The primary purpose of this note is to prove two recent conjectures concerning the \( n \) body matrix that arises in recent papers of Escobar–Ruiz, Miller, and Turbiner on the classical and quantum \( n \) body problem in \( d \)-dimensional space. First, whenever the masses are in a nonsingular configuration, meaning that they do not lie on an affine subspace of dimension \( \leq n - 2 \), the \( n \) body matrix is positive definite, and hence defines a Riemannian metric on the space coordinatized by their interpoint distances. Second, its determinant can be factored into the product of the order \( n \) Cayley–Menger determinant and a mass-dependent factor that is also of one sign on all nonsingular mass configurations. The factorization of the \( n \) body determinant is shown to be a special case of an intriguing general result proving the factorization of determinants of a certain form.

1. The \( n \) Body Matrix.

The \textit{n body problem}, meaning the motion of \( n \) point masses (or point charges) in \( d \)-dimensional space under the influence of a potential that depends solely on pairwise
distances, has a venerable history, capturing the attention of many prominent mathematicians, including Euler, Lagrange, Dirichlet, Poincaré, Sundman, etc., [12, 15]. The corresponding quantum mechanical system, obtained by quantizing the classical Hamiltonian to form a Schrödinger operator, has been of pre-eminent interest since the dawn of quantum mechanics, [7].

In three recent papers, [9, 13, 14], Escobar–Ruiz, Miller, and Turbiner made the following remarkable observation. Once the center of mass coordinates have been separated out, the quantum $n$ body Schrödinger operator separates into a “radial” component that depends only upon the distances between the masses plus an “angular” component that involves the remaining coordinates and annihilates all functions of the interpoint distances. Moreover, the radial component is gauge equivalent to the Laplace–Beltrami operator on a certain curved manifold, whose geometry is as yet not well understood. This decomposition allows one to separate out the “radial” eigenstates that depend only upon the interpoint distances from the more general eigenstates that also involve the angular coordinates. A similar separation arises in the classical $n$ body problem through the process of “dequantization”, i.e., reversion to the classical limit.

The primary goal of this paper is to prove two fundamental conjectures that were made in [9] concerning the algebraic structure of the underlying $n$ body radial metric tensor. To be precise, suppose the point masses $m_1, \ldots, m_n$ occupy positions $p_i = (p_1^i, \ldots, p_d^i)^T \in \mathbb{R}^d, \ i = 1, \ldots, n$.

**Definition 1.** The mass positions $p_1, \ldots, p_n$ will be called *singular* if they lie on a common affine subspace of dimension $\leq n - 2$.

Thus, three points are singular if they are collinear; four points are singular if they are coplanar; etc. Note that non-singularity requires that the underlying space be of sufficiently large dimension, namely $d \geq n - 1$.

Using the usual dot product and Euclidean norm, let

$$r_{ij} = r_{ji} = \| p_i - p_j \| = \sqrt{(p_i - p_j) \cdot (p_i - p_j)}, \quad i \neq j,$$

(1)

denote the interpoint distances. The subsequent formulas will slightly simplify if we express them in terms of the inverse masses

$$\alpha_i = \frac{1}{m_i}, \quad i = 1, \ldots, n.$$

(2)

The $n$ body matrix $B = B^{(n)}$ defined in [9] is the $\frac{1}{2} n(n - 1) \times \frac{1}{2} n(n - 1)$ matrix whose rows and columns are indexed by unordered pairs $\{i, j\} = \{j, i\}$ of distinct integers $1 \leq i < j \leq n$. Its diagonal entries are

$$b_{\{i\},\{i\}} = 2(\alpha_i + \alpha_j) r_{ij}^2 = 2(\alpha_i + \alpha_j) (p_i - p_j) \cdot (p_i - p_j),$$

(3)

\[1\]We work with column vectors in $\mathbb{R}^d$ throughout.
while its off diagonal entries are
\[ b_{ij,kl} = \alpha_i (r_{ij}^2 + r_{ik}^2 - r_{jk}^2) = 2 \alpha_i (p_i - p_j) \cdot (p_i - p_k), \quad i, j, k \text{ distinct}, \]
\[ b_{ij,kl} = 0, \quad i, j, k, l \text{ distinct}. \]

For example, the 3 body matrix is
\[
B^{(3)} = \begin{pmatrix}
2(\alpha_1 + \alpha_2) r_{12}^2 & \alpha_1 (r_{12}^2 + r_{13}^2 - r_{23}^2) & \alpha_2 (r_{12}^2 + r_{23}^2 - r_{13}^2) \\
\alpha_1 (r_{12}^2 + r_{13}^2 - r_{23}^2) & 2(\alpha_1 + \alpha_3) r_{13}^2 & \alpha_3 (r_{13}^2 + r_{23}^2 - r_{12}^2) \\
\alpha_2 (r_{12}^2 + r_{23}^2 - r_{13}^2) & \alpha_3 (r_{13}^2 + r_{23}^2 - r_{12}^2) & 2(\alpha_2 + \alpha_3) r_{23}^2
\end{pmatrix},
\]
where the rows and the columns are ordered as follows: \{1, 2\}, \{1, 3\}, \{2, 3\}. Our first main result concerns its positive definiteness.

**Theorem 2.** The \( n \) body matrix \( B^{(n)} \) is positive semi-definite, and is positive definite if and only if the \( n \) masses \( p_1, \ldots, p_n \) are in a non-singular position.

Thus, away from the subvariety corresponding to singular configurations, the \( n \) body matrix defines a Riemannian metric on the space coordinatized by the interpoint distances \( r_{ij} \). This implies that the identification of the radial component of the quantum \( n \) body Schrödinger operator with an elliptic Laplace–Beltrami operator, [9], is justified on the entire nonsingular component of this space.

**Remark:** Since the masses lie at distinct locations, the interpoint distances (1) are all positive, \( r_{ij} > 0 \), and are further constrained by the triangle inequalities. Thus the space they coordinatize is strictly contained in the positive orthant of \( \mathbb{R}^{n(n-1)/2} \).

