Symmetry and Explicit Solutions of Partial Differential Equations

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Dedicated to Garrett Birkhoff on the occasion of his eightieth birthday.

1. Introduction.

Garrett Birkhoff's mathematical work over the last three decades has been primarily centered on scientific computation and numerical analysis, and the other contributions to this special issue survey the broad range of his research and influence in this area. However, this subject represents only a small part of Garrett's many and diverse interests, and the number of areas of mathematics on which he has left his mark is astonishing. My own contribution to these proceedings will be devoted to another of Garrett's "applied" research areas — the applications of Lie groups to differential equations, a subject whose renaissance, especially among applied mathematicians, owes much to his pioneering efforts in the 1940's, [**2**]. In light of the subject of this conference, I find Garrett's choice of thesis topic for his last pure mathematics Ph.D. student, [**15**, **16**], slightly ironic, but his insight into its hidden potentialities has more than been fulfilled. My debt to Garrett for initiating my research career and its subsequent course cannot be understated. So it is with great

† *Supported in part by NSF Grant DMS 89–01600.*

pleasure that I am able to contribute to these proceedings a discussion of some of my own investigations into the many uses of symmetry for studying differential equations.

More specifically, this survey will cover recent (and not so recent) developments in the use of symmetry and other methods for finding explicit solutions to systems of partial differential equations. While ostensibly not directly connected with the general theme of scientific computing, this particular topic does have several points of contact with the other papers in this volume. First, and perhaps of crucial importance, is the role that explicit solutions to partial differential equations play in the design and testing of numerical integrators. Being able to reproduce an explicit solution to a complicated system of partial differential equations constitutes a reassuring check for the accuracy and reliability of one's numerical package; moreover, explicit solutions can effectively serve as benchmarks for comparison of competing packages, or in establishing practical error estimates (which might be far more reasonable than any resulting from a rigorous theoretical analysis, if such is even available). Secondly, there is a well established role played by computer algebra (symbolic manipulation) systems, such as Macsyma, Reduce, Maple and Mathematica in the determination of symmetry groups and calculation of invariant solutions. Indeed, the methods introduced by Lie in the last century for computing the symmetry groups and consequent group-invariant solutions are computationally complex, but nevertheless completely algorithmic, and hence constitute a prime candidate for automation. Finally, I should remark that the role of symmetry in the design of numerical algorithms is a subject whose importance is only slowly being recognized; see the contribution by Bill Ames in these proceedings or the book by Shokin, [**25**]. Unfortunately, space limitations preclude a broader survey of the manifold applications of symmetry groups to partial differential equations, including integration of ordinary differential equations, conservation laws, reduction of Hamiltonian systems, classification of integrable (soliton) equations, bifurcation theory, etc. (See, for example, the references in [**4**] and [**17**].)

A few brief historical remarks might be of interest to place this review in context (although I don't claim to have thoroughly researched the early history of the subject). Possibly the first type of explicit or special solution to a partial differential equation was the travelling wave solution which appears in d'Alembert's solution to the wave equation. The method of separation of variables arises in the work of Fourier on the heat equation, while similarity solutions appear quite a bit later, first in work of Weierstrass around 1870 and Boltzman around 1890. Also of interest from the last century are Bäcklund transformations, which arose originally in differential geometry, but are now of great interest in soliton theories. The classical notion of a group-invariant solution, which includes as special case travelling waves and similarity solutions was emphasized by Garrett Birkhoff in his treatise on hydrodynamics, [**2**], the general theorem being stated by Ovsiannikov in 1958. However, in the course of writing my book, [**17**], I discovered that this theorem already appeared in full generality in an 1895 paper, [**12**], by Lie himself.

In the last three decades, a number of useful extensions of the classical Lie approach to group-invariant solutions have been developed, beginning with the method of partially invariant solutions of Ovsiannikov, [**21**], and Bluman and Cole's "nonclassical method", [**3**]. In a pair of papers written with Philip Rosenau, [**18**], [**19**], these methods were further generalized to include "weak symmetries" and, even more generally, the incorporation of “side conditions” or differential constraints. The latter method incorporates all (?) known methods for determining special solutions to partial differential equations. However, our frame-work is much too general to be workable, and more recent research has concentrated on finding more specific ansatzes which can be practically applied to a wide variety of physically important examples. In this direction, the recent direct method of Clarkson and Kruskal, [**7**], and the “nonlinear separation” method of Galaktionov, [**8**] deserve mention. In this paper, I will survey these different approaches, illustrated by simple examples, and explain how they fit into a general framework based on the theory of overdetermined systems of partial differential equations.

In conclusion, I would like to express my gratitude to the organizers of the conference, especially Richard Varga, for their excellent work in putting together a fitting celebration of the eightieth birthday of one of the leading mathematical lights of the twentieth century.

2. Symmetries of Differential Equations.

We will begin by reviewing a few relevant points from Lie's theory of symmetry groups of differential equations as presented, for instance, in the textbooks [**4**], [**17**]. Consider a general system of n^{th} order (partial) differential equations

$$\Delta_\nu(x, u^{(n)}) = 0, \qquad \nu = 1, \ldots, m, \tag{1}$$

in p independent variables $x = (x^1, \ldots, x^p)$, and q dependent variables $u = (u^1, \ldots, u^q)$, with $u^{(n)}$ denoting the derivatives of the u's with respect to the x's up to order n. In general, by a *symmetry* of the system (1) we mean a transformation which takes solutions to solutions. The most basic type of symmetry is a (locally defined) invertible map on the space of independent and dependent variables:

$$(\bar{x}, \bar{u}) = g \cdot (x, u) = (\Xi(x, u), \Phi(x, u)).$$

Such transformations act on solutions $u = f(x)$ by pointwise transforming their graphs; in other words if $\Gamma_f = \{(x, f(x))\}$ denotes the graph of f, then the transformed function $\bar{f} = g \cdot f$ will have graph

$$\Gamma_{\bar{f}} = \{(\bar{x}, \bar{f}(\bar{x}))\} = g \cdot \Gamma_f \equiv \{g \cdot (x, f(x))\}. \tag{2}$$

Definition 1. A local Lie group of transformations G is called a *symmetry group* of the system of partial differential equations (1) if $\bar{f} = g \cdot f$ is a solution whenever f is.

