \widehat{\sigma}^X = u_y\, dx + \frac{1}{f_x} df_x, \qquad \widehat{\mu}_{XX} = \widehat{\mathbb{D}}_X \widehat{\mu}_X \equiv -\frac{u_{yy}}{f_x^2}\,\widehat{\sigma}^Y. \tag{10.15}$$

By (8.32) and (10.13)

$$\widehat{\sigma}^{U_Y} \equiv \widehat{U}_{XY}\,\widehat{\sigma}^X + \widehat{U}_{YY}\,\widehat{\sigma}^Y = \widehat{U}_{YY}\,\widehat{\sigma}^Y;$$

on the other hand, combining (10.14) and (10.15), we find

$$-\widehat{U}_{YY}\,\widehat{\sigma}^Y \equiv -\widehat{\sigma}^{U_Y} = \widehat{\mu}_{XX} = -\frac{u_{yy}}{f_x^2}\,\widehat{\sigma}^Y, \qquad \text{whence} \qquad \widehat{U}_{YY} = \frac{u_{yy}}{f_x^2}.$$

Assuming (for simplicity) $u_{yy} > 0$, we introduce the final normalization

$$\widehat{U}_{YY} = 1, \qquad \text{which gives} \qquad f_x = \sqrt{u_{yy}}. \tag{10.16}$$

The complete moving frame is then found by substituting (10.16) into (10.12) and their prolongations (10.8), (10.10). Up to order 3, we have

$$\begin{aligned}
&f = 0, && e = 0,\\
&f_x = \sqrt{u_{yy}}, && e_x = -u\sqrt{u_{yy}},\\
&f_{xx} = -u_y\sqrt{u_{yy}}, && e_{xx} = (u\,u_y - u_x)\sqrt{u_{yy}},\\
&f_{xxx} = (u_y^2 - u_{xy} - u\,u_{yy})\sqrt{u_{yy}}, && e_{xxx} = (u\,u_{xy} + 2u^2 u_{yy} + 2u_y\,u_x - u_{xx} - u\,u_y^2)\sqrt{u_{yy}}.
\end{aligned} \tag{10.17}$$

The operator $\widehat{\mathbb{D}}_X$ defined by the pairings

$$\widehat{\mathbb{D}}_X \lrcorner\, \widehat{\sigma}^X = \widehat{\mathbb{D}}_X \lrcorner\, \widehat{\sigma}^Y = \widehat{\mathbb{D}}_X \lrcorner\, \widehat{\sigma}^{U_{k,j}} = 0,$$

is, up to order 2, given by

$$\begin{aligned}
\widehat{\mathbb{D}}_X = {}& \frac{1}{\sqrt{u_{yy}}}\frac{\partial}{\partial x} + \frac{u}{\sqrt{u_{yy}}}\frac{\partial}{\partial y} + \frac{u_x + u\,u_y}{\sqrt{u_{yy}}}\frac{\partial}{\partial u} + \frac{u_{xy} + u\,u_{yy}}{\sqrt{u_{yy}}}\frac{\partial}{\partial u_y} + \frac{u_{xx} + u\,u_{xy}}{\sqrt{u_{yy}}}\frac{\partial}{\partial u_x},\\
& - 2u_y\sqrt{u_{yy}}\frac{\partial}{\partial u_{yy}} + \frac{u_{xxy} + 2u\,u_{xyy} + u^2 u_{yyy} + 2u\,u_y\,u_{yy}}{\sqrt{u_{yy}}}\frac{\partial}{\partial u_{xy}}\\
& + \frac{u_{xxx} + u\,u_{xxy} - u^2 u_{xyy} - u^3 u_{yyy} - 2u\,u_y\,u_{yy}}{\sqrt{u_{yy}}}\frac{\partial}{\partial u_{xx}} + \cdots .
\end{aligned}$$

Once all the pseudo-group parameters are normalized, one can verify, using (10.17), that the fully normalized versions of equations (10.8) and (10.10) still hold with $\widehat{\mathcal{D}}_x$ replaced by

$$\widehat{\mathbb{D}}_x = \sqrt{u_{yy}}\,\widehat{\mathbb{D}}_X,$$

so that

$$f_{k+1} = \widehat{\mathbb{D}}_x^k(\sqrt{u_{yy}}), \qquad e_{k+2} = \widehat{\mathbb{D}}_x e_{k+1,0} - u f_{k+2}, \qquad k \geq 1.$$