The determinant of the \( n \) body matrix
\[
\Delta^{(n)} = \det B^{(n)}
\]
will be called the \( n \) body determinant. For example, a short computation based on (5) shows that the 3 body determinant can be written in the following factored form:
\[
\Delta^{(3)} = \det B^{(3)} = -2(\alpha_1 \alpha_2 + \alpha_1 \alpha_3 + \alpha_2 \alpha_3) (\alpha_3 r_{12}^2 + \alpha_2 r_{13}^2 + \alpha_1 r_{23}^2) \\
(r_{12}^4 + r_{13}^4 + r_{23}^4 - 2r_{12}^2 r_{13}^2 - 2r_{12}^2 r_{23}^2 - 2r_{13}^2 r_{23}^2). \tag{7}
\]

Two important things to notice: ignoring the initial numerical factor, the first factor is the elementary symmetric polynomial of degree \( n - 1 = 2 \) in the mass parameters \( \alpha_i = 1/m_i \) only; further, the final polynomial factor is purely geometric, meaning that it is independent of the mass parameters, and so only depends on the configuration of their locations through their interpoint distances. Positive definiteness of \( B^{(3)} \) implies \( \Delta^{(3)} > 0 \) for nonsingular (i.e., non-collinear) configurations of the masses. In view of the sign of the initial numerical factor, this clearly implies the final geometrical factor.
is strictly negative on such configurations, a fact that is not immediately evident and in fact requires that the $r_{ij}$’s be interpoint distances; indeed, this factor is obviously positive for some non-geometrical values of the $r_{ij}$’s. Similar factorizations were found in [9] for the cases $n = 2, 3, 4$, and, in the case of equal masses, $n = 5, 6$, via symbolic calculations using both Mathematica and Maple.

The geometrical factor in each of these computed factorizations is, in fact, well known, and equal to the Cayley–Menger determinant of order $n$, a quantity that arises in the very first paper of Arthur Cayley, [2], written before he turned 20 and, apparently, was inspired by reading Lagrange and Laplace! In this paper, Cayley uses the relatively new theorem that the determinant (a quantity he calls “tolerably known”) of the product of two matrices is the product of their determinants in order to solve the problem of finding the algebraic condition (or syzygy) relating the interpoint distances among singular configurations of 5 points in three-dimensional space, as well as 4 points in a plane and 3 points on a line, each of which is expressed by the vanishing of their respective Cayley–Menger determinant.

There are two ways of constructing the Cayley–Menger determinants. What we will call the larger order $n$ Cayley–Menger matrix, due to Cayley, is the symmetric matrix

$$C^{(n)} = \begin{bmatrix} 0 & r_{12}^2 & r_{13}^2 & \ldots & r_{1n}^2 & 1 \\ r_{12}^2 & 0 & r_{23}^2 & \ldots & r_{2n}^2 & 1 \\ r_{13}^2 & r_{23}^2 & 0 & \ldots & r_{3n}^2 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ r_{1n}^2 & r_{2n}^2 & r_{3n}^2 & \ldots & 0 & 1 \\ 1 & 1 & 1 & \ldots & 1 & 0 \end{bmatrix} \quad (8)$$

of size $(n + 1) \times (n + 1)$ involving the same interpoint distances [1]. The order $n$ Cayley–Menger determinant is defined, [3], as its determinant:

$$\delta^{(n)} = \det C^{(n)}. \quad (9)$$

For example, when $n = 3$,

$$C^{(3)} = \begin{bmatrix} 0 & r_{12}^2 & r_{13}^2 & 1 \\ r_{12}^2 & 0 & r_{23}^2 & 1 \\ r_{13}^2 & r_{23}^2 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}, \quad (10)$$

$$\delta^{(3)} = \det C^{(3)} = r_{12}^4 + r_{13}^4 + r_{23}^4 - 2r_{12}^2r_{13}^2 - 2r_{12}^2r_{23}^2 - 2r_{13}^2r_{23}^2,$$

which coincides with the geometric polynomial factor in [7]. Keep in mind that both the $n$ body and Cayley–Menger determinants are homogeneous polynomials in the squared distances $r_{ij}^2$. The general form of Cayley’s result can be stated as follows.
Theorem 3. A set of (squared) interpoint distances \( r = (\ldots, r^2_{ij}, \ldots) \) for \( 1 \leq i < j \leq n \) comes from a singular point configuration if and only if the corresponding Cayley–Menger determinant vanishes: \( \delta^{(n)}(r) = 0 \).

In other words, the singular subvariety in the interpoint distance space is determined by the vanishing of a single polynomial — the Cayley–Menger determinant. Thus, Theorem 2 implies that the \( n \) body determinant, and hence the underlying metric, degenerates if and only if the Cayley–Menger determinant vanishes, and hence the masses are positioned on a lower dimensional affine subspace. See below for a modern version of Cayley’s original proof. A century later, in the hands of Karl Menger, this determinantal quantity laid the foundation of the active contemporary field of distance geometry, [1, 8]; see also [10] for further results and extensions to other geometries.

Remark: When \( n = 3 \), the Cayley–Menger determinant [10] factorizes:

\[
\delta^{(3)}(r) = (r_{12} + r_{13} + r_{23})(-r_{12} + r_{13} + r_{23})(r_{12} - r_{13} + r_{23})(r_{12} + r_{13} - r_{23}),
\]

which is Heron’s formula for the squared area of a triangle, [10]. On the other hand, when \( n \geq 4 \), the Cayley–Menger determinant is an irreducible polynomial in the distance variables \( r_{ij} \); see [3], keeping in mind that their \( n \) is our \( n - 1 \).

Based on their above-mentioned symbolic calculations, Miller, Turbiner, and Escobar–Ruiz, [9], conjectured the following result.

Theorem 4. The \( n \) body determinant factors,

\[
\Delta^{(n)} = e_{n-1}(\alpha) \delta^{(n)}(r) \sigma^{(n)}(\alpha, r)
\]

into the product of the elementary symmetric polynomial \( e_{n-1} \) of order \( n - 1 \) in the mass parameters \( \alpha = (\ldots, \alpha_i, \ldots) \) times the Cayley–Menger determinant \( \delta^{(n)} \) of order \( n \) depending on the squared interpoint distances \( r = (\ldots, r^2_{ij}, \ldots) \) times a polynomial \( \sigma^{(n)} \) that depends upon both the \( \alpha_i \) and the \( r^2_{ij} \).

Unfortunately, our proof of Theorem 4 is purely existential; it does not yield an independent formula for the non-geometrical factor, other than the obvious \( \sigma^{(n)} = \Delta^{(n)}/(e_{n-1} \delta^{(n)}) \). Thus, the problem of characterizing and understanding the non-geometric factor \( \sigma^{(n)} \) remains open, although interesting formulas involving geometric quantities — volumes of subsimplices determined by the point configuration — are known when \( n \) is small, [9]. Nor does the proof give any insight into the geometry of the Riemannian manifold whose metric tensor is prescribed by the \( n \) body matrix \( B^{(n)} \).

