

Nonlocal Symmetries and Ghosts

Peter J. Olver[†]
School of Mathematics
University of Minnesota
Minneapolis, MN 55455
U.S.A.
olver@math.umn.edu
<http://www.math.umn.edu/~olver>

1. Introduction.

The local theory of symmetries of differential equations has been well-established since the days of Sophus Lie. Generalized, or higher order symmetries can be traced back to the original paper of Noether, [32], but were not exploited until the discovery that they play a critical role in integrable (soliton) partial differential equations, cf. [30, 33, 35].

While the local theory is very well developed, the theory of nonlocal symmetries of nonlocal differential equations remains incomplete. Particular results on certain classes of nonlocal symmetries and nonlocal differential equations have been developed by several groups, including Abraham–Shrauner et. al., [1, 2, 3, 13], Bluman et. al., [5, 6, 7], Chen et. al., [8, 9, 10], Fushchich et. al., [17], Guthrie and Hickman, [20, 21, 22], Ibragimov et. al., [4], [23; Chapter 7], and many others, [11, 12, 16, 18, 19, 24, 28, 29, 31, 37]. Perhaps the most promising proposed foundation for a general theory of nonlocal symmetries is the Krasilshchik-Vinogradov theory of coverings, [25, 26, 27, 38, 39]. However, their construction relies on the a priori specification of the underlying differential equation, and so, unlike local jet space, does not form a universally valid foundation for the theory.

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One of the reasons for the lack of a proper foundation is a continuing lack of understanding of the calculus of nonlocal vector fields. Recently, [34], during an attempt to systematically investigate the symmetry properties of the Kadomtsev–Petviashvili (KP) equation, Sanders and Wang made a surprising discovery that the Jacobi identity for nonlocal vector fields appears to fail! The observed violation of the naïve version of the Jacobi identity applies to all of the preceding nonlocal symmetry calculi, and, consequently, many statements about the “Lie algebra” of nonlocal symmetries of differential equations are, by in large, not valid as stated. This indicates the need for a comprehensive re-evaluation of all earlier results on nonlocal symmetry algebras.

In this paper, I propose a new theoretical and computational basis for a nonlocal theory which, like the original jet bundle construction, does not rely on a specific differential equation, but applies equally well to a wide variety of nonlocal systems. I will also review the concept of a ghost symmetry, introduced in [34], that resolves the apparent Jacobi paradox. Applications to the classification of symmetries of the KP equation appear in [34]. Similar issues appear in the study of recursion operators by Sanders and Wang, [36].

2. Generalized Symmetries.

Let us recall the basic theory of generalized symmetries in the local jet bundle framework as presented in [33]. We specify p independent variables $x = (x^1, \dots, x^p)$ and q dependent variables $u = (u^1, \dots, u^q)$. The induced jet space coordinates are denoted by $u_J^\alpha = \partial^{\#J} u^\alpha / (\partial x^1)^{j_1} \dots (\partial x^p)^{j_p}$, in which $1 \leq \alpha \leq q$, and $J = (j_1, \dots, j_p) \in \mathbb{N}^p$ is a (non-negative) multi-index, so $j_\nu \geq 0$, of order $\#J = j_1 + \dots + j_p$. We let $u^{(\infty)} = (\dots u_J^\alpha \dots)$ denote the collection of all such local jet variables. A *differential function* is a smooth function $P[u] = P(x, u^{(\infty)})$ depending on finitely many jet variables. If $u = f(x)$ is any smooth function, we let $P[f]$ denote the evaluation of the differential function P on f .

The total derivatives D_1, \dots, D_p are defined so that $D_i P[f] = \partial_i(P[f])$ where $\partial_i = \partial/\partial x^i$. They act on the space of differential functions as derivations, and so are completely determined by their action

$$D_i(x^j) = \delta_i^j, \quad D_i(u_J^\alpha) = u_{J+e_i}^\alpha, \quad (2.1)$$

on the coordinate functions. Here $e_i \in \mathbb{N}^p$ denotes the i^{th} basis multi-index having a 1 in the i^{th} position and zeros elsewhere. If J is a multi-index, we let $D^J = D_1^{j_1} \dots D_p^{j_p}$ denote the corresponding higher order total derivative; in particular, $u_J^\alpha = D^J u^\alpha$.

We consider *generalized vector fields* in *evolutionary form*

$$\mathbf{v} = \mathbf{v}_Q = \sum_{\alpha=1}^q \sum_{J \geq 0} D^J Q^\alpha \frac{\partial}{\partial u_J^\alpha}, \quad (2.2)$$

where $Q = (Q^1, \dots, Q^q)$ is the *characteristic*, and serves to uniquely specify \mathbf{v} . We note the basic formula

$$\mathbf{v}_Q(P) = D_P(Q) \quad (2.3)$$

where D_P denotes the Fréchet derivative of the differential function P , [33], which is a total differential operator with components

$$D_P^\alpha = \sum_J \frac{\partial P}{\partial u_J^\alpha} D^J, \quad \alpha = 1, \dots, q. \quad (2.4)$$

The *Lie bracket* or *commutator* between two evolutionary vector fields is again an evolutionary vector field

$$[\mathbf{v}_P, \mathbf{v}_Q] = \mathbf{v}_{[P, Q]},$$

with characteristic

$$[P, Q] = \mathbf{v}_P(Q) - \mathbf{v}_Q(P) = D_Q(P) - D_P(Q). \quad (2.5)$$

The Lie bracket satisfies the Jacobi identity, and hence endows the space of evolutionary vector fields with the structure of a Lie algebra.

