THE CONSTRUCTION OF SPECIAL SOLUTIONS TO PARTIAL DIFFERENTIAL EQUATIONS

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Almost all the methods devised to date for constructing particular solutions to a partial differential equation can be viewed as manifestations of a single unifying method characterized by the appending of suitable "side conditions" to the equation, and solving the resulting overdetermined system of partial differential equations. These side conditions can also be regarded as specifying the invariance of the particular solutions under some generalized group of transformations.

In the study of partial differential equations, the discovery of explicit solutions has great theoretical and practical importance. In the case of linear systems, general solutions can be built up by superposition from separable solutions; for nonlinear systems, explicit solutions are used as models for physical or numerical experiments, and often reflect the asymptotic behavior of more complicated solutions. Over the years, a variety of methods for finding these special solutions by reducing the partial differential equation to one or more ordinary differential equations have been devised. Included are the method of group-invariant solutions popularized by Ovsiannikov [1], but due originally to Lie [2] (see also ref. [11]), the "nonclassical method" for group-invariant solutions due to Bluman and Cole [3], and its recent generalization by the authors [4], the method of partially invariant solutions of Ovsiannikov [1], and the general method of separation of variables, for both linear systems as well as certain nonlinear equations such as Hamilton–Jacobi equations. The one common theme of all these methods has been the appearance of some form of group invariance.

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The purpose of this note is to explain how all the above methods, as well as many others, can all be unified and significantly generalized by the concept of *a differential equation with side conditions.* By this we mean that to determine special solutions to a given system of partial differential equations one proceeds by appending one or more auxiliary differential equations, which we call "side conditions". The solutions themselves will then be found by solving the entire system of differential equations consisting of both the original system along with the prescribed side conditions. The main difference between the various special methods devised for finding explicit solutions is then only the form or complexity of the relevant side conditions. The most important conclusion to be drawn from this approach is that the unifying theme behind finding special solutions to partial differential equations is *not,* as is commonly supposed, group theory, but rather the more analytic subject of overdetermined systems of partial differential equations. Thus the key question becomes not which groups are relevant to a given system of partial differential equations, but rather *which side conditions are admissible, thereby providing genuine solutions of the system?* What is now required is an algorithmic method of determining these compatible side conditions, and then the corresponding special solutions. In this light, the group-theoretic methods alluded to above can be regarded as special techniques that allow one to construct particular classes of compatible side conditions; the general set of side conditions is much larger. This is not to say that the proposed method is meant to supplant the popular group-theoretic methods in use, but rather to be utilized as a unifying frame-work in which to compare and interpret all these different techniques. Nor does it preclude the discovery of other formalisms for constructing special solutions, although it certainly can provide motivation for the development of additional new classes of methods for determining explicit solutions. It also helps explain recent result of Kalnins and Miller [5,6] on separation of variables, in which the group-theoretic interpretation relevant to simpler systems seems to be no longer valid.

Actually, one can provide a group-theoretic explanation of the side conditions provided one uses the theory of generalized symmetries (also known as Lie–Bäcklund transformations), although in many cases this appears to us as a somewhat artificial re-interpretation of the basic issue. Using this point of view, the side conditions relevant to group-invariant solutions come from ordinary geometrical groups of transformations, those for partially invariant solutions from first order generalized symmetries which are not equivalent to geometrical symmetries and those for separable solutions from second order generalized symmetries. Higher order symmetries lead to yet more general types of ansatz for special solutions. What *has* been lost is any underlying symmetry connection with the system of partial differential equations itself – there are no a priori restrictions on the groups under consideration. (See also ref. [4] for the geometrical symmetry case.)

Rather than try to develop a general theory here, we have chosen to illustrate the basic concepts by a series of examples of the different methods for constructing solutions, each reinterpreted in the light of the unifying concept of a differential equation with side conditions. Using these as a launchpad, the astute reader will no doubt be able to envisage the form which the general theory must take. Besides, in the words of de Tocqueville, "God doesn't need general theories – He knows all the special cases!"

