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Ghost Symmetries

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Abstract

We introduce the notion of a ghost symmetry for nonlocal differential equations. Ghosts are essential for maintaining the validity of the Jacobi identity for the characteristics of nonlocal vector fields.

The local theory of symmetries of differential equations has been well-established since the days of Sophus Lie. Generalized, or higher order symmetries can be traced back to the original paper of Noether, [24], and received added importance after the discovery that they play a critical role in integrable (soliton) partial differential equations, cf. [25]. While the local theory is very well developed, the theory of nonlocal symmetries of nonlocal differential equations remains incomplete. Several groups, including Chen et. al., [5, 6, 7], Ibragimov et. al., [1], [16, Chapter 7], Fushchich et. al., [11], Guthrie and Hickman, [13, 14, 15], Kaptsov, [17], Bluman et. al., [2, 3, 4], and others, [8, 10, 12, 21, 27], have proposed a foundation for such a theory. Perhaps the most promising is the Krasilshchik-Vinogradov theory of coverings, [18, 19, 20, 28, 29], but this has the disadvantage that their construction relies on the a priori specification of the underlying differential equation, and so, unlike local jet space, does not form a universally valid foundation for the theory.

Recently, the second and third author made a surprising discovery that the Jacobi identity for nonlocal vector fields appears to fail! This observation arose during an attempt to systematically investigate the symmetry properties of the Kadomtsev–Petviashvili (KP) equation, previously studied in [6, 7, 9, 22, 23]. The observed violation of the naïve version of the Jacobi identity applies to all of the preceding nonlocal symmetry calculi, and, consequently, many statements about the "Lie algebra" of nonlocal symmetries of differential equations are, by in large, not valid as stated. This indicates the need for a comprehensive re-evaluation of all earlier results on nonlocal symmetry algebras.

In this announcement, we show how to resolve the Jacobi paradox through the introduction of what we name "ghost symmetries". Ghost symmetries are genuinely nonlocal objects that have no counterpart in the local theory, but serve to provide missing terms that resolve apparent contradictions that have, albeit unnoticed, plagued the nonlocal theory. Details of the construction and proofs will appear in a forthcoming paper.

We shall assume that the reader is familiar with the basic theory of generalized symmetries in the local jet bundle framework. We adopt the notation and terminology of [25] without further comment. We specify p independent variables $x = (x^1, \ldots, x^p)$ and q dependent variables $u = (u^1, \ldots, u^q)$, with $u^\alpha_J = D^J(u^\alpha)$ denoting the induced jet space coordinates. Here $D^J = D_1^{j_1} \cdots D_p^{j_p}$ denotes the corresponding total derivative operator. In the local version, multi-indices $J = (j_1, \ldots, j_p)$ are assumed to be non-negative, $J \geq 0$, meaning $j_\nu \geq 0$ for $\nu = 1, \ldots, p$.

We consider *generalized vector fields* in *evolutionary form*

$$\mathbf{v} = \mathbf{v}_Q = \sum_{\alpha=1}^{q} \sum_{J\geq 0} D^J Q^\alpha \frac{\partial}{\partial u^\alpha_J}\ , \tag{1}$$

where $Q = (Q^1, \ldots, Q^q)$ is the *characteristic*, and serves to uniquely specify $\mathbf{v}$. Therefore, the space of evolutionary vector fields can be identified with the space of q-tuples of differential functions. We note the basic formula

$$\mathbf{v}_Q(P) = \mathrm{D}_P(Q) \tag{2}$$

where D_P denotes the Fréchet derivative of the differential function P.

The *Lie bracket* $[\ \mathbf{v}_P,\ \mathbf{v}_Q\ ] = \mathbf{v}_{[P,Q]}$ where

$$[\ P,\ Q\ ] = \mathbf{v}_P(Q) - \mathbf{v}_Q(P) = \mathrm{D}_Q(P) - \mathrm{D}_P(Q), \tag{3}$$

satisfies the Jacobi identity, and hence endows the space of evolutionary vector fields with the structure of a Lie algebra.

Attempting to generalize the algebra of evolutionary vector fields to nonlocal variables runs into some immediate, unexpected difficulties. Intuitively, the nonlocal variables should be given by iterating the inverse total derivatives D_i^{-1}, applied to either the jet coordinates, or, more generally, to differential functions. In particular, we allow nonlocal variables $u^\alpha_J = D^J u^\alpha$ in which $J \in \mathbf{Z}^p$ is an arbitrary multi-index. The rigorous details of the construction will be deferred to a more complete exposition.