3. Counterexamples to the Jacobi identity?

Attempting to generalize the algebra of evolutionary vector fields to nonlocal variables runs into some immediate, unexpected difficulties. Intuitively, the nonlocal variables should be given by iterating the inverse total derivatives D_i^{-1} , applied to either the jet coordinates u_J^α , or, more generally, to differential functions. In particular, we allow nonlocal variables $u_J^\alpha = D^J u^\alpha$ in which $J \in \mathbb{Z}^p$ is an arbitrary multi-index. Even more generally, one might allow inversion of arbitrary total differential operators \mathcal{D}^{-1} , where $\mathcal{D} = \sum_K P_K[u] D^K$, whose coefficients P_K can be either constants, or even general differential functions.

However, the following fairly simple computation appears to indicate that the Jacobi identity does *not* hold between nonlocal vector fields.

Example 3.1. Let $p = q = 1$, with independent variable x and dependent variable u . Consider the vector fields \mathbf{v} , \mathbf{w} , and \mathbf{z} with respective characteristics 1 , u_x and $D_x^{-1}u$. The first two are local vector fields, and, in fact, correspond to the infinitesimal generators of the translation group

$$(x, u) \longmapsto (x + \delta, u + \varepsilon).$$

The Jacobi identity for these three vector fields has the form

$$[1, [u_x, D_x^{-1}u]] + [u_x, [D_x^{-1}u, 1]] + [D_x^{-1}u, [1, u_x]] = 0, \quad (3.1)$$

where we work on the level of the characteristics, using the induced commutator bracket (2.5). Since

$$[1, u_x] = D_{u_x}(1) - D_1(u_x) = D_x(1) = 0, \quad (3.2)$$

reflecting the fact that the group of translations is abelian, we only need to compute the first two terms in (3.1). First, using the definition of the Fréchet derivative, we compute

$$[u_x, D_x^{-1}u] = D_{D_x^{-1}u}(u_x) - D_{u_x}(D_x^{-1}u) = D_x^{-1}u_x - D_x(D_x^{-1}u) = u + c - u = c,$$

where c is an arbitrary constant representing the ambiguity in the antiderivative D_x^{-1} . Thus,

$$[1, [u_x, D_x^{-1}u]] = [1, c] = 0,$$

irrespective of the integration constant c . On the other hand,

$$[D_x^{-1}u, 1] = -D_x^{-1}(1) = -x + d,$$

where d is another arbitrary constant, and so

$$[u_x, [D_x^{-1}u, 1]] = [u_x, -x + d] = -D_x(-x + d) = 1.$$

Therefore, no matter how we choose the integration “constants” c, d , the left hand side of (3.1) equals 1, *not* zero, and so the Jacobi identity appears to be invalid!

This example is, in fact, the simplest of a wide variety of apparent nonlocal counterexamples to the Jacobi identity. Similar problems arise in the structure of the Lie algebra of nonlocal symmetries of the KP equation, [34], and the theory of recursion operators, [36].

4. Nonlocal Differential Algebra.

In order to keep the constructions reasonably simple, we will work entirely within the polynomial category throughout. Thus, we only consider differential polynomials with polynomial coefficients. Also we work, without any significant loss of generality, with real-valued polynomials, the complex version being an easy adaptation.

By a *multi-index* we mean a p -tuple $J = (j_1, \dots, j_p) \in \mathbb{Z}^p$ with integer entries. The *order* of J is $\#J = j_1 + \dots + j_p$. The multi-index is *positive*, written $J \geq 0$, if *all* its entries are positive: $j_\nu \geq 0$. We impose a partial order on the space of multi-indices with $J \leq K$ if and only if $K - J \geq 0$. We will also impose a total ordering $J \prec K$ on the multi-indices that respects degree, so if $\#J < \#K$, then $J \prec K$. In particular, degree lexicographic ordering is a convenient choice of total order, [14].

Let $\mathcal{A} = \mathbb{R}[x] = \mathbb{R}\{x^1, \dots, x^p\}$ denote the algebra of polynomial functions $f(x)$ depending upon p variables. The derivatives $\partial_1, \dots, \partial_p$ make \mathcal{A} into a (partial) differential algebra. Given a possibly infinite set of dependent variables $\mathcal{U} = \{ \dots u^\alpha \dots \}$, let $\mathcal{A}\{\mathcal{U}\}$ denote the differential algebra consisting of all polynomials in their local derivatives $u_j^\alpha = D^J u^\alpha$, $J \geq 0$, whose coefficients are polynomial functions in \mathcal{A} . We write $P[u] = P(\dots x^i \dots u_j^\alpha \dots)$ for a differential polynomial in $\mathcal{A}\{\mathcal{U}\}$. Even though \mathcal{U} may contain infinitely many variables, any differential polynomial $P \in \mathcal{A}\{\mathcal{U}\}$ has only finitely many summands and hence depends on only finitely many variables u_j^α . The set of polynomials that only depend on x can be identified with \mathcal{A} itself, and there is a natural decomposition

$$\mathcal{A}\{\mathcal{U}\} = \mathcal{A} \oplus \mathcal{A}_\star\{\mathcal{U}\},$$

where $\mathcal{A}_\star\{\mathcal{U}\}$ consists of all differential polynomials that vanish whenever we set all $u_j^\alpha = 0$. Any ordering $u^\alpha \prec u^\beta$ of \mathcal{U} induces an ordering of the derivatives, so $u_j^\alpha \prec u_K^\beta$ whenever $u^\alpha \prec u^\beta$ or $\alpha = \beta$ and $J \prec K$. This ordering in turn induces the degree lexicographic ordering on the differential monomials in $\mathcal{A}_\star\{\mathcal{U}\}$.