We begin with Lie's classical theory of group-invariant solutions, illustrated by the similarity solution of the heat equation

$$u_t = u_{xx} \,. \tag{1}$$

Consider the one-parameter group of scaling transformations $(x, t, u) \mapsto (\lambda x, \lambda^2 t, u)$, $\lambda > 0$. This is a classical symmetry group of the heat equation, in the sense that it takes solutions to solutions. The condition that a solution be invariant under this group (i.e. a "similarity solution") can be expressed in differential form

$$xu_x + 2tu_t = 0 \,. \tag{2}$$

Thus the similarity solutions to the heat equation can be determined as the solutions to the overdetermined system consisting of the heat equation itself (1) together with the side condition (2) reflecting the scaling invariance of the desired solutions. To solve this system (1), (2), one can proceed to first solve (2) using the method of characteristics, leading to the fact that $u = u(x/\sqrt{t})$ is a function of the similarity variable $\xi = x/\sqrt{t}$ only. Substituting into (1), we see that u satisfies the ordinary differential equation

$$u'' + \xi u'/2 = 0 \,,$$

primes denoting derivatives with respect to ξ. This leads to the general scale-invariant solutions

$$u = c_1 \operatorname{erf}(x/2\sqrt{t}) + c_2 \,,$$

where erf is the standard error function.

In the nonclassical method introduced by Bluman and Cole, one does not require that the group be a symmetry group of the original system, but, less restrictively, that it be a symmetry group of the system supplemented by the side conditions prescribing the group invariance of the desired solutions. Rather than treat the heat equation, since all the solutions obtained by the non-classical method already appear among the classical group-invariant solutions in this case, cf. ref. [3], we will use a less trivial example. Consider the nonlinear wave equation

$$u_{tt} = uu_{xx} \,, \tag{3}$$

whose classical symmetry group has been computed in ref. [7], p. 301. The one-parameter group G with infinitesimal generator $\mathbf{v} = 2t\partial_x + \partial_t + 8t\partial_u$ does not appear among the classical symmetries, and so is not a candidate for the usual method of finding group-invariant solutions. Nevertheless, we can find G-invariant solutions as follows. A function $u = f(x, t)$ is invariant under the group generated by $\mathfrak{v}$ if and only if it satisfies the side condition

$$8t = 2tu_x + u_t . \tag{4}$$

One easily checks that the combined pair of differential equations (3), (4) *is* invariant under the group G. This is precisely the requirement needed to apply the non-classical method, and hence we can find G-invariant solutions to (3) by solving an ordinary differential equation. The general solution of (4) is

$$u = 4t^2 + w(x - t^2) ,$$

where w depends on the invariant $\xi = x - t^2$. Substituting into (3) we see that w must satisfy the ordinary differential equation

$$ww'' + 2w' = 8 ,$$

where primes indicate derivatives with respect to ξ. This last equation can be integrated by Lie's method for ordinary differential equations using the obvious scaling and translational symmetries, but we will not pursue this here. For each solution $w = h(\xi)$, we obtain an explicit G-invariant solution u of the nonlinear wave equation (3); most of these do not appear among the group-invariant solutions computed using the ordinary symmetry groups of (3), and are thus genuinely new invariant solutions not obtainable by the classical method.

In ref. [4] it is shown how to generalize this method to an arbitrary group of transformations on the underlying space of independent and dependent variables. For example, returning to the heat equation (1), the one-parameter group G generated by the vector field $\mathfrak{v} = t\partial_t - x\partial_x - 3x^3\partial_u$ is not an ordinary symmetry group of the heat equation. Nor is it of the form amenable to the non-classical method given by Bluman and Cole [3]. (Indeed, using their notation on p. 1041, we would have $X = -x/t$, $U = -3x^3/t$ [cf. their eq. (90)], but these two functions do not satisfy their defining equations (94)–(96).) Nevertheless, there do exist G-invariant solutions of the heat equation, and we can construct them as follows. The relevant side condition is

$$tu_t - xu_x + 3x^3 = 0 ,$$

whose general solution is

$$u = x^3 + w(xt) ,$$

where w is a function of $\xi = xt$. Substituting into the heat equation, we obtain

$$xw' = t^2w'' + 6x , \tag{5}$$

which is *not* an ordinary differential equation for w as a function of ξ. (Indeed, if it were, then G would necessarily satisfy Bluman and Cole's non-classical conditions!) However, treating x and ξ as independent variables (valid provided $t \neq 0$), we have