Now, the recurrence relation (10.3) for U_{YY} yields

$$\widehat{\mu}_X = \frac{1}{2}\widehat{\sigma}^{U_{YY}} \equiv \frac{\widehat{U}_{XYY}}{2}\widehat{\sigma}^X + \frac{\widehat{U}_{YYY}}{2}\widehat{\sigma}^Y.$$

On the other hand, substituting (10.16) into (10.11), (10.15) yields

$$\widehat{\mu}_X \equiv \left(\frac{u_{xyy} + u\, u_{yyy} + 2u_y\, u_{yy}}{2u_{yy}^{3/2}} \right) \widehat{\sigma}^X + \frac{u_{yyy}}{2u_{yy}^{3/2}} \widehat{\sigma}^Y,$$

where

$$\widehat{\sigma}^X = \sqrt{u_{yy}}\, dx, \qquad \widehat{\sigma}^Y = \sqrt{u_{yy}}(dy - u\, dx). \tag{10.18}$$

Hence, without having to compute the prolonged action, we determine the fundamental third order differential invariants

$$\widehat{U}_{XYY} = \frac{u_{xyy} + u\, u_{yyy} + 2u_y\, u_{yy}}{u_{yy}^{3/2}}, \qquad \widehat{U}_{YYY} = \frac{u_{yyy}}{u_{yy}^{3/2}}. \tag{10.19}$$

As in [57], it can be shown, using the higher order recurrence relations, that all higher order normalized differential invariants may be expressed in terms of (10.19) and their successive derivatives using the invariant differential operators

$$\widehat{\mathcal{D}}_X = \frac{1}{\sqrt{u_{yy}}}(D_x + u\, D_y), \qquad \widehat{\mathcal{D}}_Y = \frac{1}{\sqrt{u_{yy}}} D_y,$$

dual to the invariant horizontal coframe (10.18), completing our analysis of this example.

The recursive algorithm can also be applied to local Lie group actions, as we now illustrate in the final two examples.

Example 10.2. The linear fractional transformations

$$X = x, \qquad U = f(u) = \frac{\alpha\, u + \beta}{\gamma\, u + \delta}, \qquad \alpha\, \delta - \gamma\, \beta = 1, \tag{10.20}$$

defines an intransitive, effective action of the three-dimensional projective Lie group $G = \mathrm{PSL}(2, \mathbb{R})$ on $M = \mathbb{R}^2$. The determining equations are

$$X = x, \qquad U_x = 0, \qquad U_{uuu}\, U_u - \frac{3}{2} U_{uu}^2 = 0,$$

and their differential consequences. Linearizing, the infinitesimal determining equations for a vector field

$$\mathbf{v} = \xi(x, u) \frac{\partial}{\partial x} + \varphi(x, u) \frac{\partial}{\partial u}$$

to be an infinitesimal generator of this action are

$$\xi = 0, \qquad \varphi_x = \varphi_{uuu} = 0,$$

and their differential consequences. This has the immediate consequence that a basis of Maurer–Cartan forms is provided by

$$\nu = \boldsymbol{\lambda}(\varphi), \qquad \nu_U = \boldsymbol{\lambda}(\varphi_u), \qquad \nu_{UU} = \boldsymbol{\lambda}(\varphi_{uu}),$$

their number coinciding with the dimension of G. Alternatively,

$$\Upsilon = df - f_u\, du, \qquad \Upsilon_u = df_u - f_{uu}\, du, \qquad \Upsilon_{uu} = df_{uu} + \frac{3}{2} \frac{f_{uu}^2}{f_u} du,$$

provides a complete set of independent group forms. Applying the universal formula (7.3), the recurrence relations for the action (10.20) are, up to order 2,

$$dX = \sigma^X = dx = \omega, \qquad dU_X = \sigma^{U_X} + U_X\,\nu_U, \\ dU = \sigma^U + \nu, \qquad dU_{XX} = \sigma^{U_{XX}} + U_{XX}\,\nu_U + U_X^2\,\nu_{UU}. \tag{10.21}$$