The total derivatives D_1, \dots, D_p act on $\mathcal{A}\{\mathcal{U}\}$ as derivations. Their kernels are well-known:

Lemma 4.1. *The kernel of the i^{th} total derivative is*

$$\ker D_i = \mathcal{A}_i = \{ f(x^1, \dots, x^{i-1}, x^{i+1}, \dots, x^p) \mid f \in \mathcal{A} \}. \quad (4.1)$$

In particular, the restriction of $D_i: \mathcal{A}_ \{ \mathcal{U} \} \rightarrow \mathcal{A}_* \{ \mathcal{U} \}$ has trivial kernel.*

We begin our construction with the algebra of *local differential polynomials*

$$\mathcal{B}_{loc} = \mathcal{B}^{(0)} = \mathcal{A} \oplus \mathcal{B}_*^{(0)} = \mathcal{A} \{ u^1, \dots, u^q \}.$$

Our goal is to construct a nonlocal differential algebra $\mathcal{B}^{(\infty)} = \mathcal{A} \oplus \mathcal{B}_*^{(\infty)} \supset \mathcal{B}_{loc}$ such that each total derivative $D_i: \mathcal{B}_*^{(\infty)} \rightarrow \mathcal{B}_*^{(\infty)}$ defines an invertible map everywhere except on the ordinary polynomials $f(x) \in \mathcal{A}$. The polynomials in $\mathcal{B}^{(\infty)}$ will, therefore, be polynomials involving expressions of the form $D_i^{-1}P$ where P is any local or nonlocal differential polynomial, e.g., $D_i^{-1}u$, $D_i^{-1}(u^2u_j)$, or even $D_i^{-1}(u_{ij}D_j^{-1}(u^2)D_k^{-1}(u_i^2))$, and so on. Our construction will be accomplished by inductively implementing the following construction.

At each step, we are given an infinite[†] collection of dependent variables

$$\mathcal{U}^{(m)} = \mathcal{U}_0 \cup \mathcal{U}_1 \cup \mathcal{U}_2 \cup \dots \cup \mathcal{U}_m,$$

which is the disjoint union of the subsets

$$\mathcal{U}_k = \{ u^\alpha \mid \text{depth } u^\alpha = k \}.$$

consisting of all dependent variables of a given *depth*. Roughly speaking, the depth of a variable will measure its “depth of nonlocality”. In particular, all the original variables in our local differential algebra $\mathcal{U}^{(0)} = \mathcal{U}_0 = \{ u^1, \dots, u^q \}$ have depth 0. We also assign a weight $w_\alpha = \text{wt } u^\alpha$ to each variable in $\mathcal{U}^{(m)}$. For simplicity, the original dependent variables $u^\alpha \in \mathcal{U}_0$ can have weight 1, although the initial weighting can be adapted to particular applications, as in [35].

We let $\mathcal{B}^{(m)} = \mathcal{A} \oplus \mathcal{B}_*^{(m)} = \mathcal{A} \{ \mathcal{U}^{(m)} \}$ denote the algebra of polynomials in the variables $u_j^\alpha = D^J u^\alpha$ for all $u^\alpha \in \mathcal{U}^{(m)}$ and $J \in \mathbb{Z}^p$. We define

$$\text{depth } u_j^\alpha = \text{depth } u^\alpha, \quad \text{wt } u_j^\alpha = \text{wt } u^\alpha.$$

Note that linearly nonlocal variables u_j^α for $u^\alpha \in \mathcal{U}_0$ will continue to have depth 0. The total derivatives act on $\mathcal{B}^{(m)}$ as derivations subject to the same rules (2.1).

Let $\mathcal{M}^{(m)}$ denote the set of x -independent monomials in $\mathcal{B}_*^{(m)}$, i.e., products of the form $M = u_{j_1}^{\alpha_1} \cdots u_{j_k}^{\alpha_k}$. Therefore, $\mathcal{B}^{(m)} = \mathcal{A}[\mathcal{M}^{(m)}]$ consists of finite linear combinations of monomials with coefficients in \mathcal{A} . We extend the notion of depth and weight to monomials in $\mathcal{M}^{(m)}$ by setting

$$\text{depth}(MN) = \max\{ \text{depth } M, \text{depth } N \}, \quad \text{wt}(MN) = \text{wt } M + \text{wt } N, \quad (4.2)$$

[†] Except the initial step, where we start with only finitely many dependent variables u^1, \dots, u^q .

whenever $M, N \in \mathcal{M}^{(m)}$. Thus, we write

$$\mathcal{M}^{(m)} = \mathcal{M}_0 \cup \mathcal{M}_1 \cup \dots \cup \mathcal{M}_m,$$

where \mathcal{M}_k denotes the set of monomials of depth k .