The following fairly simple computation appears to indicate that the Jacobi identity does *not* hold for characteristics of nonlocal vector fields.

Example 1. Let $p = q = 1$, with independent variable x and dependent variable u. Consider the vector fields $\mathbf{v}, \mathbf{w}$, and $\mathbf{z}$ with respective characteristics 1, u_x and $D_x^{-1}u$. The first two are local vector fields, and, in fact, correspond to the infinitesimal generators of the translation group

$$(x, u) \ \longmapsto\ (x + \delta, u + \varepsilon).$$

The Jacobi identity for these three vector fields has the form

$$[\ 1,\ [\ u_x,\ D_x^{-1}u\ ]\ ] + [\ u_x,\ [\ D_x^{-1}u,\ 1\ ]\ ] + [\ D_x^{-1}u,\ [\ 1,\ u_x\ ]\ ] = 0, \tag{4}$$

where we work on the level of the characteristics, using the induced commutator bracket (3). Since

$$[\ 1,\ u_x\ ] = \mathrm{D}_{u_x}(1) - \mathrm{D}_1(u_x) = D_x(1) = 0, \tag{5}$$

reflecting the fact that the group of translations is abelian, we only need to compute the first two terms in (4). First, using the definition of the Fréchet derivative, we compute

$$[\, u_x,\, D_x^{-1} u \,] = \mathrm{D}_{D_x^{-1} u}(u_x) - \mathrm{D}_{u_x}(D_x^{-1} u) = D_x^{-1} u_x - D_x(D_x^{-1} u) = u + c - u = c,$$

where c is an arbitrary constant representing the ambiguity in the antiderivative D_x^{-1}. Thus,

$$[\, 1,\, [\, u_x,\, D_x^{-1} u \,]\,] = [\, 1,\, c \,] = 0,$$

irrespective of the integration constant c. On the other hand,

$$[\, D_x^{-1} u,\, 1 \,] = -D_x^{-1}(1) = -x + d,$$

where d is another arbitrary constant, and so

$$[\, u_x,\, [\, D_x^{-1} u,\, 1 \,]\,] = [\, u_x,\, -x + d \,] = -D_x(-x + d) = 1.$$

Therefore, no matter how we choose the integration "constants" c, d, the left hand side of (4) equals 1, *not* zero, and so the Jacobi identity appears to be invalid!

As we shall see, if we generalize the definition of vector fields as to include terms with nonpositive J in (1), there will be no problem in verifying the Jacobi identity. Thus, the problem lies in the characteristic calculus.

This example is one of the simplest of a wide variety of apparent nonlocal counterexamples to the Jacobi identity. The main goal of this paper is to resolve these apparent paradoxes in the establishment of a proper theory and calculus for characteristics of nonlocal symmetries.

Let us begin by stating a general, abstract definition of an evolutionary vector field.

Definition 2. A *evolutionary vector field* $\mathbf{v}$ is a derivation that annihilates all the independent variables and commutes with all total derivatives. Therefore,

$$\begin{aligned} \mathbf{v}(P+Q) &= \mathbf{v}(P) + \mathbf{v}(Q), & \mathbf{v}(x^i) &= 0, \\ \mathbf{v}(P \cdot Q) &= \mathbf{v}(P) \cdot Q + P \cdot \mathbf{v}(Q), & [\, \mathbf{v},\, D_i \,] &= 0. \end{aligned} \tag{6}$$

The commutator bracket

$$[\, \mathbf{v},\, \mathbf{w} \,](P) = \mathbf{v}(\mathbf{w}(P)) - \mathbf{w}(\mathbf{v}(P))$$

between two evolutionary vector fields satisfies the usual skew symmetry and Jacobi identities, that is $[\, \mathbf{v},\, \mathbf{w} \,] + [\, \mathbf{w},\, \mathbf{v} \,] = 0$ and $[\, \mathbf{z},\, [\, \mathbf{v},\, \mathbf{w} \,]\,] + [\, \mathbf{w},\, [\, \mathbf{z},\, \mathbf{v} \,]\,] + [\, \mathbf{v},\, [\, \mathbf{w},\, \mathbf{z} \,]\,] = 0$. So, why are we obtaining a paradox in our examples?