Beginning the normalization procedure, let $\mathcal{K}^0 = \{u = 0\}$. Solving the normalization equation $U = 0$, we obtain

$$f = 0.$$

The partially normalized order 0 group form is

$$\widehat{\Upsilon} = -f_u du \equiv -f_u u_x\, dx = -f_u u_x\, \omega. \tag{10.22}$$

On the other hand, the order zero recurrence relation (10.21) for dU yields

$$\widehat{\Upsilon} = \widehat{\nu} = -\,\widehat{\sigma}^U \equiv -\,\widehat{U}_X\,\omega. \tag{10.23}$$

Combining (10.22) and (10.23), we find

$$\widehat{U}_X = f_u u_x.$$

At order 1, working under the assumption that $u_x \neq 0$, we choose the cross-section $\mathcal{K}^1 = \{u = 0,\ u_x = 1\}$ and solve the normalization equation $\widehat{U}_X = 1$ to obtain

$$f_u = \frac{1}{u_x}. \tag{10.24}$$

Substituting (10.24) in the identity $\widetilde{\boldsymbol{\tau}}^*(\widehat{\Upsilon}_u \mid \mathcal{O}_0)\ \equiv\ 0$ produces

$$0\ \equiv\ \widetilde{\boldsymbol{\tau}}^*(\widehat{\Upsilon}_u \mid \mathcal{O}_0) = d\left(\frac{1}{u_x}\right) - f_{uu}du = -\,\frac{du_x}{u_x^2} - f_{uu}du \equiv -\left(\frac{u_{xx}}{u_x^2} + f_{uu}u_x\right)dx,$$

and thus

$$f_{uu} = -\,\frac{u_{xx}}{u_x^3}. \tag{10.25}$$

Substituting the order 2 identity jet, $f = u,\ f_u = 1,\ f_{uu} = 0$, into (10.25) yields the cross-section

$$\mathcal{K}^2 = \{u = 0,\ u_x = 1,\ u_{xx} = 0\}.$$

The recurrence relations (10.21) imply

$$\widehat{\nu} = -\widehat{\sigma}^U, \qquad \widehat{\nu}_U = -\,\widehat{\sigma}^{U_X}, \qquad \widehat{\nu}_{UU} = -\,\widehat{\sigma}^{U_{XX}}, \tag{10.26}$$

where

$$\widehat{\sigma}^U = \frac{1}{u_x}du. \tag{10.27}$$

Since (10.20) is a one-to-one prolongation of the projective group of linear fractional transformations of the real line

$$U = \frac{\alpha\,u + \beta}{\gamma\,u + \delta}, \qquad \alpha\,\delta - \gamma\,\beta = 1, \tag{10.28}$$

with infinitesimal determining equation

$$\varphi_{uuu} = 0,$$

Theorem 8.12 applies, and hence

$$\widehat{\nu}_k = \widehat{\mathbb{D}}_U^k \widehat{\nu} = -\widehat{\mathbb{D}}_U^k \widehat{\sigma}^U, \qquad k = 0, 1, 2.$$

Also, since (10.28) is a translational group, $\widehat{\nu}_U$ and $\widehat{\nu}_{UU}$ are linear combinations of $\widehat{\Upsilon}_u$ and $\widehat{\Upsilon}_{uu}$. Recalling (10.26), since

$$\widehat{\mathbb{D}}_U \lrcorner \, \widehat{\nu}_U = \widehat{\mathbb{D}}_U \lrcorner \, \widehat{\nu}_{UU} = 0, \qquad \text{it follows that} \qquad \widehat{\mathbb{D}}_U \lrcorner \, \widehat{\Upsilon}_u = \widehat{\mathbb{D}}_U \lrcorner \, \widehat{\Upsilon}_{uu} = 0,$$

where

$$\widehat{\Upsilon}_u = \frac{u_{xx}}{u_x^3}\, du - \frac{1}{u_x^2}\, du_x, \qquad \widehat{\Upsilon}_{uu} = -\frac{3u_{xx}^2}{2u_x^5}\, du + \frac{3u_{xx}}{u_x^4}\, du_x - \frac{1}{u_x^3}\, du_{xx}.$$