We describe the induction step. For each monomial $M_\gamma \in \mathcal{M}_m$, and each $1 \leq i \leq p$, we introduce a new dependent variable $u^{\gamma,i} \in \mathcal{U}_{m+1}$ of depth $m+1$ such that

$$D_i u^{\gamma,i} = M_\gamma. \quad (4.3)$$

Formally, we can write

$$u^{\gamma,i} = D_i^{-1} M_\gamma, \quad (4.4)$$

but it is better, for the time being, to regard each of these as a completely new dependent variable. Once the induction procedure is complete, we shall impose the relations implied by (4.3). The weight and depth of each new variable is

$$\text{wt } u^{\gamma,i} = \text{wt } M_\gamma, \quad \text{depth } u^{\gamma,i} = m+1 > \text{depth } M_\gamma.$$

The inductive step sets

$$\mathcal{U}_{m+1} = \{u^{\gamma,i}\}, \quad \mathcal{U}^{(m+1)} = \mathcal{U}^{(m)} \cup \mathcal{U}_{m+1}, \quad \mathcal{B}^{(m+1)} = \mathcal{A}\{\mathcal{U}^{(m+1)}\}.$$

Finally, we let

$$\mathcal{U}^{(\infty)} = \bigcup_{m=0}^{\infty} \mathcal{U}_m, \quad \mathcal{B}^{(\infty)} = \mathcal{A} \oplus \mathcal{B}_*^{(\infty)} = \mathcal{A}\{\mathcal{U}^{(\infty)}\}. \quad (4.5)$$

We can identify $\mathcal{B}^{(\infty)}$ as the inductive limit $\mathcal{B}^{(0)} \hookrightarrow \mathcal{B}^{(1)} \hookrightarrow \mathcal{B}^{(2)} \hookrightarrow \dots$ of subalgebras of progressively higher and higher depth. Note that $D_i^{-1} P \in \mathcal{B}_*^{(\infty)}$ is well-defined for any $P \in \mathcal{B}_*^{(\infty)}$. A nonlocal differential polynomial is said to be *homogeneous* of weight k if all its constituent monomials have weight k . We write $\mathcal{B}_k^{(\infty)}$ for the set of all homogeneous differential polynomials of weight k , and so $\mathcal{B}^{(\infty)} = \mathcal{A} \oplus \bigoplus_{k=1}^{\infty} \mathcal{B}_k^{(\infty)}$.

Of course, the differential algebra $\mathcal{B}^{(\infty)}$ contains a huge number of redundancies, since we have not yet taken into account the defining relations (4.3) of our nonlocal variables. Thus, we need to determine which of these nonlocal expressions are trivial, meaning that they vanish when evaluated upon any smooth function. In local differential algebra, one can prove triviality by evaluating the differential polynomial on all polynomial functions $u = p(x)$. In the nonlocal case, the class of polynomial functions is not appropriate because the inverse derivatives ∂_i^{-1} include a possible integration constant, and so are not uniquely defined on the space of polynomial functions. To check the vanishing of a nonlocal differential polynomial, one needs to keep track of a consistent choice of integration constants used to evaluate the nonlocal terms, and this rapidly becomes a difficult, if not intractable issue. A more enlightened approach is to introduce the following functions.

Definition 4.2. A function of the form $f(x) = p(x) e^{n \cdot x}$ in which $p(x)$ is a polynomial and $n = (n_1, \dots, n_p) \in \mathbb{Z}^p$ will be called a *polynomial-exponential function*. It will be called *positive* if $n > 0$, meaning $n_i > 0$ for $i = 1, \dots, p$. Let $\mathcal{F} = \{p(x) e^{n \cdot x} \mid n > 0\}$ denote the algebra of all positive polynomial-exponential functions.

The key property is that, in contrast to the space of polynomials, derivatives are invertible when restricted to the positive polynomial–exponential space \mathcal{F} .

Lemma 4.3. *The derivative $\partial_i: \mathcal{F} \rightarrow \mathcal{F}$ is a one-to-one linear map, and hence its inverse $\partial_i^{-1}: \mathcal{F} \rightarrow \mathcal{F}$ is uniquely defined. Consequently, if $P \in \mathcal{B}_k^{(\infty)}$ is any homogeneous nonlocal differential polynomial, its evaluation on $f \in \mathcal{F}$ gives a uniquely defined polynomial–exponential function $P[f] \in \mathcal{F}$. Moreover, evaluation commutes with (anti-)differentiation, so $D^J P[f] = \partial^J(P[f])$ for any $J \in \mathbb{Z}^p$.*

Proof: We recall the well-known formula

$$D_i^{-1}(P Q) = \sum_{j \geq 0} (-1)^j (D_i^j P) (D_i^{-j-1} Q). \quad (4.6)$$

If $P = p(x)$ is a polynomial, then the sum terminates, and gives an explicit formula for $D_i^{-1}(p(x) e^{n \cdot x})$. *Q.E.D.*

Definition 4.4. A homogeneous nonlocal differential polynomial $P \in \mathcal{B}_k^{(\infty)}$ is *trivial* $P[f] = 0$ for all $f = (f^1, \dots, f^q) \in \mathcal{F}^q$.