We will always assume that the transformation group G is connected, thereby excluding discrete symmetry groups, which, while also of great interest for differential equations, are unfortunately not amenable to infinitesimal, constructive techniques. Connectivity implies that it suffices to work with the associated infinitesimal generators, which form a Lie algebra of vector fields

$$\mathbf{v} = \sum_{i=1}^{p} \xi^i(x, u) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \varphi^\alpha(x, u) \frac{\partial}{\partial u^\alpha}\,, \tag{3}$$

on the space of independent and dependent variables. The group transformations in G are recovered from the infinitesimal generators by the usual process of exponentiation. Thus, the one-parameter group $G = \{g_\varepsilon \mid \varepsilon \in \mathbb{R}\}$ generated by the vector field (3) is the solution $g_\varepsilon \cdot (x_0, u_0) = (x(\varepsilon), u(\varepsilon))$ to the first order system of ordinary differential equations

$$\frac{dx^i}{d\varepsilon} = \xi^i(x,u), \qquad \frac{du^\alpha}{d\varepsilon} = \varphi^\alpha(x,u), \tag{4}$$

with initial conditions (x_0, u_0) at $\varepsilon = 0$. For example, the vector field $\mathbf{v} = -u\partial_x + x\partial_u$ generates the rotation group $x(\varepsilon) = x\cos\varepsilon - u\sin\varepsilon$, $u(\varepsilon) = x\sin\varepsilon + u\cos\varepsilon$, which transforms a function $u = f(x)$ by rotating its graph.

Since the transformations in G act on functions $u = f(x)$, they also act on their derivatives, and so induce "prolonged transformations" $(\bar{x}, \bar{u}^{(n)}) = \mathrm{pr}^{(n)}\, g \cdot (x, u^{(n)})$. The explicit formula for the prolonged group transformations is rather complicated, and so it is easier to work with the prolonged infinitesimal generators, which are vector fields

$$\mathrm{pr}^{(n)}\,\mathbf{v} = \sum_{i=1}^{p} \xi^i(x,u)\,\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q}\ \sum_{\#J\le n} \varphi^\alpha_J(x,u^{(n)})\,\frac{\partial}{\partial u^\alpha_J}\,, \tag{5}$$

on the space of independent and dependent variables and their derivatives up to order n, which are denoted by $u^\alpha_J = \partial^J u^\alpha/\partial x^J$, where $J = (j_1, \ldots, j_n)$, $1 \le j_\nu \le p$. The coefficients φ^α_J of $\mathrm{pr}^{(n)}\,\mathbf{v}$ are given by the explicit formula

$$\varphi^\alpha_J = D_J Q^\alpha + \sum_{i=1}^{p} \xi^i\, u^\alpha_{J,i}, \tag{6}$$

in terms of the coefficients ξ^i, φ^α of the original vector field (3). Here D_i denotes the total derivative with respect to x^i (treating the u's as functions of the x's), and $D_J = D_{j_1}\cdot\ldots\cdot D_{j_n}$ the corresponding higher order total derivative. Furthermore, the q-tuple $Q = (Q^1, \ldots, Q^q)$ of functions of x's, u's and first order derivatives of the u's defined by

$$Q^\alpha(x,u^{(1)}) = \varphi^\alpha(x,u) - \sum_{i=1}^{p} \xi^i(x,u)\,\frac{\partial u^\alpha}{\partial x^i}\,, \qquad \alpha = 1, \ldots, q, \tag{7}$$

is known as the *characteristic* of the vector field (3), and plays a significant role in our subsequent discussion. The main point the reader should glean from this paragraph is not the particular complicated expressions in (5), (6), (7) (although, of course, these *are* required when performing any particular calculation), but rather that there *are* known, explicit formulas which can, in a relatively straightforward manner, be computed. See [**17**] for details.

Theorem 2. *A connected group of transformations G is a symmetry group of the (nondegenerate) system of differential equations* (1) *if and only if the classical infinitesimal symmetry criterion*

$$\mathrm{pr}^{(n)}\,\mathbf{v}(\Delta_\nu) = 0, \qquad \nu = 1, \ldots, r, \qquad \textit{whenever} \quad \Delta = 0. \tag{8}$$

holds for every infinitesimal generator $\mathbf{v}$ *of* G.

The equations (8) are known as the *determining equations* of the symmetry group for the system. They form a large over-determined linear system of partial differential equations for the coefficients ξ^i, φ^α of $\mathbf{v}$, and can, in practice, be explicitly solved to determine the complete (connected) symmetry group of the system (1). There are now a wide variety of computer algebra packages available which will automate most of the routine steps in the calculation of the symmetry group of a given system of partial differential equations. See [**6**], [**23**], [**10**], [**24**] for examples in MACSYMA, MAPLE, REDUCE and SCRATCHPAD. (Conspicuously lacking are packages in MATHEMATICA.) Reference [**6**] gives a good survey of the different packages available at present, and a discussion of their strengths and weaknesses.

Example 3. The classic example illustrating the basic techniques is the linear heat equation

$$u_t = u_{xx}. \tag{9}$$

(See [**3**], although the result was known long before this, [**9**].) An infinitesimal symmetry of the heat equation will be a vector field $\mathbf{v} = \xi\partial_x + \tau\partial_t + \varphi\partial_u$, where ξ, τ, φ are functions of x, t, u. To determine which coefficient functions ξ, τ, φ yield genuine symmetries, we need to solve the symmetry criterion (8), which, in this case, is

$$\varphi^t = \varphi^{xx} \qquad \text{whenever} \qquad u_t = u_{xx}. \tag{10}$$

Here, utilizing the characteristic $Q = \varphi - \xi u_x - \tau u_t$ given by (7),

$$\varphi^t = D_t Q + \xi u_{xt} + \tau u_{tt}, \qquad \varphi^{xx} = D_x^2 Q + \xi u_{xxt} + \tau u_{xtt}, \tag{11}$$

are the coefficients of the terms $\partial_{u_t}, \partial_{u_{xx}}$ in the second prolongation of $\mathbf{v}$, cf. (6). Substituting the formulas (11) into (10), and replacing u_t by u_{xx} wherever it occurs, we are left with a polynomial equation involving the various derivatives of u whose coefficients are certain derivatives of ξ, τ, φ. Since ξ, τ, φ only depend on x, t, u we can equate the individual coefficients to zero, leading to the complete set of *determining equations*:

Coefficient	*Monomial*
$0 = -2\tau_u$	$u_x u_{xt}$
$0 = -2\tau_x$	u_{xt}
$0 = -\tau_{uu}$	$u_x^2 u_{xx}$
$-\xi_u = -2\tau_{xu} - 3\xi_u$	$u_x u_{xx}$
$\varphi_u - \tau_t = -\tau_{xx} + \varphi_u - 2\xi_x$	u_{xx}
$0 = -\xi_{uu}$	u_x^3
$0 = \varphi_{uu} - 2\xi_{xu}$	u_x^2
$-\xi_t = 2\varphi_{xu} - \xi_{xx}$	u_x
$\varphi_t = \varphi_{xx}$	1

The general solution to these elementary differential equations is readily found:

$$\xi = c_1 + c_4 x + 2c_5 t + 4c_6 xt, \quad \tau = c_2 + 2c_4 t + 4c_6 t^2, \quad \varphi = (c_3 - c_5 x - 2c_6 t - c_6 x^2)u + \alpha(x,t),$$

where c_i are arbitrary constants and $\alpha_t = \alpha_{xx}$ is an arbitrary solution to the heat equation. Therefore, the symmetry algebra of the heat equation is spanned by the vector fields

$$\begin{aligned} &\mathbf{v}_1 = \partial_x, \qquad \mathbf{v}_2 = \partial_t, \qquad \mathbf{v}_3 = u\partial_u, \qquad \mathbf{v}_4 = x\partial_x + 2t\partial_t, \\ &\mathbf{v}_5 = 2t\partial_x - xu\partial_u, \qquad \mathbf{v}_6 = 4xt\partial_x + 4t^2\partial_t - (x^2+2t)u\partial_u, \\ &\mathbf{v}_\alpha = \alpha(x,t)\partial_u, \qquad \text{where} \qquad \alpha_t = \alpha_{xx}. \end{aligned}$$

The corresponding one-parameter groups are, respectively, x and t translations, scaling in u, the combined scaling $(x,t) \mapsto (\lambda x, \lambda^2 t)$, Galilean boosts, an "inversional symmetry", and the addition of solutions stemming from the linearity of the equation. See [**3**] or [**17**] for more details.

Example 4. The Boussinesq equation

$$u_{tt} + uu_{xx} + u_x^2 + u_{xxxx} = 0, \tag{12}$$

is a well-known soliton equation, and arises as a model equation for the unidirectional propagation of solitary waves in shallow water, [**14**]. The basic symmetry condition (8) now takes the form

$$\varphi^{tt} + u\varphi^{xx} + u_{xx}\varphi + 2u_x\varphi^x + \varphi^{xxxx} = 0,$$

which must hold whenever (12) is satisfied. Here $\varphi^{tt}, \varphi^x, \varphi^{xx}, \varphi^{xxxx}$, are the coefficients of the terms $\partial_{u_{tt}}, \partial_{u_x}, \partial_{u_{xx}}, \partial_{u_{xxxx}}$, respectively, in the fourth prolongation of $\mathbf{v}$, with formulae similar to (11). A straightforward calculation eventually yields the complete symmetry algebra, which is spanned by

$$\mathbf{v}_1 = \partial_x, \qquad \mathbf{v}_2 = \partial_t, \qquad \mathbf{v}_3 = x\partial_x + 2t\partial_t - 2u\partial_u. \tag{13}$$

In this example, the classical symmetry group is disappointingly trivial, consisting of easily guessed translations and scaling symmetries. Theorem 2 guarantees that these are the only continuous classical symmetries of the equation. (There are, however, higher order generalized symmetries, cf. [**17**], which account for the infinity of conservation laws of this equation.) Sometimes the complicated calculation of the symmetry group of a system of differential equations yields only rather trivial symmetries; however, there are numerous examples where this is not the case and new and physically and/or mathematically important symmetries have arisen from a complete group analysis.

3. Group Invariant Solutions.

We begin by discussing the classical notion of a group-invariant solution, which includes many of the common special solutions to partial differential equations, such as similarity solutions, travelling wave solutions, etc.

Definition 5. Assume that G is a symmetry group of a system of differential equations (1). A solution $u = f(x)$ is called G-invariant if $g \cdot f = f$ for all $g \in G$.

In other words, a solution is G-invariant if does not change under the action of the transformations in the given symmetry group G. This is equivalent to the requirement that the graph Γ_f is a (locally) G-invariant set, i.e., $g \cdot \Gamma_f = \Gamma_f$, for all $g \in G$, cf. (2). For example, if G is the group of rotations in the independent variables x, then a G-invariant solution f is one whose graph is rotationally invariant, which is the same as requiring that $u = f(|x|)$ be a function of the radius alone. Similarity solutions arise when the group is a group of scaling transformations; travelling wave solutions correspond to translational symmetry groups.

Theorem 6. *Suppose that the symmetry group G acts regularly and has r dimensional orbits. Then all the G-invariant solutions to $\Delta = 0$ can be found by solving a reduced system of differential equations $\Delta/G = 0$ in r fewer independent variables.*

Thus, for example, if we have a system of partial differential equations in 2 independent variables, then the solutions invariant under a one-parameter symmetry group can all be found by integrating a system of ordinary differential equations. Of course, for Theorem 6 to be applicable, the orbit dimension r must be strictly less than the number of independent variables p. (Often the orbit dimension coincides with the dimension or number of independent generators of the group. See [**17**] for the precise definition of regular.) If $r = p$, the invariant solutions can be found from a system of algebraic equations, while if $r > p$ there are no invariant solutions.