We shall, in fact, prove Theorem 4 as a special case of a much more general determinantal factorization Theorem 9, which replaces the squared distances \( r^2_{ij} \) by \( n^2 \) arbitrary elements \( s_{ij} \), not necessarily satisfying \( s_{ij} = s_{ji} \) and \( s_{ii} = 0 \). We shall also generalize the dependence on the inverse mass parameters \( \alpha_i \) using the following elementary observation.
Lemma 5. Given the parameters $\alpha_1, \ldots, \alpha_n$, consider the following $(n+1) \times (n+1)$ matrix

$$C_A = \begin{pmatrix}
\alpha_1 & 0 & \cdots & 0 & 1 \\
0 & \alpha_2 & \cdots & 0 & 1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & \alpha_n & 1 \\
1 & 1 & \cdots & 1 & 0
\end{pmatrix}. \quad (13)$$

Then,

$$\det C_A = -e_{n-1}(\alpha). \quad (14)$$

To establish this formula, one can simply expand the determinant along its last row. Thus, the two initial factors in the $n$ body determinant factorization formula (12) are both realized by determinants of $(n+1) \times (n+1)$-matrices whose final row and column are of a very particular form. In our further generalization of the $n$ body determinant factorization formula (12), we will replace the diagonal $n \times n$ block in (8) by a general matrix depending on $n^2$ arbitrary elements $s_{i,j}$ and the diagonal $n \times n$ block in (13) by a general matrix depending on an additional $n^2$ arbitrary elements $t_{k,l}$. See below for details.

Combining Theorems 2 and 4 allows us to resolve another conjecture in [9], that for nonsingular point configurations, the mass-dependent factor $\sigma(n)$ is of one sign.

Theorem 6. All factors in the $n$ body determinant factorization (12) are of one sign, namely

$$\Delta(n) > 0, \quad e_{n-1} > 0, \quad (-1)^n \delta(n) > 0, \quad (-1)^n \sigma(n) > 0, \quad (15)$$

provided the mass parameters $\alpha_i = 1/m_i$ are positive and their positions $p_i$ do not all lie in an affine subspace of dimension $\leq n - 2$.

Proof: Since the determinant of a positive definite matrix is positive, [11], Theorem 2 immediately implies the first inequality in (15). The positivity of the elementary symmetric polynomial for $\alpha_i > 0$ is trivial. The sign of the Cayley–Menger determinant $\delta(n)$ on nonsingular configurations is well known; see (24) below for a proof. The final inequality follows immediately from the factorization (12). $Q.E.D.$

2. Positive Definiteness.

In this section, we present a proof of Theorem 2 as well as the well known results concerning the vanishing and the sign of the Cayley–Menger determinants. These results will, modulo the proof of the Factorization Theorem 4, establish the Sign Theorem 6.
Let us first introduce, for each \( k = 1, \ldots, n \), the smaller Cayley–Menger matrix \( M_k^{(n)} \) of order \( n \) based at the point \( \mathbf{p}_k \). It is defined as the \( (n - 1) \times (n - 1) \) matrix with entries

\[
m_{ij} = 2(\mathbf{p}_i - \mathbf{p}_k) \cdot (\mathbf{p}_j - \mathbf{p}_k) = \|\mathbf{p}_i - \mathbf{p}_k\|^2 + \|\mathbf{p}_j - \mathbf{p}_k\|^2 - \|\mathbf{p}_i - \mathbf{p}_j\|^2
\]

\[
= r_{ik}^2 + r_{jk}^2 - r_{ij}^2, \quad i, j \neq k,
\]

where the indices \( i, j \) run from 1 to \( n \) omitting \( k \) (and where \( r_{ii} = 0 \)). Note that its diagonal entries are \( m_{ii} = 2r_{ik}^2 \). Thus, in particular, \( M_n^{(n)} \) is explicitly given by

\[
M_n^{(n)} = \begin{pmatrix}
2r_{1n}^2 & r_{1n}^2 + r_{2n}^2 - r_{12}^2 & r_{1n}^2 + r_{3n}^2 - r_{13}^2 & \cdots & r_{1n}^2 + r_{n-1,n}^2 - r_{1,n-1}^2 \\
2r_{2n}^2 & r_{2n}^2 + r_{3n}^2 - r_{23}^2 & r_{2n}^2 + r_{4n}^2 - r_{24}^2 & \cdots & r_{2n}^2 + r_{n-1,n}^2 - r_{2,n-1}^2 \\
r_{1n}^2 + r_{3n}^2 - r_{13}^2 & r_{2n}^2 + r_{3n}^2 - r_{23}^2 & r_{3n}^2 + r_{4n}^2 - r_{34}^2 & \cdots & r_{3n}^2 + r_{n-1,n}^2 - r_{3,n-1}^2 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
r_{1n}^2 + r_{n-1,n}^2 - r_{1,n-1}^2 & r_{2n}^2 + r_{n-1,n}^2 - r_{2,n-1}^2 & r_{3n}^2 + r_{n-1,n}^2 - r_{3,n-1}^2 & \cdots & 2r_{n-1,n}^2
\end{pmatrix},
\]

with evident modifications for the general case \( M_k^{(n)} \). For example, when \( n = 3 \),

\[
M_1^{(3)} = \begin{pmatrix}
2r_{12}^2 & r_{12}^2 + r_{13}^2 - r_{23}^2 \\
2r_{12}^2 & r_{12}^2 + r_{23}^2 - r_{13}^2
\end{pmatrix},
\]

\[
M_2^{(3)} = \begin{pmatrix}
2r_{12}^2 & r_{12}^2 + r_{23}^2 - r_{13}^2 \\
2r_{13}^2 & r_{13}^2 + r_{23}^2 - r_{12}^2
\end{pmatrix},
\]

\[
M_3^{(3)} = \begin{pmatrix}
2r_{13}^2 & r_{13}^2 + r_{23}^2 - r_{12}^2 \\
2r_{23}^2 & r_{23}^2 + r_{12}^2 - r_{13}^2
\end{pmatrix}
\]

Proposition 7. The Cayley–Menger determinant is also given by

\[
\delta^{(n)} = (-1)^n \det M_k^{(n)}
\]

for any value of \( k = 1, \ldots, n \).

Proof: Let us concentrate on the case \( k = n \), noting that all formulas are invariant under permutations of the mass positions, and hence it suffices to establish this particular case. We perform the following elementary row and column operations on the larger Cayley–Menger matrix \( C^{(n)} \), cf. (8), that do not affect its determinant. We subtract its \( n \)-th row from the first through \( (n - 1) \)-st rows, and then subtract its \( n \)-th column, which has not changed, from the resulting first through \( (n - 1) \)-st columns. The result is the \((n + 1) \times (n + 1)\)-matrix

\[
\tilde{C}^{(n)} = \begin{pmatrix}
-M_n^{(n)} & * & 0 \\
* & 0 & 1 \\
0 & 1 & 0
\end{pmatrix}
\]
where the upper left \((n - 1) \times (n - 1)\) block is \(-M_n^{(n)}\), the \(n\)-th row and column of \(\tilde{C}^{(n)}\) are the same as the \(n\)-th row and column of \(C^{(n)}\) (the stars indicate the entries), and the last row and column have all zeros except for their \(n\)-th entry. We can further subtract suitable multiples of the last row and column from the first \(n - 1\) rows and columns in order to annihilate their \(n\)-th entries, leading to

\[
\tilde{C}^{(n)} = \begin{pmatrix}
-M_n^{(n)} & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix}.
\]

It is then easy to see that

\[
\delta^{(n)} = \det C^{(n)} = \det \tilde{C}^{(n)} = \det \hat{C}^{(n)} = (-1)^n \det M_n^{(n)}.
\]

Q.E.D.