Each evolutionary vector field is uniquely specified by its action $\mathbf{v}(u_J^\alpha)$ on the coordinate variables. Given an evolutionary vector field $\mathbf{v}$, we define its *characteristic* $Q = (Q^1, \ldots, Q^q)$ to have components $\mathbf{v}(u^\alpha) = Q^\alpha$. Note that

$$\mathbf{v}(u_J^\alpha) = \mathbf{v}(D_J u^\alpha) = D_J \mathbf{v}(u^\alpha) = D_J Q^\alpha$$

for all positive multi-indices $J \geq 0$. Thus, in the local situation, an evolutionary vector field is uniquely determined by its characteristic. This basic fact is *not* true in nonlocal differential algebras — there are nonzero evolutionary vector fields with zero characteristic! This crucial observation motivates the following key definition.

Definition 3. An evolutionary vector field γ is called a a *K-ghost* for some $K \in \mathbf{Z}^p$ if $\gamma(u_L^\alpha) = 0$ for all $L \geq K$ and $\alpha = 1, \ldots, q$.

There are no negative ghost vector fields in a local differential algebra because each evolutionary vector field is uniquely determined by its characteristic Q. There are, however, positive ghost vector fields; for example the vector field with characteristic $Q = 1$ is a K-ghost for any positive multi-index $K > 0$.

Definition 4. Given a multi-index $K \in \mathbf{Z}^p$, define

$$\pi_K = D^K(1) = \begin{cases} \dfrac{x^{-K}}{(-K)!}, & K \leq 0, \\ 0 & \text{otherwise}. \end{cases} \tag{7}$$

Example 5. Let us see how the existence of ghost vector fields serves to resolve the Jacobi identity paradox in (4). Surprisingly, the problem is *not* with the nonlocal vector field $\mathbf{z}$ with characteristic $D_x^{-1}u$, but rather the local commutator $[\,\mathbf{v},\, \mathbf{w}\,]$ corresponding to the vector fields with characteristics 1 and u_x, respectively. While $[\,\mathbf{v},\, \mathbf{w}\,] = 0$ on a local differential algebra, it is, in fact, a ghost vector field when extended to nonlocal differential algebras!

First, the action of the vector fields on the local variables does not uniquely specify their action on the nonlocal variables, due to the presence of possible integration constants. However, as we have seen, the integration constants do not play a significant role in the resolution of the Jacobi identity paradox. We shall fix all the integration constants to be zero by default. Therefore, we set

$$\mathbf{v}(u_k) = D_x^k(1) = \pi_k \tag{8}$$

where $u_k = D_x^k u$ for any $k \in \mathbf{Z}$. Since $\mathbf{v}(u_k)$ only depends on x, we have $\mathbf{w}(\mathbf{v}(u_k)) = 0$, and so $[\,\mathbf{v},\, \mathbf{w}\,](u_k) = \mathbf{v}(u_{k+1}) = \pi_{k+1}$. Therefore, $[\,\mathbf{v},\, \mathbf{w}\,] = \gamma_1$ is a ghost vector field that satisfies $\gamma_1(u_k) = \pi_{k+1}$. This ghost provides the missing term in the Jacobi identity (4). Indeed,

$$[\,\mathbf{z},\, \gamma_1\,](u) = -\,\gamma_1(\mathbf{z}(u)) = -\,\gamma_1(D_x^{-1}u) = -1.$$

Thus, the fact that the local commutator is a nonlocal ghost resolves the preceding Jacobi paradox.

This and subsequent computations can simplified by introducing a "ghost calculus" for general nonlocal evolutionary vector fields. The first remark is that only evolutionary vector fields that depend purely on the independent variables can be ghosts.

Definition 6. Define the basis ghost vector field γ_J^α for $J \in \mathbf{Z}^p$ to satisfy

$$\gamma_J^\alpha(u_K^\beta) = \delta_{\alpha\beta}\pi_{J+K}, \qquad \gamma_J^\alpha(x^I) = 0, \qquad I \geq 0. \tag{9}$$

Note that γ_J^α is a K-ghost for any $K + J \not\leq 0$.

Theorem 7. *Every polynomial ghost vector field is a linear combination of the basis ghosts,*

$$\gamma = \sum_{\alpha,J} c_J^\alpha \gamma_J^\alpha, \tag{10}$$

where the $c_J^\alpha \in \mathbf{R}$ are constants.

Remark: The summation in (10) can be infinite. However, only certain "configurations" of the nonzero coefficients c_J^α are allowed in order that γ map (nonlocal) differential polynomials to differential polynomials.