There are several approaches to finding the reduced system of differential equations Δ/G. We begin by recalling the following convenient characterization of functions $u = f(x)$ which are invariant under a given group of transformations, a result based on the (classical) theory of systems of first order partial differential equations

Proposition 7. *Let G be a local Lie group of transformations with infinitesimal generators $\mathbf{v}_1, \ldots, \mathbf{v}_r$. Let $Q_1, \ldots, Q_r$ be the associated characteristics of these vector fields, cf.* (7). *Then a function is G-invariant if and only if it is a solution to the system of first order partial differential equations*

$$Q_\kappa^\alpha(x, u^{(1)}) = 0, \qquad \kappa = 1, \ldots, r, \quad \alpha = 1, \ldots q. \tag{14}$$

Thus, the group-invariant solutions to the system of partial differential equations (1) will consist of those functions $u = f(x)$ which satisfy both the original system (1) *and* a collection of "differential constraints" which are the first order partial differential equations (14) characterizing the G-invariant functions. In other words, to determine the group-invariant solutions to the system (1), we must solve the following overdetermined system of partial differential equations:

$$\begin{aligned} \Delta_\nu(x, u^{(n)}) &= 0, \qquad \nu = 1, \ldots, m, \\ Q_\kappa^\alpha(x, u^{(1)}) &= 0, \qquad \kappa = 1, \ldots, r, \quad \alpha = 1, \ldots q. \end{aligned} \tag{15}$$

Theorem 6 implies that if G is a symmetry group of (1), then the overdetermined system (15) can be reduced to a system of differential equations in fewer independent variables. The classical implementation of this theorem is to first solve the invariance constraints (14) by utilizing invariant coordinates; these will form the new variables for the reduced system. (Other approaches have recently been introduced, cf. [**4**].)

Definition 8. A function $\eta(x,u)$ is called an *invariant* of the transformation group G if it is unaffected by the transformations in G; in other words

$$\eta\big(g\cdot(x,u)\big)=\eta(x,u),\qquad \text{forall}\quad g\in G.$$

Invariants of a transformation group can be, in many cases, be determined by inspection. (More complicated cases can often be handled by solving the infinitesimal invariance criterion $\mathbf{v}_\kappa[\eta]=0$, $\kappa=1,\dots,r$, using the method of characteristics; $\mathbf{v}_1,\dots,\mathbf{v}_r$ are the infinitesimal generators of G.) A basic result states that, locally, any regular transformation group with r dimensional orbits has a complete set of $p+q-r$ functionally independent invariants

$$y^1=\eta^1(x,u),\dots,\ y^{p-r}=\eta^{p-r}(x,u),\ w^1=\zeta^1(x,u),\dots,\ w^q=\zeta^q(x,u). \tag{16}$$

The y's and w's will serve as new independent and dependent variables respectively. The splitting of the invariants into the two classes is more or less arbitrary; however, in many cases (including all those we discuss below), we can choose $p-r$ of the invariants to be independent of u, in which case these will naturally serve as the new independent variables, the remaining invariants being dependent variables. A function $u=f(x)$ will be invariant under G, or, equivalently, a solution to the system (14), if and only if it can be rewritten in terms of the basic invariants (16), i.e., in the form

$$w=h(y),\qquad \text{or, explicitly,}\qquad \zeta(x,u)=h[\eta(x,u)]. \tag{17}$$

Formula (17) can be differentiated using the chain rule to deduce formulae for the derivatives of the u's with respect to the x's in terms of the derivatives of the w's with respect to the y's. These expressions are then substituted into the original system (1). The crucial point in the proof is that, because G is a symmetry group of (1), the resulting equations are necessarily equivalent to a reduced system of differential equations $\Delta/G(y,w^{(n)})=0$ involving only y, w and derivatives of w. Solutions of the reduced system are in one-to-one correspondence with G-invariant solutions to the original system, via (17).

Example 9. Consider the heat equation (9). For each one of the symmetry generators (or any linear combination thereof) the reduction algorithm will lead to an ordinary differential equation for the associated group-invariant solutions. We just present the classical example of the general similarity solution, which corresponds to the scaling symmetry $x\partial_x+2t\partial_t+2au\partial_u$, where a is a constant, which generates the one-parameter group $(x,t,u)\to(\lambda x,\lambda^2 t,\lambda^{2a}u)$. The associated characteristic is, according to (7), $Q=au-xu_x-2tu_t$, hence the overdetermined system (15) in this particular case is

$$u_t=u_{xx},\qquad xu_x+2tu_t-au=0.$$

The independent invariants of this group are $y=x/\sqrt{t}, w=t^{-a}u$. Thus, by (17), every scaling invariant solution can be written in the form $w=w(y)$, or, explicitly, $u=t^a w(y)=t^a w(x/\sqrt{t}\,)$, which is just the general solution to the characteristic equation $Q=0$. Differentiating this formula, we find $u_t=t^{a-1}(-\frac{1}{2}yw'+aw)$, $u_{xx}=t^a w''$. Substituting these into the heat equation and cancelling a power of t immediately yields the reduced equation: $w''+\frac{1}{2}yw'-aw=0$, whose general solution is $w(y)=e^{-y^2/8}\,\mathrm{U}\big(2a+\frac{1}{2},y/\sqrt{2}\big)$, where U denotes a parabolic cylinder function. Therefore, the general similarity solution to the heat equation is

$$u(x,t) = t^a e^{-x^2/8t}\, \mathrm{U}\left(2a + \frac{1}{2}, \frac{x}{\sqrt{2t}}\right).$$

Example 10. Consider the Boussinesq equation (12). In the case of the scaling symmetry group, generated by the vector field $\mathbf{v}_3$, the invariants are $y = x/\sqrt{t}$, $w = tu$, with the general invariant solution $u = t^{-1} w(x/\sqrt{t})$. Differentiating and substituting into (12) leads to the reduced equation for the similarity solutions of the Boussinesq equation:

$$w'''' + ww'' + w'^2 + \tfrac{1}{4} y^2 w'' + \tfrac{7}{4} y w' + 2w = 0. \tag{18}$$

Unfortunately, this ordinary differential equation does not have an elementary solution, so it is not possible to write down the explicit form for the general similarity solution to the Boussinesq equation in closed form; nevertheless, numerical integration of (18) is certainly an option. It should also be noted that, in accordance with the original Painlevé conjecture for soliton equations, [**1**], the ordinary differential equation (18) is of Painlevé type, having only poles for moveable singularities.