$$x^3(w' - 6) = \xi^2w'' .$$

Since w is a function of ξ only, this latter equation is satisfied for all x and ξ if and only if w satisfies the pair of ordinary differential equations

$$w'' = 0, \quad w' = 6 .$$

These are compatible, with solution $w = 6\xi + c$, where c is an arbitrary constant. Thus we obtain a one-parameter family of G-invariant solutions to the heat equation

$$u = x^3 + 6xt + c ,$$

which do not appear among the classical group-invariant solutions (although they are, of course, linear combinations of two such solutions).

Before leaving side conditions arising from group-invariance of solutions, it is worth pointing out that if one has found a solution to a system of differential equations, one can always devise a group that will lead to the given solution by an application of the above method. This is because any function $u = f(x, t)$ is invariant under a multitude of groups acting on the space of variables (x, t, u), and, as the generalized non-classical method of ref. [4] does not impose *any* conditions on the group, any one of these groups will lead to the given solution. However, this reasoning is more than likely done *a posteriori*, and therefore of limited practical importance. Of more interest is the question of which groups G lead to actual solutions; in other words, when are the side conditions expresing the G-invariance of a solution compatible with the system under consideration. Obviously not any group will do, but the problem of determining precisely which ones are valid is no doubt very difficult, even for the simplest systems. (Note that it is even possible to devise simple examples of classical symmetry groups of a system, to which no G-invariant solution can be found. For instance the equation $u_t + u_x = 1$ admits the one-parameter group $(x, t, u) \mapsto (x + \epsilon, t + \epsilon, u)$, $\epsilon \in \mathbb{R}$, but there are no solutions to it which are invariant under this group.)

Partially invariant solutions can also be treated by this general approach, using side conditions in the form of first order differential equations, but which are not of the form prescribing group-invariance in the classical sense. (Indeed the general method of Ovsiannikov rests on the theory of over-determined systems of differential equations!) As an illustrative example we consider the system

$$u_y = v_x \,, \quad uu_x = v_y \,, \tag{6}$$

describing the transonic flow of a gas, which is treated by Ovsiannikov (ref. [1], p. 286); note that these are equivalent to the nonlinear wave equation $u_{yy} = \frac{1}{2}(u^2)_{xx}$ (see below). For the two-parameter symmetry group of translations $(x, y, u, v) \mapsto (x + \epsilon, y, u, v + \tilde{\epsilon})$ generated by ∂_x and ∂_v, a partially invariant solution of rank 1 and defect 1 has general form $u = \varphi(y)$, $v = \psi(x, y)$. Equivalently, we can prescibe this class of solutions by appending the single side condition

$$u_x = 0 \,. \tag{7}$$

The resulting over-determined system (6), (7) has the general solution

$$u = c_1 y + c_0 \,, \quad v = c_1 x + c_2 \,,$$

which is the most general such partially invariant solution. Note that we could reinterpret (7) as expressing the invariance of the desired solutions under the one-parameter generalized group generated by the generalized vector field $\mathbf{v} = u_x \partial_u$; the corresponding group transformations are obtained by solving the system of evolution equations

$$\partial u/\partial\epsilon = u_x \,, \quad \partial v/\partial\epsilon = 0 \,,$$

with corresponding group law

$$(u(x,y), v(x,y)) \mapsto (u(x + \epsilon, y), v(x,y)) \,.$$

(See refs. [2,8] for the general theory of generalized symmetries.) Note that this group is not equivalent to a geometrical group acting locally on the variables (x, y, u, v), but is truly "non-local". Also, the group is not a symmetry group of the system (6) per se, so we are in a "generalized version" of the non-classical symmetry group method. The partially invariant solutions considered by Ovsiannikov (ref. [4], p. 286), have a similar interpretation using the side condition $xu_x + yu_y = 0$ reflecting their invariance under the generalized vector field $(xu_x + yu_y)\partial_u$. It is also a relatively easy matter to extend Ovsiannikov's method to include solutions which are partially invariant under non-classical or even more general transformation groups.