The fact that testing a nonlocal differential polynomial on all polynomial–exponentials is sufficient to detect triviality is a consequence of the fact that polynomial–exponential functions are sufficiently extensive to match any finite nonlocal jet. We let

$$\mathcal{I}^{(\infty)} = \bigoplus_{k=1}^{\infty} \mathcal{I}_k^{(\infty)}, \quad \text{where} \quad \mathcal{I}_k^{(\infty)} = \left\{ P \in \mathcal{B}_k^{(\infty)} \mid P[f] = 0 \text{ for all } f \in \mathcal{F} \right\} \quad (4.7)$$

denote the ideal of all trivial nonlocal differential polynomials. If $P \in \mathcal{I}^{(\infty)}$ then, by the last remark in Lemma 4.3, $D^J P \in \mathcal{I}^{(\infty)}$ for any $J \in \mathbb{Z}^p$, and hence $\mathcal{I}^{(\infty)} \subset \mathcal{B}_\star^{(\infty)}$ is a homogeneous nonlocal differential ideal. In particular, the defining relations (4.3) of our nonlocal variables belong to the ideal, meaning $D_i u^{\gamma, i} - M_\gamma \in \mathcal{I}^{(\infty)}$.

Finally, we define our *nonlocal differential algebra* to be the quotient algebra

$$\pi: \mathcal{B}^{(\infty)} \longrightarrow \mathcal{Q}^{(\infty)} = \mathcal{A} \oplus \mathcal{Q}_\star^{(\infty)} = \mathcal{B}^{(\infty)} / \mathcal{I}^{(\infty)}. \quad (4.8)$$

This algebra incorporates all the relations implied by (4.3) and their (anti-)derivatives. We easily check that D_i^{-1} is uniquely defined on all of $\mathcal{Q}^{(\infty)}$, and moreover forms an inverse to D_i when restricted to $\mathcal{Q}_\star^{(\infty)}$.

Theorem 4.5. *If $P \in \mathcal{Q}_\star^{(\infty)}$ and $1 \leq i \leq p$, then there exists a unique $S_i \in \mathcal{Q}_\star^{(\infty)}$ such that $P = D_i S_i$. We write $S_i = D_i^{-1} P$.*

In practical applications, the key issue is whether we can perform effective computations in the nonlocal differential algebra $\mathcal{Q}^{(\infty)}$. The main question is how to recognize whether a given differential polynomial $P \in \mathcal{B}^{(\infty)}$ lies in the differential ideal $\mathcal{I}^{(\infty)}$. We assume, without loss of generality, that $P \in \mathcal{I}_k^{(\infty)}$ is homogeneous. Roughly speaking, differentiating the polynomial P sufficiently often will eventually (and in a finite number of steps) produce a purely local differential polynomial $P^\star \in \mathcal{B}_{loc}$ with the property that

$P \in \mathcal{I}^{(\infty)}$ if and only if $P^* \in \mathcal{I}^{(\infty)}$. However, the latter will occur if and only if $P^* = 0$ in \mathcal{B}_{loc} , which is trivial to check.

In order to implement an algorithm, we extend our original term ordering to include all the nonlocal variables $\mathcal{U}^{(\infty)}$. We set $u^{\alpha,i} \prec u^{\beta,j}$ if and only if $\text{depth } u^{\alpha,i} < \text{depth } u^{\beta,j}$, or if $\text{depth } u^{\alpha,i} = \text{depth } u^{\beta,j} = m \geq 1$ and the corresponding monomials satisfy $M_\alpha \prec M_\beta$ in the induced term ordering on $\mathcal{M}^{(m)}$.

Although the full differential algebra $\mathcal{B}^{(\infty)}$ contains a gigantic number of different variables, any given polynomial $P(\dots x^i \dots u_J^\alpha \dots u_K^{\gamma,i} \dots) \in \mathcal{B}_k^{(\infty)}$ only depends on finitely many of them, and so all computations are finite in extent. Let $u_K^{\gamma,i} = D_K u^{\gamma,i}$ be the highest order variable occurring in P . We can assume $k_i > 0$, since otherwise we replace $u_K^{\gamma,i} \mapsto D_{K-e_i} M_\gamma$, in accordance with (4.3), which has smaller depth and hence appears earlier in the term ordering. We write out

$$P = P_n (u_K^{\gamma,i})^n + \sum_{\ell=0}^{n-1} P_\ell (u_K^{\gamma,i})^\ell, \quad (4.9)$$

where each coefficient $P_\ell \in \mathcal{B}^{(\infty)}$ depends on lower order variables $u_J^\alpha \prec u_K^\gamma$ and we assume $P_n \notin \mathcal{I}^{(\infty)}$. Since P_n is of lower order than P , the latter condition can be checked by the same algorithm. The derivative

$$D_i P = D_i P_n (u_K^{\gamma,i})^n + \sum_{\ell=0}^{n-1} [D_i P_\ell + (\ell+1) P_{\ell+1} D_K M_\gamma] (u_K^{\gamma,i})^\ell. \quad (4.10)$$

does not appear earlier in the term ordering. However, the combination

$$P_n D_i P - P D_i P_n = Q = \sum_{\ell=0}^{n-1} Q_\ell (u_K^{\gamma,i})^\ell \quad (4.11)$$

is of lower order than P in $u_K^{\gamma,i}$. The induction step claims that $P \in \mathcal{I}^{(\infty)}$ if and only if $Q \in \mathcal{I}^{(\infty)}$, and hence the same algorithm can be used on Q . Since Q is of lower order, we use the same algorithm on Q , and so eventually — but in a finite number of steps — reducing to a purely local differential polynomial, as desired.