Therefore,

$$\widehat{\mathbb{D}}_U = u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + \frac{3}{2} \frac{u_{xx}^2}{u_x} \frac{\partial}{\partial u_{xx}} + \cdots. \tag{10.29}$$

Lie differentiating (10.27) with respect to (10.29) gives

$$\widehat{\nu}_U = -\frac{du_x}{u_x} + \frac{u_{xx}}{u_x^2}\, du, \qquad \widehat{\nu}_{UU} = -\frac{du_{xx}}{u_x} + \frac{2u_{xx}}{u_x^2}\, du_x - \frac{u_{xx}^2}{2u_x^3}\, du.$$

It follows that, modulo contact forms,

$$\left(\frac{u_{xxx}}{u_x} - \frac{3u_{xx}^2}{2u_x^2} \right) \omega \equiv -\widehat{\nu}_{UU} = \widehat{\sigma}^{U_{XX}} \equiv \widehat{U}_{XXX}\, \omega.$$

We conclude that the basic differential invariant is the well-known Schwarzian, [48],

$$\widehat{U}_{XXX} = \frac{u_{xxx}}{u_x} - \frac{3u_{xx}^2}{2u_x^2}.$$

All higher order normalized differential invariants can be expressed in terms of the Schwarzian invariant and its invariant derivative with respect to D_x. Again, observe that we never needed to compute any of the more complicated formulas for the prolonged action of the projective group to obtain the fundamental differential invariant.

Example 10.3. Finally, we present an example where the non-zero correction terms in (8.28) play an essential role in the recursive construction. Consider the transitive four-parameter Lie group action

$$X = x + k, \qquad U = a\, u + b\, e^x + c\, e^{-x}, \tag{10.30}$$

where the group G is parametrized by $a, b, c, k \in \mathbb{R}$, with $a \neq 0$. The two-dimensional abelian subgroup $X = x + k$, $U = a\, u$, acts transitively and freely on the upper and lower half planes $u > 0$ and $u < 0$, and so G is a translational group on these subdomains, with translational coordinates $x, v = \log |u|$. However, it is not translational on all of $\mathbb{R}^2$. Nevertheless, we can still apply our recursive algorithm there. *Note*: it would not be difficult to construct and analyze similar but more complicated actions that are non-translational even on subdomains.

The infinitesimal generator of the group action is

$$\mathbf{v} = \xi \frac{\partial}{\partial x} + \varphi \frac{\partial}{\partial u} = c_0 \frac{\partial}{\partial x} + (c_1 e^x + c_2 e^{-x} + c_3 u) \frac{\partial}{\partial u}, \tag{10.31}$$

where c_0, c_1, c_2, c_3 are arbitrary constants. The corresponding infinitesimal determining equations are

$$\xi_x = \xi_u = \varphi_{xu} = \varphi_{uu} = 0, \qquad \varphi_{xx} = \varphi - u\,\varphi_u. \tag{10.32}$$

As in Corollary 2.9, the fact that the action is non-translational is indicated by the occurrence of the order zero vector field jet φ in the infinitesimal determining equations. Nor is it the one-to-one prolongation of a one-dimensional translational action, since the action on u is not uniquely determined by the action on x and vice versa.

Let

$$\mu_{X^k U^\ell} = \boldsymbol{\lambda}(\xi_{x^k u^\ell}), \qquad \nu_{X^k U^\ell} = \boldsymbol{\lambda}(\varphi_{x^k u^\ell}), \qquad k, \ell \geq 0,$$

be the corresponding restricted diffeomorphism Maurer–Cartan forms. Then the lift of (10.32) gives the linear relations

$$\mu_X = \mu_U = \nu_{XU} = \nu_{UU} = 0, \qquad \nu_{XX} = \nu - U\nu_U.$$

It follows that a basis of Maurer–Cartan forms is given by μ, ν, ν_U, ν_X. Their associated structure equations are

$$\begin{gathered} d\mu = 0, \qquad d\nu = \sigma^X \wedge \nu_X + \sigma^U \wedge \nu_U, \qquad d\nu_U = 0, \\ d\nu_X = \sigma^X \wedge \nu_{XX} + \nu_U \wedge \nu_X = \sigma^X \wedge (\nu - U\,\nu_U) + \nu_U \wedge \nu_X, \end{gathered} \tag{10.33}$$

where

$$\sigma^X = \boldsymbol{\lambda}(dx) = dx = \omega, \qquad \sigma^U = \boldsymbol{\lambda}(du) = (b\,e^x - c\,e^{-x})\,dx + a\,du. \tag{10.34}$$