The third major class of special solutions to partial differential equations are those obtained through separation of variables. The simplest form of separation of variables is *additive separation*, and indeed all other modes of separation known to date can, by a suitable transformation, be reduced to additive separation. For example, in the case of the heat equation (1), we look for solutions of the form

$$u(x, t) = f(x) + g(t) \,. \tag{8}$$

Substituting (8) into (1), we are left with two ordinary differential equations

$$f'' = \lambda \,, \quad g' = \lambda \,,$$

in terms of a separation constant λ. This immediately leads to the three-parameter family of additively separable solutions of the heat equation:

$$u(x, t) = c_2 x^2 + c_1 x + 2c_2 t + c_0 \,,$$

where $\lambda = 2c_2$. As with partially invariant solutions, the ansatz (8) does not arise from the invariance of the separable solutions under some geometrical group of transformations on the space of variables (x, t, u). (Indeed, this would lead to only a single ordinary differential equation for the invariant solutions, and here we have two ordinary differential equations.) However, we can recover (8) from the second-order side condition

$$u_{xt} = 0 \,, \tag{9}$$

in other words, additively separable solutions of the heat equation are found by solving the system consisting of the heat equation (1) along with the second order side condition (9). As with (7), we can interpret (9) as requiring the desired solutions to be invariant under the generalized symmetry group with infinitesimal generator $\mathbf{v} = u_{xt}\partial_u$; the corresponding non-local group transformations are found by solving the associated evolution equation

$$\partial u/\partial \epsilon = u_{xt}, \quad u|_{\epsilon=0} = u_0(x,t). \tag{10}$$

If the initial data u_0 is separable, then the solution to (10) does not depend on the group parameter ϵ, reflecting its invariance under the group. The recent work of Kalnins and Miller [5,6] on additive separation for linear equations is, essentially, a detailed analysis of these special types of side conditions.

Multiplicative separation for the heat equation comes from the ansatz

$$u(x,t) = f(x)g(t). \tag{11}$$

This can be reduced to additive separation by rewriting the heat equation in terms of $v = \log u$. Alternatively, one can characterize (11) via the side condition

$$uu_{xt} = u_x u_t. \tag{12}$$

(See ref. [9] for a more precise statement.) Combining (1) and (12), we immediately deduce that a common solution $u(x,t)$ must satisfy the pair of ordinary differential equations

$$uu_{xxx} = u_x u_{xx}, \quad uu_{tt} = u_t^2,$$

the latter following from differentiating (12) with respect to t. These can both be integrated once, leading to the more familiar separation equations

$$u_{xx} + \lambda u = 0, \quad u_t + \lambda u = 0,$$

where (1) has required both integration constants to be the same separation constant λ. For $\lambda > 0$ we recover the standard solutions

$$u = e^{-\lambda t}\cos(\sqrt{\lambda}\,x), \quad u = e^{-\lambda t}\sin(\sqrt{\lambda}\,x),$$

to the heat equation. Again, we could reinterpret (12) as specifying the invariance of these solutions under the generalized symmetry group generated by $\mathbf{v} = (uu_{xt} - u_x u_t)\,\partial_u$.

A more interesting, "nonclassical" form of multiplicative separation can be found in the equation

$$u_{tt} = u_{xx} + \beta(u^2)_{xx} + \gamma u_{xxtt}, \quad \beta, \gamma \text{ constant},$$

which arises in the vibration of rods. If we set

$$u(x,t) = v(t)\,w(x) - 1/2\beta,$$

arising from the side condition

$$uu_{xt} - u_x u_t + u_{xt}/2\beta = 0,$$

then we find that v and w satisfy the pair of ordinary differential equations involving a separation constant λ

$$v_{tt} = \lambda v^2, \quad \beta(w^2)_{xx} = \lambda(w - \gamma w_{xx}),$$

both of whose solutions can be explicitly written in terms of elliptic functions. (To integrate the second equation, set $\tilde{w} = w + \lambda\gamma/2\beta$, and multiply by $(\tilde{w}^2)_{xx} = 2\tilde{w}\tilde{w}_x$.) Included among these are the elementary rational solutions