4. The Non-Classical Method.

In the proof of Theorem 6, the key point is that the group in question is a symmetry group of the original system of partial differential equations (1). Actually, a closer look at the proof reveals that one can relax this condition, by only requiring that the overdetermined system (15), consisting of the original system of partial differential equations and the characteristic invariance system, is invariant under the group G. Bluman and Cole, [**3**], were the first to note that this relaxed condition could lead to new types of groups and new types of explicit solutions which cannot be obtained by the classical group invariance approach, hence the name "non-classical method". One draw back is that the determining equations for the invariance of the combined system (15) are nonlinear and hence usually impossible to solve explicitly, because the coefficient functions ξ^i, φ^α of the vector field (3) occur also in the equations themselves via the characteristics (7). Nevertheless, any solution to the nonclassical determining equations will lead to a "non-classical symmetry group" and hence to invariant solutions which can be determined by the same basic algorithm as the classical ones.

Example 11. Consider the Boussinesq equation (12). An explicit example of a non-classical group is the Galilean group generated by the vector field $\mathbf{v} = t\partial_x + \partial_t - 2t\partial_u$. This is not a symmetry of the Boussinesq equation, since it does not appear in the complete classical symmetry group found in Example 4. Nevertheless, the combined overdetermined system of partial differential equations

$$u_{tt} + \tfrac{1}{2}(u^2)_{xx} + u_{xxxx} = 0, \qquad -Q = tu_x + u_t + 2t = 0,$$

does admit $\mathbf{v}$ as a symmetry, as can be checked from the basic infinitesimal symmetry criterion (8). The determination of Galilean-invariant solutions to the Boussinesq equation then proceeds as in the classical case. The invariants of the group are provided by the functions $y = x - \frac{1}{2}t^2$ and $w = u + t^2$, so the general invariant solution will have the form

$$u(x,t) = w(y) - t^2 = w\big(x - \tfrac{1}{2}t^2\big) - t^2. \tag{19}$$

Differentiating and substituting into (12) yields the reduced ordinary differential equation

$$w'''' + ww'' + (w')^2 - w' + 2 = 0.$$

As in the classical similarity solution case, Example 10, this equation is not explicitly solvable, but is of Painlevé type.

Example 12. Another intriguing example, noted by Vorob'ev, [**27**], shows how the Bäcklund transformation (see [**14**]) for the sine-Gordon equation

$$u_{xt} = \tfrac{1}{2}\sin 2u, \tag{20}$$

can be characterized using the non-classical symmetry method. Let $v(x,t)$ be any particular solution to (20). Then the two vector fields

$$\mathbf{v}_1 = \partial_x + \big(\sin(u-v) - v_x\big)\partial_u \qquad \mathbf{v}_2 = \partial_t + \big(\sin(u+v) + v_t\big)\partial_u,$$

generate a non-classical symmetry group. In this case the invariant solutions $u(x,t)$ are those obtained from $v(x,t)$ by the classical Bäcklund transformation:

$$u_x = \sin(u-v) - v_x, \qquad u_t = \sin(u+v) + v_t.$$

5. The Clarkson-Kruskal Direct Approach.

In an attempt to understand Example 11 and similar constructions, Clarkson and Kruskal, [**7**], introduced a direct method for reducing partial differential equations to ordinary differential equations. The basic idea is to make a similarity-like ansatz

$$u(x,t) = U(x,t,w(z)), \qquad \text{where} \quad z = \zeta(x,t), \tag{21}$$

and choose the functions U and ζ in such a manner that the partial differential equation will reduce to an ordinary differential equation for $w(z)$.

Example 13. For the Boussinesq equation (12), it can be proved that there is no loss of generality if we assume that the function U in the ansatz (21) is linear in w, of the form

$$u(x,t) = \alpha(x,t) + \beta(x,t)w(z), \qquad z = \zeta(x,t). \tag{22}$$

Plugging (22) into (12) leads to a rather complicated polynomial expression involving various monomial products of derivatives of w whose coefficients depend on the partial derivatives of α, β, ζ. For this expression to be an ordinary differential equation for $w(z)$, the coefficients of the different monomials must be functions of z alone. Rather than reproduce the lengthy analysis required, we refer the reader to [**7**] and just quote the final result.

Theorem 14. *The most general ansatz* (21) *reducing the Boussinesq equation to an ordinary differential equation is*

$$u(x,t) = \theta(t)^2 w(z) - \theta(t)^{-2}(x\theta'(t) + \sigma'(t))^2, \qquad z = x\theta(t) + \sigma(t),$$

where θ, σ *are, in general, elliptic functions, satisfying*

$$\theta'' = A\theta^5, \qquad \sigma'' = \theta^4(A\sigma + B),$$

for constants A, B*. The corresponding function* $w(z)$ *satisfies a reduced ordinary differential equation of Painlevé type*

$$w'''' + ww'' + w'^2 + (Az + B)w' + 2Aw = 2(Az + B)^2. \tag{23}$$

Levi and Winternitz, [**11**], have shown how all the Clarkson-Kruskal reductions of the Boussinesq equation come from either classical or non-classical group reductions. For example, the case $A = 0$, $B = -1$ includes our earlier nonclassical ansatz (19), while the case $A = \frac{3}{4}, B = 0$ includes the classical similarity reduction (18) (with w replaced by $w - \frac{1}{4}y^2$) as special cases. However, the connection between these two approaches for general partial differential equations remains hazy.

6. Weak Symmetry Groups.

In [**18**] Philip Rosenau and I proposed a further generalization of the non-classical method. Since the combined system (15) is an overdetermined system of partial differential equations, one should, in treating it, take into account any integrability conditions given by equating mixed partials. (The Cartan-Kuranishi Theorem, [**5**], assures us that, under mild regularity conditions, the integrability conditions can all be found in a finite number of steps; Gröbner basis methods, as in [**22**], [**26**], provide a practical means to compute them.) Therefore, one should compute the symmetry group not of just the system (15) but also any associated integrability conditions. Thus, we define a *weak symmetry group* of the system (1) to be any symmetry group of the overdetermined system (15) and all its integrability conditions.