Now, dropping the \(^{(n)}\) superscript and \(n\) subscript from here on to avoid cluttering the formulas, the first formula in (16) implies that, up to a factor of 2, the smaller Cayley–Menger matrix \(M = M_n^{(n)}\) is a Gram matrix, cf. [11], namely

\[
M = 2A^TA,
\]

where

\[
A = \begin{pmatrix}
p_1 - p_n, & \ldots, & p_{n-1} - p_n
\end{pmatrix}
\]

(20)
is the \(d \times n\) matrix with the indicated columns. We know that \(\delta^{(n)} = (-1)^n \det M = 0\) if and only if \(\ker M \neq \{0\}\), meaning there exists \(\hat{x} \neq 0\) such that

\[
M \hat{x} = 0.
\]

(21)

Multiplying the left hand side by \(\hat{x}^T\) and using (20), we find

\[
\hat{x}^T M \hat{x} = 2 \hat{x}^T A^T A \hat{x} = 2 \|A \hat{x}\|^2 \geq 0
\]

for all \(\hat{x} \in \mathbb{R}^{n-1}\).

(22)

This identity establishes the known result that the smaller Cayley-Menger matrix \(M\) is positive semi-definite, and is positive definite if and only if \(\ker A = \{0\}\). Consequently, (21) holds if and only if

\[
A \hat{x} = 0.
\]

(23)

Since \(\hat{x} \neq 0\), this is equivalent to the linear dependence of the columns of \(A\), meaning the vectors \(p_1 - p_n, \ldots, p_{n-1} - p_n\) span a subspace of dimension \(\leq n - 2\), which requires that \(p_1, \ldots, p_n\) lie in an affine subspace of dimension \(\leq n - 2\), i.e., they form a singular point configuration. We conclude that this occurs if and only if the Cayley–Menger determinant vanishes, \(\delta = 0\), which thus establishes Cayley’s Theorem 3.

Moreover, positive (semi-)definiteness of \(M\) implies non-negativity of its determinant, hence, by (19),

\[
(-1)^n \delta^{(n)} \geq 0,
\]

with equality if and only if \(\ker A \neq \{0\}\),

(24)

thus establishing the last inequality in (15). Replacing \(p_n\) by \(p_k\) does not change the argument, and hence we have established the following known result.
**Theorem 8.** The smaller Cayley–Menger matrices $M^{(n)}_k$ are positive semi-definite, and are positive definite if and only if the $n$ masses are in a nonsingular configuration.

Let us next prove Theorem 2 establishing the positive definiteness of the $n$ body matrix for nonsingular point configurations. Observe that if we let the mass parameter $m_k = 1$ and send all other $m_j \to \infty$, or, equivalently, $\alpha_k = 1$ and $\alpha_j = 0$ for $j \neq k$, then the $n$ body matrix $B = B^{(n)}$ reduces to the matrix $B^{(n)}_k = \hat{M}^{(n)}_k$ obtained by placing the $(i,j)$-th entry of the smaller Cayley–Menger matrix $M^{(n)}_k$ based at the point $p_k$ in the position labelled by the unordered index pairs \{i,k\} and \{j,k\}, and setting all other entries, i.e., those with one or both labels not containing $k$, to zero. Let us call the resulting matrix the $k$-th expanded Cayley–Menger matrix. We have thus shown that the $n$ body matrix decomposes into a linear combination thereof:

$$B^{(n)} = \sum_{k=1}^n \alpha_k \hat{M}^{(n)}_k.$$  \hspace{1cm} (25)

For example when $n = 3$, we write (5) as

$$B^{(3)} = \alpha_1 \hat{M}^{(3)}_1 + \alpha_2 \hat{M}^{(3)}_2 + \alpha_3 \hat{M}^{(3)}_3 = \alpha_1 \begin{pmatrix} 2r_{12}^2 & r_{12}^2 + r_{13}^2 - r_{23}^2 & 0 \\ r_{12}^2 + r_{13}^2 - r_{23}^2 & 2r_{13}^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 2r_{12}^2 & 0 & r_{12}^2 + r_{23}^2 - r_{13}^2 \\ 0 & 0 & 0 \\ r_{12}^2 + r_{23}^2 - r_{13}^2 & 0 & 2r_{23}^2 \end{pmatrix} + \alpha_3 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2r_{13}^2 & r_{13}^2 + r_{23}^2 - r_{12}^2 \\ 0 & r_{13}^2 + r_{23}^2 - r_{12}^2 & 2r_{23}^2 \end{pmatrix},$$

and recognize the nonzero entries of its three matrix summands as smaller order 3 Cayley–Menger matrices (18).

Now, to prove positive definiteness of $B = B^{(n)}$, we need to show positivity of the associated quadratic form:

$$z^T B z > 0 \quad \text{for all} \quad 0 \neq z = (\ldots, z_{\{i,j\}}, \ldots)^T \in \mathbb{R}^{n-1}/2.$$  \hspace{1cm} (26)

Using (25), we can similarly expand this quadratic form

$$z^T B z = \sum_{k=1}^n \alpha_k z^T \hat{M}^{(n)}_k z = \sum_{k=1}^n \alpha_k z_k^T M^{(n)}_k z_k,$$  \hspace{1cm} (27)

where

$$z_k = (z_{\{1,k\}}, \ldots, z_{\{k-1,k\}}, z_{\{k+1,k\}}, \ldots, z_{\{n,k\}})^T \in \mathbb{R}^{n-1}, \quad \text{for} \quad k = 1, \ldots, n,$$

so $z_{\{k,k\}}$ is omitted from the vector, and keeping in mind that the indices are symmetric, so $z_{\{i,j\}} = z_{\{j,i\}}$. The final identity in (27) comes from eliminating all the terms
involving the zero entries in $\tilde{M}_k^{(n)}$. Now, Theorem 8 implies positive semi-definiteness of the smaller Cayley–Menger matrices $M_k^{(n)}$, and hence

$$z_k^T M_k^{(n)} z_k \geq 0,$$

which, by (27), establishes positive semi-definiteness of the $n$ body matrix. Moreover, if the masses $p_1, \ldots, p_n$ are in a nonsingular configuration, Theorem 8 implies positive definiteness of the smaller Cayley–Menger matrices, and hence (28) becomes an equality if and only if $z_k = 0$. Moreover, if $z \neq 0 \in \mathbb{R}^{n(n-1)/2}$, then at least one $z_k \neq 0 \in \mathbb{R}^{n-1}$, and hence at least one of the summands on the right hand side of (27) is strictly positive, which thus establishes the desired inequality (26), thus proving positive definiteness of the $n$ body matrix. On the other hand, if the masses are in a singular configuration, their Cayley–Menger determinant vanishes, and so the Factorization Theorem 4, to be proved below, implies that the $n$ body determinant also vanishes, which means that the $n$ body matrix cannot be positive definite.