$$u = (1/2\beta)\{[(x+c)^2 + 3\gamma]/(t+d)^2 - 1\},$$

where c and d are arbitrary constants. An interesting question is whether these simple solutions, which appear among the solutions invariant under the nonlocal group generated by the vector field $\mathbf{v} = (uu_{xt} - u_x u_t + u_{xt}/2\beta)\,\partial_u$, could, in fact, have been found by *local* (but possibly non-classical) group methods. Of course, according to ref. [4], any *one* of these solutions could be found by use of some local group, but it is not too hard to see that the entire two-parameter family could not have come from a single local group. Indeed, suppose they were simultaneously invariant under some one-parameter group with infinitesimal generator $\mathbf{v} = \xi\partial_x + \tau\partial_t + \varphi\partial_u$ (where ξ, τ, φ depend just on x, t and u). Solving for the parameter d, we would have

$$d = h(x,t,u,c)$$

$$\equiv -t + \{[(x+c^2) + 3\gamma]/(u + 1/2\beta)\}^{1/2},$$

where h would necessarily be invariant under the group, and hence satisfy

$$\mathbf{v}(h) = \xi h_x + \tau h_t + \varphi h_u = 0$$

for all x, t, u, c. Since ξ, τ, φ do not depend on the parameter c, this would only be possible if, for each fixed x, t, u, the derivatives h_x, h_t, h_u were linearly dependent functions of c. A tedious computation using wronskians shows that this is *not* the case, hence the given two-parameter family of solutions cannot come from a single local group of transformations. (The same argument proves that any *one*-parameter family of solutions *does* come from a local one-parameter group via the non-classical method of (8). Moreover, if $\gamma = 0$, so we are reduced to a nonlinear wave equation, any one-parameter family of rational solutions can be determined using the classical symmetry groups, of which there are several more in this case.) A similar non-classical separation of variables is discussed in ref. [10] for the equation of a nonlinear string.

For general differential equations, one can now easily envisage more general second order, and even higher order side conditions to append in the hopes of determining more general classes of solutions. Two questions are apparent: (1) Which side conditions are *admissible* in the sense that there do exist solutions to the combined system of equations plus side conditions? (2) Which side conditions are *soluble* in the sense that the combined system is in some way easier to solve than the original partial differential equation? We will not attempt to answer these questions here, but merely provide two final examples to illustrate the possibilities which lie beyond simple group-invariance and separability. First consider the two-dimensional heat equation

$$u_t = u_{xx} + u_{yy} . \tag{13}$$

We append the third order side condition

$$u_{xyt} = 0 , \tag{14}$$

(implying invariance under the group generated by $\mathbf{v} = u_{xyt}\partial_u$), the general solution of which has the "semi-separable" form

$$u(x,y,t) = f(x,y) + g(x,t) + h(y,t) . \tag{15}$$

Differentiating (13) and using (14), we see that u must satisfy three partial differential equations

$$0 = u_{xxxy} + u_{xyyy} , \quad u_{xtt} = u_{xxxt} , \quad u_{ytt} = u_{yyyt} ,$$

each of which has one fewer independent variable than the original eq. (13). A little manipulation shows that the functions in (15) satisfy the equations

$$f_{xx} + f_{yy} = \alpha(x) - \beta(y), \quad g_t - g_{xx} = \gamma(t) - \alpha(x),$$

$$h_t - h_{yy} = \beta(y) - \gamma(t) ,$$

involving *separation functions* (as opposed to separation constants) α, β, and γ. One can now write down a host of such solutions, much more general than the additively separable solutions. The multiplicative analog of this "semi-separation" proceeds similarly (and, indeed, is perhaps of even greater interest!).

Secondly, for the nonlinear wave equation

$$u_{tt} = \tfrac{1}{2}(u^2)_{xx} ,$$

equivalent to the system (6), we can generate a more interesting class of solutions *not* obtainable by partial invariance by appending the second order side condition

$$u_{tt} = 2\alpha ,$$

where α is a constant. The resulting over-determined system is easy to solve, leading to the new explicit solutions

$$u = \alpha t^2 \pm \sqrt{2\alpha}\, x + at + b , \quad \alpha > 0 ,$$

$$= \pm (t + a)\sqrt{x + b} , \quad \alpha = 0 ,$$

where a and b are arbitrary constants.

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