Example 15. An example of a weak symmetry group for the Boussinesq equation (12) is the scaling group generated by the vector field $\mathbf{v} = x\partial_x + t\partial_t$. This is not a symmetry of the Boussinesq equation, nor is it a symmetry of the combined system

$$u_{tt} + uu_{xx} + u_x^2 + u_{xxxx} = 0, \qquad Q = xu_x + tu_t = 0. \tag{24}$$

Nevertheless, if we append the integrability conditions to (24), we do find that $\mathbf{v}$ satisfies the weak symmetry conditions. To compute the invariant solutions, we begin as before by introducing the invariants, $y = x/t$, and $w = u$. Differentiating the formula $u = w(y) = w(x/t)$ and substituting into the Boussinesq equation, we come to the following equation

$$t^{-4}w'''' + t^{-2}[(y^2 + w)w'' + (w')^2 + 2yw'] = 0.$$

At this point the crucial difference between the weak symmetries and the nonclassical (or classical) symmetries appears. In the latter case, any non-invariant coordinate, e.g., the t here, will factor out of the resulting equation and thereby leave a single ordinary differential equation for the invariant function $w(y)$. For weak symmetries this is no longer true, since we have yet to incorporate the integrability conditions for (24). However, we can separate out the coefficients of the various powers of t in the above equation, leading to an overdetermined system of ordinary differential equations,

$$w'''' = 0, \qquad (y^2 + w)w'' + (w')^2 + 2yw' = 0,$$

for the unknown function w. In this particular case, the resulting overdetermined system does have solutions, namely $w(y) = -y^2$, or $w(y) =$ constant. The latter are trivial, but the former yield a nontrivial similarity solution: $u(x,t) = -x^2/t^2$. Thus we have the intriguing phenomenon of an equation with a similarity solution which does not come from a classical scaling symmetry group! This leads to interesting speculations concerning the asymptotics or blow up behavior of solutions, often governed by classical similarity solutions.

Weak symmetry groups, while at the outset quite promising, have some critical drawbacks. It can be shown that every group is a weak symmetry group of a given system of partial differential equations, and, moreover, every solution to the system can be derived from some weak symmetry group, cf. [**18**]. Therefore, the generalization is too severe. Nevertheless, it gives some hints as to how to proceed in any practical analysis of such solution methods. What is required is an appropriate theory of overdetermined systems of partial differential equations which will allow one to write down reasonable classes of groups for which the combined system (15) is compatible, in the sense that is has solutions, or, more restrictively, has solutions that can be algorithmically computed. For example, restricting to scaling groups, or other elementary classes of groups, might be a useful starting point.

7. Partially Invariant Solutions.

The concept of a partially invariant solution was introduced by Ovsiannikov, [**21**], as a generalization of the classical concept of group-invariant solution. The basic remark is the following: suppose G is a symmetry group of some system of partial differential equations which acts regularly with r dimensional orbits. Let $u = f(x)$ be a solution of the system (1), whose graph Γ_f will be a p dimensional submanifold of the space of independent and dependent variables. Consider the set $G \cdot \Gamma_f = \{ g \cdot (x,u) \mid (x,u) \in \Gamma_f \}$ obtained by transforming the graph of f by all possible group elements in G, cf. (2). As remarked earlier, a solution f is a G-invariant if and only if $G \cdot \Gamma_f = \Gamma_f$. On the other hand, if f is a "generic" solution, then one expects the submanifold $G \cdot \Gamma_f$ to have dimension $p + s$, where $s = \min\{r, q\}$. A *partially invariant solution* is one such that the dimension of $G \cdot \Gamma_f$ has some intermediate value, as made precise by the next definition.

Definition 16. The *defect* δ of a solution f with respect to the group G is given by $\delta = \dim(G \cdot \Gamma_f) - p$. A solution f is *G-invariant* if $\delta = 0$, *generic* if $\delta = s = \min\{r, q\}$, and *partially invariant* if $0 < \delta < s$.

Ovsiannikov introduced a complicated algorithm for calculating partially invariant solutions of a given defect, requiring the solution to an associated overdetermined system of partial differential equations. A significant simplification, based on suggestions in [**19**], was introduced in the thesis of Ondich, [**20**]. From the characteristics $Q_1, \ldots, Q_r$ of the infinitesimal generators $\mathbf{v}_1, \ldots, \mathbf{v}_r$ of G, we form the $r \times q$ "characteristic matrix"

$$\mathbf{Q}(x, u^{(1)}) = \Big(Q^\alpha_\kappa(x, u^{(1)}) \Big), \qquad \alpha = 1, \ldots, q, \quad \kappa = 1, \ldots, r. \tag{25}$$

Proposition 17. *The function $u = f(x)$ is a partially invariant solution to* (1) *of rank δ if and only if*

$$\operatorname{rank} \mathbf{Q}(x, u^{(1)}) = \delta. \tag{26}$$

Note that the corresponding inequality, $\operatorname{rank} \mathbf{Q}(x, u^{(1)}) \le \delta$*, corresponding to partially invariant solutions of rank at most δ, is prescribed by a collection of first order differential constraints found by setting all $\delta \times \delta$ subdeterminants of $\mathbf{Q}$ to zero.*

Example 18. (Ovsiannikov) Consider the equations for steady trans-sonic gas flow

$$u_t = v_x, \qquad v_t + u u_x = 0, \tag{27}$$

(where $t = y$ is the vertical coordinate) which are equivalent to the nonlinear hyperbolic equation $u_{tt} + \frac{1}{2}(u^2)_{xx} = 0$, a dispersionless limit of the Boussinesq equation (12). As an example, we find the partially invariant solutions of defect 1 for the symmetry group consisting of scalings in x, t and translations in v, which is generated by the two vector fields $x\partial_x + t\partial_t$ and ∂_v. The characteristic matrix (25) is

$$\mathbf{Q} = \begin{pmatrix} -x u_x - t u_t & -x v_x - t v_t \\ 0 & 1 \end{pmatrix}.$$

A defect 1 solution will satisfy the condition $\operatorname{rank} \mathbf{Q} = 1$, which requires $x u_x + t u_t = 0$. Therefore we need to solve the combined system of partial differential equations

$$u_t = v_x, \qquad v_t + u u_x = 0, \qquad x u_x + t u_t = 0.$$

(Note that the extra constraint is not the same as a group invariance constraint, (14), which would require two additional equations; for instance, invariance under the scaling subgroup generated by $x\partial_x + t\partial_t$ gives the constraints $x u_x + t u_t = x v_x + t v_t = 0$.) The solution to the constraint equation is $u = \varphi(z)$, where $z = x/t$. Substituting into (27) yields the system $v_x = u_t = -t^{-1} z \varphi'(z)$, $v_t = -u u_x = -t^{-1}\varphi(z)\varphi'(z)$, which has the integrability condition $[(z^2 + \varphi)\varphi']' = 0$. This can be reduced to the first order ordinary differential equation $(\varphi + z^2)\varphi' = k$, which, however, does not have an elementary general solution.