Q.E.D.

3. **Factorization of Certain Determinants.**

In order to prove the Factorization Theorem 4, we will, in fact, significantly generalize it. A proof of the generalization will establish the desired result as a special case.

**Notation:** For each nonnegative $m \in \mathbb{Z}$, we let $[m]$ be the set $\{1, 2, \ldots, m\}$.

Let us now define a class of matrices that includes the larger Cayley–Menger matrix $C^{(n)}$ in (8) and the matrix $C_A$ in (13).

Let $\mathcal{R}$ be a ring. Fix an integer $n \geq 1$. If $H = (h_{ij})_{1 \leq i \leq n, 1 \leq j \leq n} \in \mathcal{R}^{n \times n}$ is an $n \times n$-matrix over $\mathcal{R}$, then we define an $(n+1) \times (n+1)$-matrix $C_H \in \mathcal{R}^{(n+1) \times (n+1)}$ by

$$C_H = \begin{pmatrix}
1 & & & & & & & & & & & & & & & & \\
& h_{i,j} & & & & & & & & & & & & & & & \\
& & 1 & & & & & & & & & & & & & & & \\
& & & 0 & & & & & & & & & & & & & & & \\
1 & 1 & \cdots & 1 & 0
\end{pmatrix}
$$

(29)

Observe that our earlier matrices $C^{(n)} = C_R$, as in (8), and $C_A$, as in (13), are both of
this form based respectively on the $n \times n$ matrices

$$
R = 
\begin{pmatrix}
0 & r_{12}^2 & r_{13}^2 & \cdots & r_{1n}^2 \\
r_{12}^2 & 0 & r_{23}^2 & \cdots & r_{2n}^2 \\
r_{13}^2 & r_{23}^2 & 0 & \cdots & r_{3n}^2 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
r_{1n}^2 & r_{2n}^2 & r_{3n}^2 & \cdots & 0
\end{pmatrix},
A = 
\begin{pmatrix}
\alpha_1 & 0 & 0 & \cdots & 0 \\
0 & \alpha_2 & 0 & \cdots & 0 \\
0 & 0 & \alpha_3 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \alpha_n
\end{pmatrix}.
\tag{30}
$$

We will work in the polynomial ring

$$
\mathcal{R} = \mathbb{Z} \left[ \{ s_{ij} \mid i, j \in [n] \} \cup \{ t_{k,l} \mid k, l \in [n] \} \right],
\tag{31}
$$

consisting of polynomials with integer coefficients depending on the $n^2 + n^2 = 2n^2$ independent variables $s_{ij}, t_{k,l}$. Define the corresponding pair of $n \times n$-matrices

$$
S = (s_{ij})_{1 \leq i \leq n, 1 \leq j \leq n} \in \mathcal{R}^{n \times n},
T = (t_{ij})_{1 \leq i \leq n, 1 \leq j \leq n} \in \mathcal{R}^{n \times n},
\tag{32}
$$

which we use to construct the $(n + 1) \times (n + 1)$ matrices $C_S$ and $C_T$ via (29).

Next, let $E$ be the set of all 2-element subsets of $[n]$; we regard these subsets as unordered pairs of distinct elements of $[n]$. Note that $|E| = n(n - 1)/2$. Our generalization of the $n$ body matrix will be the matrix $W_{S,T} \in \mathcal{R}^{E \times E}$ — that is, a matrix whose rows and columns are indexed by elements of $E$ — whose entries are given by

$$
\omega_{\{i,j\},\{k,l\}} = (t_{j,k} + t_{i,l} - t_{i,k} - t_{j,l}) \left( s_{j,k} + s_{i,l} - s_{i,k} - s_{j,l} \right).
\tag{33}
$$

It is easy to see that (33) is well defined for any $\{i,j\}, \{k,l\} \in E$, since the right hand side is unchanged when $i$ is switched with $j$, and is also unchanged when $k$ is switched with $l$. We also remark that the factor $s_{j,k} + s_{i,l} - s_{i,k} - s_{j,l}$ on the right hand side of (33) can be rewritten as

$$
\det
\begin{pmatrix}
s_{j,l} & s_{j,k} & 1 \\
s_{i,l} & s_{i,k} & 1 \\
1 & 1 & 0
\end{pmatrix} = \det \left( C_{S[j,l][k]} \right),
\text{ where } S[j,i \mid l,k] = 
\begin{pmatrix}
s_{j,l} & s_{j,k} \\
s_{i,l} & s_{i,k}
\end{pmatrix},
$$

and similarly for the first factor $t_{j,k} + t_{i,l} - t_{i,k} - t_{j,l}$. Thus, each entry of $W_{S,T}$ is the product of the determinants of a pair of $3 \times 3$ matrices that also have our basic form (29).

Since $W_{S,T}$ is a square matrix of size $|E| \times |E|$, it has a determinant $\det W_{S,T} \in \mathcal{R}$. The main result of this sections is its divisibility:

**Theorem 9.** We have $(\det C_S) (\det C_T) \mid \det W_{S,T}$ in $\mathcal{R}$.
Notice that Theorem 9 is a divisibility in \( R \). Thus, the quotient is a polynomial \( Z_{S,T} = \det W_{S,T} / (\det C_S \det C_T) \), with integer coefficients, in the independent variables \( s_{i,j}, t_{i,j} \). Thus,

\[
\det W_{S,T} = (\det C_S) (\det C_T) Z_{S,T}. \tag{34}
\]

Observe that, whereas the left hand side of (34) depends on all \( 2n^2 \) variables, the first factor depends only on the \( s_{i,j} \) and the second factor only on the \( t_{k,l} \), while the final factor is, in general, a “mixed” function of both sets of variables. As in Theorem 4 the factorization (34) is existential, and we do not have a direct formula for the mixed factor \( Z_{S,T} \). Finding such a formula and giving it an algebraic or geometric interpretation is an outstanding and very interesting problem.