To prove the claim, equation (4.11) implies that, for any $f \in \mathcal{F}$,

$$P_n[f] \partial_i P[f] - P[f] \partial_i P_n[f] = Q[f] = 0.$$

Therefore,

$$\frac{\partial}{\partial x^i} \left(\frac{P[f]}{P_n[f]} \right) = 0,$$

and hence $P[f] = 0$ or $P_n[f] = 0$. Now, if $P_n \notin \mathcal{I}^{(\infty)}$ is nontrivial, then the jets of polynomial-exponential functions that solve the nonlocal differential equation $P_n[f] = 0$ forms a proper subvariety, and $P[f] = 0$ everywhere outside this subvariety, which, by continuity, implies $P[f] = 0$ for all $f \in \mathcal{F}$, and proves the claim. Of course, the implementation of this algorithm might be quite lengthy, and so developing more efficient algorithms would be an interesting research topic.

5. Evolutionary Vector Fields and Symmetries.

In this section we extend the space of evolutionary vector fields to our nonlocal differential algebra. Since they are defined as commutators, the Jacobi Identity will be automatically valid.

Definition 5.1. A *evolutionary vector field* \mathbf{v} on a differential algebra $\mathcal{B} = \mathcal{A} \oplus \mathcal{B}^{(0)}$ is a derivation $\mathbf{v}: \mathcal{B} \rightarrow \mathcal{B}$, with $\mathcal{A} \subset \ker \mathbf{v}$, while $[\mathbf{v}, D_i] = 0$ commutes with all total derivatives.

Remark: If we drop the hypothesis $\mathcal{A} \subset \ker \mathbf{v}$ then the only additional derivations that commute with the total derivatives are the partial derivatives $\partial/\partial x^j$; see [33].

Therefore, an evolutionary vector field \mathbf{v} must satisfy

$$\begin{aligned} \mathbf{v}(P + Q) &= \mathbf{v}(P) + \mathbf{v}(Q), & \mathbf{v}(x^i) &= 0, \\ \mathbf{v}(PQ) &= \mathbf{v}(P)Q + P\mathbf{v}(Q), & [\mathbf{v}, D_i] &= 0, \end{aligned} \tag{5.1}$$

for all $P, Q \in \mathcal{B}$ and $i = 1, \dots, p$. Each evolutionary vector field is uniquely specified by its action $\mathbf{v}(u_j^\alpha)$ on the coordinate variables. We denote the space of evolutionary vector fields by $\mathcal{V} = \mathcal{V}(\mathcal{B})$. The commutator bracket

$$[\mathbf{v}, \mathbf{w}](P) = \mathbf{v}(\mathbf{w}(P)) - \mathbf{w}(\mathbf{v}(P)), \quad P \in \mathcal{B}.$$

between two evolutionary vector fields endows \mathcal{V} with the structure of a Lie algebra, satisfying the usual skew symmetry and Jacobi identities. The proof of the latter is elementary.

Warning: The space of evolutionary vector fields is *not* a \mathcal{B} module. The product $P\mathbf{v}$ of $P \in \mathcal{B}$ and $\mathbf{v} \in \mathcal{V}$ does not commute with total differentiation.

Given an evolutionary vector field \mathbf{v} , we define its *characteristic* $Q \in \mathcal{B}^q$ to have components

$$\mathbf{v}(u^\alpha) = Q^\alpha, \quad \alpha = 1, \dots, q.$$

The commutation condition implies

$$\mathbf{v}(u_J^\alpha) = \mathbf{v}(D_J u^\alpha) = D_J \mathbf{v}(u^\alpha) = D_J Q^\alpha$$

for all positive multi-indices $J \geq 0$. Thus, in the local situation, an evolutionary vector field is uniquely determined by its characteristic. This basic fact is *not* true in nonlocal differential algebras — there are nonzero evolutionary vector fields with zero characteristic — and this observation motivates the following key definition.

Definition 5.2. An evolutionary vector field γ is called a *K-ghost* for some $K \in \mathbb{Z}^p$ if $\gamma(u_L^\alpha) = 0$ for all $L \geq K$ and $\alpha = 1, \dots, q$.

There are no ghost vector fields in a local differential algebra \mathcal{B}_{loc} because each evolutionary vector field is uniquely determined by its characteristic Q . There are, however, positive ghost vector fields; for example the vector field with characteristic $Q = 1$ is a K -ghost for any positive multi-index $K > 0$.

Example 5.3. Let us see how the existence of ghost vector fields serves to resolve the Jacobi identity paradox in (3.1). Surprisingly, the problem is *not* with the nonlocal vector field \mathbf{z} with characteristic $D_x^{-1}u$, but rather the local commutator $[\mathbf{v}, \mathbf{w}]$ corresponding to the vector fields with characteristics 1 and u_x , respectively. While $[\mathbf{v}, \mathbf{w}] = 0$ on the local differential algebra \mathcal{B}_{loc} , it is, in fact, a ghost vector field on a nonlocal differential algebra. thus, surprisingly, in a nonlocal setting, the group of translations is not abelian!