Example 19. (Ondich) A more substantial example is provided by the Prandtl boundary layer equations

$$u_{yy} = u_t + u u_x + v u_y + p_x, \qquad u_x + v_y = 0, \qquad p_y = 0. \tag{28}$$

The classical symmetry generators are

$$\mathbf{v}_1 = \partial_t, \qquad \mathbf{v}_2 = 2t\partial_t + 2x\partial_x + y\partial_y - v\partial_v, \qquad \mathbf{v}_3 = x\partial_x + u\partial_u + 2p\partial_p,$$
$$\mathbf{v}_\alpha = \alpha(t)\partial_x + \alpha'(t)\partial_u - \alpha''(t)x\partial_p, \qquad \mathbf{v}_\beta = \beta(t)\partial_y + \beta'(t)\partial_v,$$

As a particular example, we consider partially invariant solutions of defect 2 for the subgroup generated by

$$\mathbf{v}_0 = \partial_t, \qquad \mathbf{v}_{\beta=1} = \partial_y, \qquad \mathbf{v}_3 = x\partial_x + u\partial_u + 2p\partial_p.$$

The characteristic matrix (25) is

$$-\mathbf{Q} = \begin{pmatrix} u_t & v_t & p_t \\ u_y & v_y & 0 \\ xu_x - u & xv_x & xp_x - 2p \end{pmatrix}.$$

For defect $\delta = 2$, we require $\operatorname{rank}\mathbf{Q} = 2$. We concentrate on the subcase when the first two columns of $\mathbf{Q}$ are linearly dependent, which is equivalent to the condition $v = \phi(u/x)$. Substituting this ansatz into the system yields

$$u_x + \frac{1}{x}\phi'\left(\frac{u}{x}\right)u_y = 0, \qquad u_{yy} = u_t + uu_x + \phi\left(\frac{u}{x}\right)u_y + p_x, \qquad p_y = 0.$$

For example, if $\phi(s) = s$, then $u = \psi(xe^{-y}, t)$, where ψ satisfies

$$\psi_{zt} = \psi_z + 3z\psi_{zz} + z^2\psi_{zzz}.$$

Thus we have reduced the nonlinear system (28) to a linear equation. A particular solution found by standard separation of variables methods is

$$u = \cos\left(\sqrt{k}\left[\,\log x - y\right]\right)e^{-kt}, \qquad v = \frac{1}{x}\cos\left(\sqrt{k}\left[\,\log x - y\right]\right)e^{-kt}, \qquad p = 0.$$

There are two principal reasons why the partially invariant solution method has not been as extensively applied as other approaches. First, the algorithm is quite complicated, although the above approach is a significant simplification. Second, in many cases, all partially invariant solutions are invariant under some subgroup, and so could have been determined by the classical, simple Lie approach. For example, Ondich, [**20**] has proved that this is always the case if the system is elliptic and the defect is 1. There appears to be an intimate connection between the existence of non-trivial partially invariant solutions and the classical characteristic directions of the system; in particular elliptic systems have no real characteristic directions, which explains Ondich's result. Nevertheless, there are, I believe, many potentially important applications of this technique to physically interesting systems which remain to be fully developed.

8. Differential Constraints.

The ultimate generalization, proposed in [**19**], which includes all of the preceding methods, and many others, is the introduction of general "side conditions" or *differential constraints*. (See also the earlier work of Yanenko, [**28**], and Meleshko, [**13**].) Now we generalize the combined system (15) by allowing the possibility of constraints which depend on higher order derivatives, leading to an overdetermined system of the form

$$\begin{aligned} \Delta_\nu(x, u^{(n)}) &= 0, \qquad \nu = 1, \ldots, m, \\ Q_\kappa(x, u^{(k)}) &= 0, \qquad \kappa = 1, \ldots, r. \end{aligned} \tag{29}$$

The differential constraints $Q_\kappa(x, u^{(k)}) = 0$ are no longer restricted to be first order partial differential equations provided by the characteristics of (weak, non-classical or classical) symmetries, or partial invariance conditions. The method of differential constraints includes (almost) all known methods for determining special solutions to partial differential equations, such as group invariant solutions, non-classical and weak symmetries, partially invariant solutions, separation of variables, as well as many others. Besides the first order constraints discussed above, higher order constraints include the additive separation of variables ansatz $u(x,t) = \alpha(x) + \beta(t)$, corresponding to the constraint $u_{xt} = 0$, and the more usual multiplicative separation of variables $u(x,t) = \alpha(x)\beta(t)$ corresponding to the constraint $uu_{xt} - u_x u_t = 0$. (Of course, these two ansatzes are easily related by the change of variables $u \mapsto \log u$.) Differential constraints do have a group theoretic interpretation as "generalized weak symmetries" although this does not appear to be an overly useful observation. The main difficulty is that this approach, while providing a completely general framework which includes many special approaches, is much too general to be of direct computation use. The crucial questions now are: Which constraints are compatible? What are reasonable ansatzes that will (in many cases) yield computable solutions?

Example 20. Consider once again the Boussinesq equation (12). We impose the differential constraint $uu_{xt} - u_x u_t = 0$, which is equivalent to the "separation of variables" ansatz $u = \alpha(x)\beta(t)$. Substituting this expression into (12) and separating the terms that depend on either x or t results in a "reduced system"

$$\lambda\alpha + \alpha\alpha'' + \alpha'^2 + \alpha'''' = 0, \qquad \beta'' = \lambda\beta^2.$$

The general solution to this system includes our previous anomalous similarity solution $u(x,t) = -x^2/t^2$.