If we now specialize \( S \mapsto R \) and \( T \mapsto A \), where \( R, A \) are the matrices (30), then \( W_{S,T} \) reduces to the \( n \) body matrix \( W_{R,A} = B^{(n)} \) defined by (3), (4), and thus the left hand side of formula (34) reduces to the \( n \) body determinant \( \det W_{R,A} = \det B^{(n)} = \Delta^{(n)} \). On the other hand, we use (9) to identify \( \det C_R \) with the Cayley–Menger determinant \( \delta^{(n)} \), and (14) to identify \( \det C_A \) with the negative of the elementary symmetric polynomial \( -e_{n-1}(a) \). Thus, the general factorization formula (34) reduces to the \( n \) body determinant factorization formula (12) where the mass-dependent factor \( \sigma^{(n)} = -Z_{R,A} \) is identified with the corresponding reduction of the mixed factor in (34). Thus, Theorem 9 immediately implies the Factorization Theorem 4 upon specialization. Again, we do not have a direct formula for constructing either factor \( Z_{S,T} \) or \( Z_{R,A} \).

Our proof of Theorem 9 will rely on basic properties of UFDs (unique factorization domains), which are found in most texts on abstract algebra, e.g., [6, Section VIII.4]. We shall also use the fact that any polynomial ring (in finitely many variables) over \( \mathbb{Z} \) is a UFD. (This follows, e.g., from [6, Corollary 8.21] by induction on the number of variables.) Moreover, we shall use the fact (obvious from degree considerations) that the only units (i.e., invertible elements) of a polynomial ring are constant polynomials. Hence, a polynomial \( p \) in a polynomial ring \( \mathbb{Z} \{x_1, x_2, \ldots, x_k\} \) is irreducible if its content, i.e., the gcd of its coefficients, is 1 and \( p \) is irreducible in the ring \( \mathbb{Q} \{x_1, x_2, \ldots, x_k\} \) (since any constant factor of \( p \) in \( \mathbb{Z} \{x_1, x_2, \ldots, x_k\} \) must divide the content of \( p \)).

Before we prove Theorem 9, we require a technical lemma:

**Lemma 10.** Assume that \( n > 1 \). Then, \( \det C_S \) is a prime element of the UFD \( R \).

**Proof of Lemma 10** Expanding \( \det C_S \) as a sum over all \( (n+1)! \) permutations \( \pi \) of \([n+1]\), we observe that the permutations \( \pi \) satisfying \( \pi(n+1) = n+1 \) give rise to summands that equal 0, whereas all the other permutations \( \pi \) contribute pairwise distinct monomials to the sum. This shows that the polynomial \( \det C_S \) has content 1; \footnote{Why pairwise distinct? The monomial corresponding to such a permutation \( \pi \) is \( \prod_{i \in [n]} s_{i,\pi(i)} \). Knowing this monomial, we can reconstruct the value of \( \pi \) at the unique \( i \) satisfying \( \pi(i) = n+1 \) (namely, this value is the unique \( k \in [n] \) for which no entry from the \( k \)-th row of \( S \) appears in the monomial), as well as the remaining values of \( \pi \) on \([n]\) (by inspecting the corresponding \( s_{i,j} \) in the...}
indeed, each of its nonzero coefficients is 1 or −1. Moreover, it shows that \( \det \mathcal{C}_S \) is a polynomial of degree 1 in each of the indeterminates \( s_{i,j} \) (not 0 because \( n > 1 \)). Furthermore, in the expansion of \( \det \mathcal{C}_S \) into monomials, each monomial contains at most one variable from each row and at most one from each column. Thus, the same argument that is used in [4, proof of Lemma 5.12] to prove the irreducibility of \( \det \mathcal{S} \) can be used to see that \( \det \mathcal{C}_S \) is an irreducible element of the ring \( Q \left[ s_{i,j} \mid i, j \in [n] \right] \). Hence, \( \det \mathcal{C}_S \) to see that \( \det \mathcal{C}_S \) is also an irreducible element of the ring \( \mathcal{R} \) (which differs from \( \mathcal{R}_0 \) merely in the introduction of \( n^2 \) new variables \( t_{i,j} \), which clearly do not contribute any possible divisors to \( \det \mathcal{C}_S \)). Since \( \mathcal{R} \) is a UFD, we thus conclude that \( \det \mathcal{C}_S \) is a prime element of \( \mathcal{R} \).

**Proof of Theorem** [9] If \( n = 1 \), then Theorem [9] is clear, since \( \det \mathcal{C}_S = -1 \) and \( \det \mathcal{C}_T = -1 \) in this case. Thus, without loss of generality assume that \( n > 1 \).

Since \( \mathcal{R} \) is a polynomial ring over \( \mathbb{Z} \), it is a UFD. Moreover, Lemma [10] yields that \( \det \mathcal{C}_S \) is a prime element of \( \mathcal{R} \). Similarly, \( \det \mathcal{C}_T \) is a prime element of \( \mathcal{R} \).

Let \( \mathcal{Q} = \mathcal{R} / \det \mathcal{C}_T \) be the quotient ring, which is an integral domain since \( \det \mathcal{C}_S \) is a prime element of \( \mathcal{R} \). Since, by construction, \( \det \mathcal{C}_S = 0 \) in \( \mathcal{Q} \), the matrix \( \mathcal{C}_S \) is singular over \( \mathcal{Q} \) and hence has a nontrivial kernel because \( \mathcal{Q} \) is an integral domain. In other words, there exists a nonzero vector \( \mathbf{0} \neq \mathbf{x}^* = (x_1, x_2, \ldots, x_n, v)^T \in \mathcal{Q}^{n+1} \) such that

\[
\mathcal{C}_S \mathbf{x}^* = \mathbf{0}.
\]

The entries of the vector identity (35) imply\(^3\)

\[
\sum_i s_{i,j} x_i + v = 0, \quad \text{for all } i \in [n], \quad \text{and} \quad \sum_i x_i = 0.
\] (36)

Given such an \( \mathbf{x}^* \), let \( \mathbf{x} = (x_1, x_2, \ldots, x_n)^T \in \mathcal{Q}^n \) be the vector obtained by omitting the last entry. If \( \mathbf{x} = \mathbf{0} \), then, according to the first equations in (36), this would require \( v = 0 \), which would contradict the fact that \( \mathbf{x}^* \neq \mathbf{0} \). Thus, \( \mathbf{x} \neq \mathbf{0} \), which, by the last equation in (36), implies that \( \mathbf{x} \) has at least two nonzero entries, so \( x_i x_j \neq 0 \) for some \( i \neq j \), since \( \mathcal{Q} \) is an integral domain.