Alternatively, a basis of group forms is given by

$$\begin{aligned} \Upsilon &= dX - X_x dx - X_u du = dk, & \alpha &= dU - U_x dx - U_u du = u da + e^x db + e^{-x} dc, \\ \alpha_u &= dU_u - U_{xu} dx - U_{uu} du = da & \alpha_x &= dU_x - U_{xx} dx - U_{xu} du = e^x db - e^{-x} dc. \end{aligned}$$

Prolonging the infinitesimal generator (10.31), we construct the recurrence formulas, which, up to order 2, are

$$\begin{aligned} dX &= \sigma^X + \mu = \omega + \mu, \\ dU &= \sigma^U + \nu \equiv U_X\,\omega + \nu, \\ dU_X &= \sigma^{U_X} + \nu_X + U_X\,\nu_U \equiv U_{XX}\,\omega + \nu_X + U_X \nu_U, \\ dU_{XX} &= \sigma^{U_{XX}} + \nu_{XX} + U_{XX}\nu_U = \sigma^{U_{XX}} + \nu + (U_{XX} - U)\nu_U \\ &\qquad \equiv U_{XXX}\,\omega + \nu + (U_{XX} - U)\nu_U. \end{aligned} \tag{10.35}$$

We begin the recursive algorithm by performing the order 0 normalizations. Setting[10]

$$X = U = 0, \qquad \text{yields} \qquad k = -x, \qquad c = -a\,e^x\,u - b\,e^{2x}.$$

[10]Note that this normalization does not work in the aforementioned translational coordinates.

The resulting partially normalized order zero jet forms (10.34) are

$$\widehat{\sigma}^X = \omega = dx, \quad \widehat{\sigma}^U = (a\,u + 2\,b\,e^x)dx + a\,du \equiv (a\,u + a\,u_x + 2\,b\,e^x)dx = \widehat{U}_X\,\omega, \tag{10.36}$$

from which we deduce the partially normalized order 1 prolonged action

$$\widehat{U}_X = a\,u + a\,u_x + 2\,b\,e^x.$$

On the other hand, the partially normalized group forms are

$$\widehat{\Upsilon} = -\widehat{\sigma}^X, \qquad \widehat{\alpha} = -\widehat{\sigma}^U, \qquad \widehat{\alpha}_u = da, \qquad \widehat{\alpha}_x = 2e^x db + u da + \widehat{\sigma}^U,$$

so that

$$\widehat{\mathbb{D}}_X = \frac{\partial}{\partial x} - \left(\frac{a\,u + 2\,b\,e^x}{a}\right)\frac{\partial}{\partial u} + \cdots, \qquad \widehat{\mathbb{D}}_U = \frac{1}{a}\frac{\partial}{\partial u} - \frac{e^{-x}}{2}\frac{\partial}{\partial b}. \tag{10.37}$$

From the structure equations (10.33), we have

$$d\widehat{\nu} = \widehat{\sigma}^X \wedge \widehat{\nu}_X + \widehat{\sigma}^U \wedge \widehat{\nu}_U.$$

Since the partially normalized Maurer–Cartan forms $\widehat{\nu}_X$, $\widehat{\nu}_U$ are linear combinations of the partially normalized group forms $\widehat{\alpha}_u$ and $\widehat{\alpha}_x$, combined with the fact that their inner products with $\widehat{\mathbb{D}}_X$ and $\widehat{\mathbb{D}}_U$ are zero, Cartan's formula for Lie differentiation yields

$$\widehat{\mathbb{D}}_U\widehat{\nu} = \widehat{\mathbb{D}}_U \lrcorner\, d\widehat{\nu} + d(\widehat{\mathbb{D}}_U \lrcorner\, \widehat{\nu}) = \widehat{\nu}_U, \qquad \widehat{\mathbb{D}}_X\widehat{\nu} = \widehat{\mathbb{D}}_X \lrcorner\, d\widehat{\nu} + d(\widehat{\mathbb{D}}_X \lrcorner\, \widehat{\nu}) = \widehat{\nu}_X.$$