The preceding example and the Clarkson-Kruskal direct approach have both been recently extended by a method of "nonlinear separation" developed by Galaktionov and others in a detailed study of the development of singularities in the solutions of certain kinds of quasi-linear parabolic equations; see [**8**] and the references therein.

Example 21. Consider the nonlinear parabolic evolution equation

$$u_t = u_{xx} + u_x^2 + u^2, \tag{30}$$

arising as a model for combustion. The differential constraint $u_x u_{xxt} - u_{xx} u_{xt} = 0$ corresponds to a "separation" ansatz

$$u(x,t) = \alpha(t) + \beta(t)\theta(x), \tag{31}$$

where α, β, θ are functions of a single variable. Plugging (31) into (30) gives

$$(\alpha_t - \alpha^2) + (\beta_t - 2\beta\alpha)\theta - \beta\theta_{xx} - \alpha^2(\theta_x^2 + \theta^2) = 0. \tag{32}$$

Since α, β depend only on t, whereas θ depends only on x, equation (32) will be incompatible unless θ satisfies

$$\theta'' = a\theta + b, \qquad \theta'^2 + \theta^2 = c\theta + d,$$

for constants a, b, c, d. This works if, for example, $a = -1$, $b = c = 0$, $d = 1$, whereby $\theta(x) = \cos x$. In this case, (32) reduces to a coupled system of first order ordinary differential equations

$$\alpha' = \alpha^2 + \beta^2, \qquad \beta' = 2\beta\alpha - \beta,$$

for the as yet undetermined functions $\alpha(t), \beta(t)$. The latter first order system of ordinary differential equations does not have an explicit general solution; nevertheless, numerical and analytical investigations lead to important applications to the study of blow-up of solutions to the original equation (30), cf. [**8**].

References

[**1**] Ablowitz, M.J., Ramani, A., and Segur, H., A connection between nonlinear evolution equations and ordinary differential equations of P-type, *J. Math. Phys.* **21** (1980), 715–721.

[**2**] Birkhoff, G., *Hydrodynamics — A Study in Logic, Fact and Similitude*, 1st ed., Princeton University press, Princeton, 1950.

[**3**] Bluman, G.W., and Cole, J.D., The general similarity solution of the heat equation, *J. Math. Mech.* **18** (1969), 1025–1042.

[**4**] Bluman, G.W., and Kumei, S., *Symmetries and Differential Equations*, Springer–Verlag, New York, 1989.

[**5**] Bryant, R.L., Chern, S.S., Gardner, R.B., Goldschmidt, H.L., and Griffiths, P.A., *Exterior Differential Systems*, Mathematical Sciences Research Institute Publications, vol. 18, Springer-Verlag, New York, 1991.

[**6**] Champagne, B. Winternitz, P.,,, and Hereman, W. Winternitz, P.,,, The computer calculation of Lie point symmetries of large systems of differential equations, *Comp. Phys. Comm.*, to appear.

[**7**] Clarkson, P., and Kruskal, M., New similarity reductions of the Boussinesq equation, *J. Math. Phys.* **30** (1989), 2201–2213.

[**8**] Galaktionov, V.A., On new exact blow–up solutions for nonlinear heat conduction equations with source and applications, *Diff. Int. Eqs.* **3** (1990), 863–874.

[**9**] Goff, J.A., Transformations leaving invariant the heat equation of physics, *Amer. J. Math.* **49** (1927), 117–122.

[**10**] Kersten, P.H.M., Software to compute infinitesimal symmetries of exterior differential systems, with applications, *Acta Appl. Math.* **16** (1989), 207–229.

[**11**] Levi, D., and Winternitz, P., Nonclassical symmetry reduction: example of the Boussinesq equation, *J. Phys. A* **22** (1989), 2915–2924.

[**12**] Lie, S., Zur allgemeinen Theorie der partiellen Differentialgleichungen beliebeger Ordnung, *Leipz. Berich.* **1** (1895), 53–128; also *Gesammelte Abhandlungen*, vol. 4, B.G. Teubner, Leipzig, 1929, 320–384.

[**13**] Meleshko, S.V., Differential constraints and one-parameter Lie-Bäcklund groups, *Sov. Math. Dokl.* **28** (1983), 37–41.

[**14**] Newell, A., *Solitons in Mathematics and Physics*, SIAM, Philadelphia, 1985.

[**15**] Olver, P.J., *Symmetry Groups of Partial Differential Equations*, Thesis, Harvard University, 1976.

[**16**] Olver, P.J., Symmetry groups and group invariant solutions of partial differential equations, *J. Diff. Geom.* **14** (1979), 497–542.

[**17**] Olver, P.J., *Applications of Lie Groups to Differential Equations*, Springer-Verlag, New York, 1986.

[**18**] Olver, P.J., and Rosenau, P., Group-invariant solutions of differential equations, *SIAM J. Appl. Math.* **47** (1987), 263–278.

[**19**] Olver, P.J., and Rosenau, P., The construction of special solutions to partial differential equations, *Phys. Lett.* **114A** (1986), 107–112.

[**20**] Ondich, J., *Partially Invariant Solutions of Partial Differential Equations*, Thesis, University of Minnesota, 1989.

[**21**] Ovsiannikov, L.V., *Group Analysis of Differential Equations*, Academic Press, New York, 1982.

[**22**] Pankrat'ev, E.V., Computations in differential and difference modules, *Acta Appl. Math.* **16** (1989), 167–189.

[**23**] Reid, G.J., Finding symmetries of differential equations without integrating determining equations, *Eur. J. Appl. Math.*, to appear.

[**24**] Schwarz, F., Automatically determining symmetries of partial differential equations, *Computing* **34** (1985), 91–106.

[**25**] Shokin, Yu. I., *The Method of Differential Approximation*, Springer-Verlag, New York, 1983.

[**26**] Topunov, V.L., Reducing systems of linear differential equations to a passive form, *Acta Appl. Math.* **16** (1989), 181–206.

[**27**] Vorob'ev, E. M., Reduction and quotient equations for differential equations with symmetries, *Acta Appl. Math.* **23** (1991), 1–24.

[**28**] Yanenko, N.N., Theory of consistency and methods of integrating systems of nonlinear partial differential equations, in: *Proceedings of the Fourth All-Union Mathematics Congress*, Leningrad, 1964, pp. 247–259. *(in Russian).*