Define the vector \( \mathbf{z} \in \mathcal{Q}^E \) whose entries are indexed by unordered pairs \( \{i, j\} \in E \) and given by the products of distinct entries of \( \mathbf{x} \), so

\[
z_{\{i,j\}} = x_i x_j, \quad \{i, j\} \in E.
\]

Hence, \( \mathbf{z} \neq \mathbf{0} \) (since \( x_i x_j \neq 0 \) for some \( i \neq j \)).

Let us abbreviate \( \mathcal{W} = \mathcal{W}_{S,T} \). We shall prove that \( \mathbf{0} \neq \mathbf{z} \in \ker \mathcal{W} \). To this end, for any \( 1 \leq i < j \leq n \), the \( \{i, j\} \)-th entry of the vector \( \mathcal{W} \mathbf{z} \) is

---

\(^3\)Here and in the following, “\( \sum \)” always means “\( \sum_{i=1}^{n} \).”
\[
\sum_{\{k,l\} \in E} \sum_{i,l} w_{ij} x_k x_l = \sum_{k<l} \left( t_{jk} + t_{jl} - t_{ik} - t_{il} \right) (s_{jk} + s_{jl} - s_{ik} - s_{il}) x_k x_l \\
= \sum_{k<l} (t_{jk} + t_{jl} - t_{ik} - t_{il}) \left( s_{jk} + s_{jl} - s_{ik} - s_{il} \right) x_k x_l \\
= \sum_{k<l} \left( (t_{jk} - t_{ik}) - (t_{jl} - t_{il}) \right) \left( s_{jk} + s_{jl} - s_{ik} - s_{il} \right) x_k x_l \\
= \sum_{k<l} (t_{jk} - t_{ik}) \left( s_{jk} + s_{jl} - s_{ik} - s_{il} \right) x_k x_l - \sum_{k<l} (t_{jl} - t_{il}) \left( s_{jk} + s_{jl} - s_{ik} - s_{il} \right) x_l x_k \\
= \sum_{k<l} (t_{jk} - t_{ik}) \left( s_{jk} + s_{jl} - s_{ik} - s_{il} \right) x_k x_l + \sum_{k<l} (t_{jl} - t_{il}) \left( s_{jk} + s_{jl} - s_{ik} - s_{il} \right) x_l x_k \\
= \sum_{k<l} \left( t_{jk} - t_{ik} \right) \left( s_{jk} + s_{jl} - s_{ik} - s_{il} \right) x_k x_l + \sum_{k<l} \left( t_{jl} - t_{il} \right) \left( s_{jk} + s_{jl} - s_{ik} - s_{il} \right) x_l x_k \\
\quad \text{(here, we switched the roles of } k \text{ and } l \text{ in the second sum)} \\
= \sum_{k,l \in [n]} (t_{jk} - t_{ik}) \left( s_{jk} + s_{jl} - s_{ik} - s_{il} \right) x_k x_l \\
\quad \text{(here, we have combined the two sums, while also including extra terms for } k = l \text{ (which don’t change the sum since they are 0)} \\
= \sum_{k,l \in [n]} (t_{jk} - t_{ik}) x_k \sum_{l \in [n]} \left( s_{jk} + s_{jl} - s_{ik} - s_{il} \right) x_l \\
= \sum_{k \in [n]} (t_{jk} - t_{ik}) x_k \left( s_{jk} \sum_{l \in [n]} x_l + \sum_{l \in [n]} s_{jl} x_l - s_{ik} \sum_{l \in [n]} x_l - s_{il} \sum_{l \in [n]} x_l \right) \\
\quad \text{(here, we have used the equations in } (36) \text{ on each set of terms)} \\
= \sum_{k \in [n]} (t_{jk} - t_{ik}) x_k \left( s_{jk} 0 + (-v) - s_{ik} 0 - (-v) \right) = 0.
\]

Hence, \( Wz = 0 \). Since \( z \) is a nonzero vector, this shows that \( W \) has a nontrivial kernel over \( Q \). Since \( Q \) is an integral domain, we thus conclude that \( \det W = 0 \) in \( Q \). In other words, \( \det C_S \mid \det W \). The same argument shows that \( \det C_T \mid \det W \) also, since \( S \) and \( T \) play symmetric roles in the definition of the matrix \( W \).

Finally, we note that the two prime elements \( \det C_S \) and \( \det C_T \) of \( R \) are distinct — indeed, they are polynomials in disjoint sets of indeterminates \( s_{i,j} \) and \( t_{k,l} \), respectively, so they could only be equal if they were both constant, which they are not. Thus, they are coprime. Hence, an element of the UFD \( R \) divisible both by \( \det C_S \) and by \( \det C_T \) must also be divisible by their product \( \det C_S \det C_T \). Applying this to the element \( \det W \) completes the proof of the General Factorization Theorem. 

\[ \text{Q.E.D.} \]
4. A Biquadratic Form Identity.

In this section we establish a striking identity involving the matrix $W_{S,T}$, which naturally defines a biquadratic form that factorizes over a particular pair of hyperplanes. The reduction of this formula to the $n$ body matrix is also of note.

As above, let $W = W_{S,T} \in \mathcal{R}^{E \times E}$ be the $|E| \times |E|$ matrix whose entries are given by (33). Let $A$ be a commutative $\mathcal{R}$-algebra. Define the biquadratic form

$$q_W(x, y) = \sum_{\{i,j\} \in E} \sum_{\{k,l\} \in E} w_{\{i,j\},\{k,l\}} x_i x_j y_k y_l,$$

(37)

where $x = (x_1, x_2, \ldots, x_n)^T$ and $y = (y_1, y_2, \ldots, y_n)^T$ are vectors in $A^n$.