Therefore,

$$\widehat{\nu}_U = \widehat{\mathbb{D}}_U\widehat{\nu} = \frac{1}{a}\,da, \qquad \widehat{\nu}_X = \widehat{\mathbb{D}}_X\widehat{\nu} = 2e^x db - \frac{be^x}{a}\,da + \widehat{\sigma}^U. \tag{10.38}$$

Normalizing

$$\widehat{U}_X = 0 \qquad \text{yields} \qquad b = -\tfrac{1}{2}\,a\,e^{-x}(u_x + u).$$

The order 0 partially normalized jet forms (10.36) then become

$$\widehat{\sigma}^X = dx, \qquad \widehat{\sigma}^U = -\,a\,u_x\,dx + a\,du = a\,\theta,$$

and the only unnormalized group form is $\widehat{\alpha}_u = da$. From the recurrence relations (10.35),

$$\widehat{\mu} = -\,\widehat{\sigma}^X, \qquad \widehat{\nu} = -\,\widehat{\sigma}^U,$$

and the formula

$$-\,\widehat{U}_{XX}\omega \equiv -\,\widehat{\sigma}^{U_X} = \widehat{\nu}_X = a\,u\,dx - a\,du_x \equiv a(u - u_{xx})\,dx = a(u - u_{xx})\,\omega,$$

we deduce that

$$\widehat{U}_{XX} = a(u_{xx} - u).$$

When $u_{xx} \neq u$, we can normalize

$$\widehat{U}_{XX} = 1 \qquad \text{so that} \qquad a = \frac{1}{u_{xx} - u}.$$

From the second order recurrence relation (10.35) we find

$$\widehat{\sigma}^{U_{XX}} = -\widehat{\nu}_U + \widehat{\sigma}^U = \frac{du_{xx} - du}{u_{xx} - u} + \frac{du - u_x dx}{(u_{xx} - u)^2} \equiv \frac{u_{xxx} - u_x}{u_{xx} - u}\,\omega, \tag{10.39}$$

and conclude that the lowest order differential invariant is

$$\widehat{U}_{XXX} = \frac{u_{xxx} - u_x}{u_{xx} - u}. \tag{10.40}$$

Higher order invariants are obtained by repeatedly differentiating (10.40) with respect to the invariant differential operator D_x.

Once all the group parameters have been normalized, the operators (10.37) reduce to

$$\widehat{\mathbb{D}}_X = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u \frac{\partial}{\partial u_x} + u_x \frac{\partial}{\partial u_{xx}} + \cdots, \qquad \widehat{\mathbb{D}}_U = (u_{xx} - u)\frac{\partial}{\partial u}.$$

Also, we note that the first equality in (10.38) no longer holds. From the structure equations (10.33), we have

$$d\widehat{\nu} = \widehat{\sigma}^X \wedge \widehat{\nu}_X + \widehat{\sigma}^U \wedge \widehat{\nu}_U = \widehat{\sigma}^{U_X} \wedge \widehat{\sigma}^X + \widehat{\sigma}^{U_{XX}} \wedge \widehat{\sigma}^U.$$

Therefore, the Cartan formula

$$\widehat{\mathbb{D}}_U \widehat{\nu} = \widehat{\mathbb{D}}_U \lrcorner\, d\widehat{\nu} + d(\widehat{\mathbb{D}}_U \lrcorner\, \widehat{\nu}) = -\widehat{\sigma}^{U_{XX}}$$

and (10.39) imply that

$$\widehat{\nu}_U = -\widehat{\sigma}^{U_{XX}} + \widehat{\sigma}^U = \widehat{\mathbb{D}}_U \widehat{\nu} + \widehat{\sigma}^U.$$

The latter identity illustrates the occurrence of the correction term $L^u_{,U} = \widehat{\sigma}^U$ in the general formula (8.28), which stems from the appearance of the order zero Maurer–Cartan form ν in the recurrence relation for U_{XX} in (10.35).

The group action (10.30) also shows that the hypothesis of Proposition 8.20 is too restrictive since its conclusion remains valid despite the appearance of order zero vector field jet φ in the infinitesimal determining equations (10.32). On the other hand, one can verify the validity of Proposition 8.19, which has weaker hypotheses.

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