Corollary 8. *Any polynomial evolutionary vector field can be written a linear combination of basis ghosts and a u-dependent vector field:*

$$\mathbf{v} = \mathbf{v}_Q + \gamma, \qquad \text{whereby} \qquad \mathbf{v}(u_K^\beta) = D^K Q^\beta + \sum_J c_J^\beta \, \pi_{K+J} \, . \tag{11}$$

To implement a calculus of evolutionary vector fields, we identify a vector field with its "characteristic". The characteristic of the evolutionary vector field $\mathbf{v}_Q$ is, as usual, $Q = (Q^1, \ldots, Q^q)$. We define the characteristic of the ghost vector field γ_J^α to be, formally,

$$\chi_J^\alpha = \chi_J \mathbf{e}^\alpha = (0, \ldots, 0, \chi_K, 0, \ldots, 0),$$

where $\mathbf{e}^\alpha \in \mathbf{R}^q$ is the α-th standard basis vector. In this manner, every nonlocal vector field (11) has a unique characteristic

$$S = Q + \sum_{\alpha, J} c_J^\alpha \chi_J \mathbf{e}^\alpha, \qquad \text{with components} \qquad S^\alpha = Q^\alpha + \sum_J c_J^\alpha \chi_J . \tag{12}$$

Note that in the one dependent variable case $q = 1$, we can drop the irrelevant basis vector $\mathbf{e}^1$ to further simplify the notation.

When we evaluate the ghost vector field on a differential polynomial, the formal characteristic χ_J will be replaced by the function π_J in formula (7), and hence will vanish if $J \not\geq 0$. However, in the calculus of ghost vector fields, we cannot make this replacement in advance since this would lead back to the original Jacobi identity contradiction.

For computational purposes, it helps to use a uniform notation. We can in the u–independent terms unambiguously replace all polynomials

$$x^K \quad \longmapsto \quad K! \, \chi_{-K}$$

appearing in the characteristic of our vector field by their equivalent ghost characteristics. In particular, we can write the local vector field $x^K \frac{\partial}{\partial u^\alpha}$ with polynomial characteristic $x^K \mathbf{e}^\alpha$ as $K! \, \chi_{-K} \mathbf{e}^\alpha$. For the u–dependent terms, it is convenient to change

$$x^K \quad \longmapsto \quad K! \, \pi_{-K} .$$

We retain a different notation to remind ourselves that the u-dependent terms are never ghosts, and so $\pi_K = 0$ for $K \not\leq 0$. For instance, with $p = q = 1$, one translates $x^2 + xu + u_1$ into $2\chi_{-2} + \pi_{-1} u_0 + u_1$. In this calculus, the product rule $x^J x^K = x^{J+K}$ becomes the ghost product rule

$$\pi_K \, \chi_J = \binom{-K - J}{-K} \chi_{K+J} . \tag{13}$$

The product makes sense, provided we adopt the Pochhammer definition

$$\binom{L}{I} = \frac{1}{I!} \prod_{\nu=1}^{p} \prod_{k=0}^{i_\nu - 1} (\, l_\nu - k \,), \qquad\qquad I \geq 0, \tag{14}$$

for the multinomial symbol. And, indeed, only such products will appear when we evaluate commutators and apply vector fields to nonlocal differential polynomials.

We now describe the precise ghost calculus rules for computing the commutators of ghost characteristics. The commutators of ordinary characteristics $[\,Q,\,R\,]$ follow the same rules (3) as in the local case, where we replace the multiplication of monomials by the ghost multiplication rule (13). Secondly, since ghosts do not involve the dependent variables, they mutually commute:

$$[\,\chi_J \mathbf{e}^\alpha,\, \chi_K \mathbf{e}^\beta\,] = 0. \tag{15}$$

Here and in the sequel we write the brackets as brackets between characteristics. These are, by definition, equal to the brackets between the evolutionary vector fields. No attempt should be made to substitute for instance $\chi_1 \mapsto \pi_1 = 0$ (using equation (8)) inside the bracket (this would create exactly the paradox we try to resolve here!). The only way to do any valid computations with ghost characteristics is to replace them by the corresponding basis ghost fields and then apply the rules for the ghost fields. On the other hand, since only u-independent vector fields are true ghosts, whenever a ghost characteristic χ_K with $K \not\leq 0$ multiplies any terms involving a u^α_J, it can be replaced by 0 without affecting the final outcome of the computation.