**Theorem 11.** When $x_1 + x_2 + \cdots + x_n = 0$ and $y_1 + y_2 + \cdots + y_n = 0$, the biquadratic form $q_W(x, y)$ factors into a product of two elementary bilinear forms\(^4\) based on the matrices $S, T$ given in (32):

$$q_W(x, y) = (x^T S y) (x^T T y).$$

(38)

**Proof.** We calculate

$$q_W(x, y) = \sum_{\{i,j\} \in E} \sum_{\{k,l\} \in E} w_{\{i,j\},\{k,l\}} x_i x_j y_k y_l$$

$$= \sum_{i \neq j} \sum_{k \neq l} t_{i,j} \left( s_{i,j} + s_{i,k} - s_{i,l} - s_{j,k} \right) x_i x_j y_k y_l$$

$$+ \sum_{i \neq j} \sum_{l \neq k} t_{i,l} \left( s_{i,j} + s_{i,l} - s_{i,k} - s_{j,l} \right) x_i x_j y_k y_l$$

$$- \sum_{i \neq j} \sum_{l \neq k} t_{i,l} \left( s_{i,j} + s_{i,l} - s_{i,k} - s_{j,l} \right) x_i x_j y_k y_l$$

$$= \sum_{i \neq j} \sum_{k \neq l} t_{i,j} \left( s_{i,j} + s_{i,k} - s_{i,l} - s_{j,k} \right) x_i x_j y_k y_l$$

$$+ \sum_{i \neq j} \sum_{l \neq k} t_{i,l} \left( s_{i,j} + s_{i,l} - s_{i,k} - s_{j,l} \right) x_i x_j y_k y_l$$

$$- \sum_{i \neq j} \sum_{l \neq k} t_{i,l} \left( s_{i,j} + s_{i,l} - s_{i,k} - s_{j,l} \right) x_i x_j y_k y_l$$

$$- \sum_{i \neq j} \sum_{k \neq l} t_{i,j} \left( s_{i,j} + s_{i,k} - s_{i,l} - s_{j,k} \right) x_i x_j y_k y_l$$

$$= \sum_{i \neq j} \sum_{k \neq l} t_{i,j} \left( s_{i,j} + s_{i,k} - s_{i,l} - s_{j,k} \right) x_i x_j y_k y_l$$

$$+ \sum_{i \neq j} \sum_{l \neq k} t_{i,l} \left( s_{i,j} + s_{i,l} - s_{i,k} - s_{j,l} \right) x_i x_j y_k y_l$$

$$- \sum_{i \neq j} \sum_{k \neq l} t_{i,j} \left( s_{i,j} + s_{i,k} - s_{i,l} - s_{j,k} \right) x_i x_j y_k y_l$$

$$- \sum_{i \neq j} \sum_{l \neq k} t_{i,l} \left( s_{i,j} + s_{i,l} - s_{i,k} - s_{j,l} \right) x_i x_j y_k y_l$$

$$= \sum_{i \neq j} \sum_{k \neq l} t_{i,j} \left( s_{i,j} + s_{i,k} - s_{i,l} - s_{j,k} \right) x_i x_j y_k y_l$$

$$+ \sum_{i \neq j} \sum_{l \neq k} t_{i,l} \left( s_{i,j} + s_{i,l} - s_{i,k} - s_{j,l} \right) x_i x_j y_k y_l$$

$$- \sum_{i \neq j} \sum_{k \neq l} t_{i,j} \left( s_{i,j} + s_{i,k} - s_{i,l} - s_{j,k} \right) x_i x_j y_k y_l$$

$$- \sum_{i \neq j} \sum_{l \neq k} t_{i,l} \left( s_{i,j} + s_{i,l} - s_{i,k} - s_{j,l} \right) x_i x_j y_k y_l$$

\(^4\)The $^T$ superscripts are transposes of column vectors, and have nothing to do with the matrix $T$. 
\[ P = \sum_{i<j<k<l} t_{j,k} \left( s_{j,k} + s_{i,l} - s_{j,l} - s_{i,k} \right) x_i x_j y_k y_l + \sum_{i>j<k<l} t_{j,k} \left( s_{j,k} + s_{i,l} - s_{i,k} - s_{j,l} \right) x_i x_j y_k y_l + \sum_{i>j<k<l} t_{j,k} \left( s_{i,j} + s_{i,k} - s_{j,k} - s_{i,l} \right) x_i x_j y_k y_l \]

\[ = \sum_{i<j<k} t_{j,k} \left( s_{j,k} + s_{i,l} - s_{i,k} - s_{j,l} \right) x_i x_j y_k y_l \]

(here, we have combined all four sums into a single one)

\[ = \sum_{i,j} t_{j,k} \left( s_{j,k} + s_{i,l} - s_{i,k} - s_{j,l} \right) x_i x_j y_k y_l \]

Given the \( n \) body matrix \( B \) reduce, respectively, to \( C \) the \( n \) body case, so that \( C_S, C_T \) reduce, respectively, to \( C_A, C_R \). We further set \( x = y \) to obtain the following intriguing result.

**Theorem 12.** Define the quadratic forms:

\[ r(x) = \sum_{i=1}^{n} \alpha_i x_i^2 = \sum_{i=1}^{n} \frac{x_i^2}{m_i}, \quad p(x) = \sum_{i,j} (p_i \cdot p_j) x_i x_j = \| P x \|^2, \]

where \( P = (p_1, p_2, \ldots, p_n) \) is the matrix whose columns are the locations \( p_i \) of the masses. Given the \( n \) body matrix \( B = B^{(n)} \) with entries (3, 4), define the corresponding homogeneous
quartic form

\[ q_B(x) = \sum_{\{i,j\},\{k,l\}} b_{\{i,j\},\{k,l\}} x_i x_j x_k x_l, \quad x = (x_1, \ldots, x_n)^T. \quad (40) \]

Then \( q_B(x) \) factors as the product of the preceding quadratic forms (39):

\[ q(x) = r(x) p(x) \quad \text{when} \quad x_1 + x_2 + \cdots + x_n = 0. \quad (41) \]

Note that because \( b_{\{i,j\},\{k,l\}} = 0 \) when \( i, j, k, l \) are distinct, the \( n \) body matrix \( B^{(n)} \) is, in fact uniquely determined by its associated quartic form \( q_B(x) \).

If rank \( P = n \), meaning that the masses are in a nonsingular configuration, then the right hand side of (41) is clearly positive whenever \( x \neq 0 \), and hence \( q_B(x) > 0 \) whenever \( x \neq 0 \) and \( x_1 + x_2 + \cdots + x_n = 0 \). However, this does not lead to the conclusion that the \( n \) body matrix, which forms the coefficients of \( q_B(x) \), is itself positive definite, and hence we needed a different approach to establish this result.

5. Future Directions.

As noted above, the challenge now is to determine an explicit geometrical formula for the mass-dependent factor \( \sigma^{(n)} \) in the \( n \) body determinant factorization formula (12) or, more generally, the mixed factor \( Z_{S,T} \) in our generalized factorization formula (34), to ascertain its significance. Is there some as yet undetected interesting determinantal identity or algebraic structure, perhaps representation-theoretic, that will provide some insight into this problem? Do the biquadratic and quartic form identities we found in (38), (41) provide any additional insight into these issues?

Another important problem is to understand the geometric structure of the associated Riemannian manifold that prescribes the radial \( n \) body Laplace–Beltrami operator constructed in [9].

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References


\(^5\)The sum is over all ordered pairs \( \{\{i,j\},\{k,l\}\} \) of unordered pairs \( \{i,j\} \) and \( \{k,l\} \).