The action of the vector fields on the local variables does not uniquely specify their action on the nonlocal variables, due to the presence of possible integration constants. However, as we have seen, the integration constants do not play a significant role in the resolution of the Jacobi identity paradox, and so we shall fix all the integration constants to be zero by default. Therefore,

$$\mathbf{v}(u_k) = D_x^k(1) = \chi_k(x) \equiv \begin{cases} 0 & k > 0, \\ \frac{x^{-k}}{(-k)!} & k \leq 0, \end{cases} \quad \mathbf{w}(u_k) = D_x(u_k) = u_{k+1}. \quad (5.2)$$

Since $\mathbf{v}(u_k)$ only depends on x , we have $\mathbf{w}(\mathbf{v}(u_k)) = 0$, and so

$$[\mathbf{v}, \mathbf{w}](u_k) = \mathbf{v}(u_{k+1}) = \chi_{k+1}(x).$$

Therefore, $[\mathbf{v}, \mathbf{w}] = \gamma$ is a ghost vector field that satisfies

$$\gamma(u_k) = \chi_{k+1}(x) = \begin{cases} 0 & k \geq 0, \\ \frac{x^{-k-1}}{(-k-1)!} & k < 0. \end{cases}$$

This ghost provides the missing term in the Jacobi identity (3.1). Indeed,

$$[\mathbf{z}, \gamma](u) = -\gamma(\mathbf{z}(u)) = -\gamma(D_x^{-1}u) = -1.$$

In [34], we introduced a “ghost calculus” for general nonlocal evolutionary vector fields. The first remark is that only evolutionary vector fields that depend on the independent variables can be ghosts. Indeed, if γ is a K -ghost, then

$$\gamma(u_J) = D^{J-K}\gamma(u_K) = 0, \quad J \geq K.$$

Therefore, if $\gamma(u_I) = P_I$ and $J \geq 0$ is any positive multi-index such that $J + I \geq K$, then

$$0 = \gamma(u_{J+I}) = D^J\gamma(u_I) = D^J P_I,$$

and we know that $\ker D^J \subset \mathcal{A}$, so that P_I is a function of x only.

Lemma 5.4. *An evolutionary vector field γ is a K -ghost for some $K \in \mathbb{Z}^p$ if and only if $\gamma(u_J^\alpha) = p_J^\alpha(x)$ is a polynomial function of x^1, \dots, x^p .*

Definition 5.5. Given a multi-index $K \in \mathbb{Z}^p$, define

$$\chi_K = D^K(1) = \begin{cases} \frac{x^{-K}}{(-K)!}, & K \leq 0, \\ 0 & \text{otherwise,} \end{cases} \quad \text{where} \quad (-K)! = \prod_{\nu=1}^p (-k_\nu)!. \quad (5.3)$$

Definition 5.6. Given a multi-index $J \in \mathbb{Z}^p$, define the basis ghost vector field γ_J so that $\gamma_J(u_K) = \chi_{J+K}$, which is a K -ghost for any $K + J > 0$.

Proposition 5.7. *Every ghost vector field is a linear combination of the basis ghosts, $\gamma = \sum_J c_J \gamma_J$, where the $c_J \in \mathbb{R}$ are constants.*

The summation in Proposition 5.7 can be infinite. However, only certain “configurations” of the nonzero coefficients c_J are allowed in order that γ map $\mathcal{Q}^{(\infty)}$ to $\mathcal{Q}^{(\infty)}$.

Let us formulate the results in the one dependent variable case where $u \in \mathbb{R}$, and so $q = 1$. (The multi-variable case can be found in [34].) Let us split the space of evolutionary vector fields $\mathcal{V} = \mathcal{V}_x \oplus \mathcal{V}_*$ where \mathcal{V}_x denotes the space of purely x -dependent vector fields, so $\mathbf{v}(u_K) = p_K(x) \in \mathcal{A}$. In the polynomial category, every $\mathbf{v} \in \mathcal{V}_x$ is a ghost vector field. The remainder, \mathcal{V}_* , consists of u -dependent vector fields, where $\mathbf{v}(u_K) = D^K Q \in \mathcal{B}_*^{(\infty)}$. Since $\ker D^K = \{0\}$ on $\mathcal{B}_*^{(\infty)}$, the evolutionary vector fields in \mathcal{V}_* are uniquely determined by their characteristics $Q = \mathbf{v}(u)$, and we write $\mathbf{v} = \mathbf{v}_Q$ as in the local category. Thus, to re-emphasize: only the x -dependent vector fields can be ghosts and hence cause any difficulty in the non-local category.

Corollary 5.8. *If $\mathcal{B}^{(\infty)}$ is a polynomial differential algebra, then any evolutionary vector field $\mathbf{v} \in \mathcal{V}$ can be written a linear combination of basis ghosts and a u -dependent vector field $\mathbf{v}_Q \in \mathcal{V}_*$:*

$$\mathbf{v} = \mathbf{v}_Q + \sum_J c_J \gamma_J, \quad \text{whereby} \quad \mathbf{v}(u_K) = D^K Q + \sum_J c_J \chi_{K+J}. \quad (5.4)$$

To implement a calculus of evolutionary vector fields, we identify a vector field with its “characteristic”. The characteristic of the evolutionary vector field \mathbf{v}_Q is, as usual, Q . The characteristic of the ghost vector field γ_J will be formally written as χ_J . In this manner, every nonlocal vector field (5.4) has a unique characteristic

$$S = Q + \sum_J c_J \chi_J. \quad (5.5)$$

In particular, a local vector field with polynomial characteristic x^K becomes a ghost characteristic $K! \chi_{-K}$. Indeed, one can, again in the one dependent variable case, replace all polynomials $x^K \mapsto K! \chi_{-K}$ wherever they appear in the characteristic (5.5). The only place true ghosts appear, i.e., χ_J with $J \not\leq 0$, is in the u -independent terms in the summation. Only when the vector field has been evaluated on a nonlocal differential polynomial are we allowed to replace the ghost functions χ_J by their actual formulas (5.3).