Finally, let Q be a characteristic and H a function multiplying it. The ghost characteristics χ^α_J acts on this product as follows:

$$[\,\chi_J \mathbf{e}^\alpha,\, H\,Q\,] = H\,[\,\chi_J \mathbf{e}^\alpha,\, Q\,] + \gamma^\alpha_J(H) Q. \tag{16}$$

Thus, we only need to know how to commute ghosts and derivative coordinates,

$$[\,\chi_J \mathbf{e}^\alpha,\, u^\beta_K \mathbf{e}^\mu\,] = \delta_{\alpha\beta}\chi_{J+K}\mathbf{e}^\mu, \tag{17}$$

in order to compute in the ghost characteristic space.

Example 9. Let us revisit Example 5. The three ghost characteristics are

$$1 = \chi_0, \qquad u_x = u_1, \qquad D_x^{-1} u = u_{-1}.$$

Then the three terms are

$$\begin{aligned}
&[\,\chi_0,\,[\,u_1,\,u_{-1}\,]\,] = 0,\\
&[\,u_1,\,[\,u_{-1},\,\chi_0\,]\,] = -[\,u_1,\,\chi_{-1}\,] = \chi_0,\\
&[\,u_{-1},\,[\,\chi_0,\,u_1\,]\,] = [\,u_{-1},\,\chi_1\,] = -\chi_0.
\end{aligned}$$

The sum of these three terms is 0, and so the Jacobi paradox is resolved.

Example 10. The first Jacobi identity paradox that was found while working on the symmetry algebra of the KP equation, and was more complicated than (4). Here $p = 2$, with independent variables x, y, and $q = 1$, with dependent variable u. Consider the vector fields with characteristics y, yu_x and $u_x D_x^{-1} u_y$. A similar computation as in Example 1 shows that without the introduction of ghost terms, the Jacobi sum

$$[\,u_x D_x^{-1} u_y,\,[\,yu_x,\,y\,]\,] + [\,y,\,[\,u_x D_x^{-1} u_y,\,yu_x\,]\,] + [\,yu_x,\,[\,y,\,u_x D_x^{-1} u_y\,]\,] \tag{18}$$

equals $-2yu_x$ instead of zero. In this case, the three ghost characteristics are

$$y = \chi_{0,-1}, \qquad yu_x = \pi_{0,-1}\, u_{1,0}, \qquad u_x D_x^{-1} u_y = u_{1,0}\, u_{-1,1},$$

where $u_{i,j} = D_x^i D_y^j u$. Then, using (13), (17), the three terms are

$$\begin{aligned} [\, \chi_{0,-1},\, \pi_{0,-1}\, u_{1,0} \,] &= \pi_{0,-1}\, \chi_{1,-1} = 2\, \chi_{1,-2}, \\ [\, u_{1,0}\, u_{-1,1},\, 2\, \chi_{1,-2} \,] &= -2\, \pi_{0,-1}\, u_{1,0} = -2yu_x, \end{aligned}$$

and

$$\begin{aligned} [\, \pi_{0,-1}\, u_{1,0},\, u_{1,0}\, u_{-1,1} \,] &= \mathrm{D}_{u_{1,0}\, u_{-1,1}}(\pi_{0,-1}\, u_{1,0}) - \pi_{0,-1}\, D_x(u_{1,0}\, u_{-1,1}) = u_{0,0} u_{1,0}, \\ [\, \chi_{0,-1},\, u_{0,0} u_{1,0} \,] &= \pi_{0,-1}\, u_{1,0} = yu_x, \end{aligned}$$

and, finally,

$$\begin{aligned} [\, u_{1,0}\, u_{-1,1},\, \chi_{0,-1} \,] &= -\pi_{-1,0}\, u_{1,0}, \\ [\, \pi_{0,-1}\, u_{1,0},\, -\pi_{-1,0}\, u_{1,0} \,] &= \pi_{0,-1}\, u_{1,0} = yu_x. \end{aligned}$$

The sum of these three terms is 0, and so the Jacobi identity is valid at the level of characteristics in the ghost framework.

In conclusion, we have seen that the Jacobi identity for the characteristics of nonlocal vector fields remains valid provided one pays proper attention to the ghost terms in the commutators. The appearance of such ghost terms is surprising at first, but, in hindsight, quite natural. These results indicate that a complete re-evaluation of earlier work on nonlocal symmetries (using characteristics) of local and non-local partial differential equations is required. A complete understanding of the hitherto undetected ghost terms needs to be properly incorporated into earlier results, including the study of recursion operators and master symmetries, all of which typically involve nonlocal operations, cf. [26]. Implementation of the ghost calculus in standard computer algebra packages would help a lot in these investigations.

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