In this calculus, the product rule $x^J x^K = x^{J+K}$ becomes the ghost product rule

$$\chi_K \chi_J = \binom{-K - J}{-K} \chi_{K+J}, \quad J \geq 0. \quad (5.6)$$

The product makes sense as long as one of the multi-indices is non-negative, provided we adopt the Pochhammer definition

$$\binom{J + K}{J} = \frac{1}{J!} \prod_{\nu=1}^p \prod_{i=0}^{j_\nu-1} (j_\nu + k_\nu - i) \quad (5.7)$$

for the multinomial symbol. And, indeed, only such products will appear when we evaluate commutators and apply vector fields to nonlocal differential polynomials.

The precise ghost calculus rules for computing the commutators of ghost characteristics will now be described. The commutators of ordinary characteristics $[Q, R]$ for $Q, R \in \mathcal{B}_*^{(\infty)}$ follow the same rules (2.5) as in the local case, where we replace the multiplication of monomials by the ghost multiplication rule (5.6). Secondly, since ghosts do not involve the dependent variables, they mutually commute:

$$[\chi_J, \chi_K] = 0. \quad (5.8)$$

Finally, the ghost characteristics χ_J act as derivations on the ordinary characteristics:

$$[\chi_K, QR] = Q \chi_K(R) + R \chi_K(Q).$$

Thus, we only need to know how to commute ghosts and derivative coordinates,

$$[\chi_J, u_K] = \chi_{J+K} \quad (5.9)$$

in order to compute in the ghost characteristic space.

Example 5.9. Let us revisit Example 5.3. The three ghost characteristics are

$$1 = \chi_0, \quad u_x = u_1, \quad D_x^{-1}u = u_{-1}.$$

Then the three terms are

$$\begin{aligned} [\chi_0, [u_1, u_{-1}]] &= 0, & [u_1, [u_{-1}, \chi_0]] &= -[u_1, \chi_{-1}] = \chi_0, \\ [u_{-1}, [\chi_0, u_1]] &= [u_{-1}, \chi_1] = -\chi_0. \end{aligned}$$

The sum of these three terms is 0, and so the Jacobi paradox is resolved.

Example 5.10. The first Jacobi identity paradox that was found in [34], while working on the symmetry algebra of the KP equation, [9, 10, 15, 29], was more complicated than (3.1). Here $p = 2$, with independent variables x, y , and $q = 1$, with dependent variable u . Consider the vector fields with characteristics y, yu_x and $u_x D_x^{-1}u_y$. As in Section 3, without the introduction of ghost terms, the Jacobi sum

$$[y, [u_x D_x^{-1}u_y, yu_x]] + [yu_x, [y, u_x D_x^{-1}u_y]] + [u_x D_x^{-1}u_y, [yu_x, y]] \quad (5.10)$$

equals $-2yu_x$, not zero. In this case, the three ghost characteristics are

$$y = \chi_{0,-1}, \quad yu_x = \chi_{0,-1} u_{1,0}, \quad u_x D_x^{-1}u_y = u_{1,0} u_{-1,1}.$$

Then,

$$\begin{aligned} [\chi_{0,-1}, \chi_{0,-1} u_{1,0}] &= 2\chi_{1,-2}, & [\chi_{0,-1}, u_{0,0}u_{1,0}] &= \chi_{0,-1} u_{1,0}, \\ [\chi_{0,-1} u_{1,0}, u_{1,0} u_{-1,1}] &= D_{u_{1,0} u_{-1,1}}(\chi_{0,-1} u_{1,0}) - \chi_{0,-1} D_x(u_{1,0} u_{-1,1}) = u_{0,0}u_{1,0}, \end{aligned}$$

and so,

$$\begin{aligned} [u_{1,0} u_{-1,1}, 2\chi_{1,-2}] &= -2\chi_{0,-1} u_{1,0}, & [u_{1,0} u_{-1,1}, \chi_{0,-1}] &= -\chi_{-1,0} u_{1,0}, \\ [\chi_{0,-1} u_{1,0}, -\chi_{-1,0} u_{1,0}] &= \chi_{0,-1} u_{1,0}. \end{aligned}$$

The latter three terms add up to 0, and so the Jacobi identity is valid in the ghost framework.

6. Conclusions.

In this paper, I have introduced a general framework for a nonlocal differential algebra that will handle quite general nonlocal polynomial expressions. Several further topics of investigation are now of importance:

- (a) A complete re-evaluation of earlier work on nonlocal symmetries of local and nonlocal partial differential equations is required. A proper understanding of the hitherto undetected ghost terms needs to be properly incorporated into earlier results, including the study of recursion operators and master symmetries, all of which typically involve nonlocal operations.
- (b) The framework for the geometric and algebraic study of nonlocal symmetries and nonlocal differential equations requires further development. The establishment of a complete nonlocal variational calculus on the nonlocal differential algebra $\mathcal{Q}^{(\infty)}$, including nonlocal conservation laws, [40] and a nonlocal form of Noether's Theorem, [32, 33], would be a very worthwhile project for both theoretical developments and practical applications.
- (c) Implementation of the nonlocal ghost calculus in standard computer algebra packages would help a lot in these investigations.

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