Higher Order Symmetries of Underdetermined Systems of Partial Differential Equations and Noether’s Second Theorem

Peter J. Olver
School of Mathematics
University of Minnesota
Minneapolis, MN 55455
olver@umn.edu
http://www.math.umn.edu/~olver

Abstract. Every underdetermined system of partial differential equations arising from a variational principle admits an infinite hierarchy of higher order generalized symmetries. These symmetries are a consequence of the Noether dependencies among the Euler–Lagrange equations that follow from Noether’s Second Theorem. This result is a consequence of a more general theorem on the existence of higher order generalized symmetries for any system of differential equations that admits an infinitesimal symmetry generator depending on an arbitrary function of the independent variables.

There are two well known classes of partial differential equations that admit infinite hierarchies of higher order generalized symmetries, [10, 12]. The first consists of linear systems of partial differential equations that admit a nontrivial point symmetry group, as well as systems that can be linearized into one of these by a change of variables. A second consists of integrable nonlinear partial differential equations such as the Korteweg–deVries equation, the nonlinear Schrödinger equation, Burgers’ equation, etc. Indeed, an interesting question, [4, 5, 7, 10], is whether, under certain conditions, the existence of higher order symmetries or, more generally, an infinite hierarchy of higher order symmetries, implies integrability. The purpose of this note is to introduce a third general class: underdetermined systems of partial differential equations that admit an infinite-dimensional

March 6, 2021
symmetry group depending on one or more arbitrary functions of the independent variables. A cautionary consequence of the latter fact is that any integrability or linearizability criterion that is based on higher order symmetries will not be applicable to such systems.

An important subclass of the third category are the underdetermined systems arising from a variational principle that admits an infinite-dimensional variational symmetry group depending on one or more arbitrary functions of the independent variables. Noether’s Second Theorem, [11, 12], tells us that the conservation laws associated with such a symmetry group, as provided by the Noether integration by parts identity, are all trivial. On the other hand, the associated Euler–Lagrange equations are underdetermined, meaning that they admit a Noether dependency: a nontrivial linear combination of them and their derivatives that vanishes identically. This result, when specialized to Einstein’s equations of general relativity based on Hilbert’s variational principle, allowed Noether to explain why the energy conservation law in relativistic theories is trivial: it is a consequence of the fact that the corresponding time translational symmetry group is contained in such an infinite-dimensional variational symmetry group, whose associated Noether dependencies are the relativistic Bianchi identities. Noether’s remarkable result resolved an issue that perplexed both Einstein and Hilbert in the early days of general relativity. We refer the reader to Kosmann-Schwarzbach’s book, [9], for an in depth presentation of the historical details surrounding both of Noether’s Theorems.

The main theorems in this paper are relatively easy consequences of the basic calculus of symmetry groups and conservation laws of differential equations, as developed, for example, in [12], whose results and notation we will use throughout. In particular, all functions are assumed to be smooth, i.e., $C^\infty$. Since the first main result appears not to have been noticed previously, the author deemed it worth setting it down in print. On the other hand, a result similar to our second theorem for variational problems can be found in a paper by Fulp, Lada, and Stasheff, [8], in which they allow arbitrary differential functions in their definition of gauge symmetries, and then apply Noether’s method to derive the Noether dependencies. Later, in a recent survey paper, [1], Anco proved that an underdetermined system of differential equations possesses adjoint symmetries that depend on an arbitrary differential function; as with Noether’s result, the associated conservation laws are trivial. In the case of a system of Euler–Lagrange equations, adjoint symmetries coincide with ordinary symmetries, and hence Anco’s result includes that in [8]. Apparently, the first place in the literature where generalized symmetries depending upon an arbitrary differential function appear is in a paper of I. Anderson and C. Torre, [2, 3], in which they prove that the only generalized symmetries of Einstein’s vacuum field equations are scalings of the metric tensor and what they call “infinitesimal generalized diffeomorphisms”. The latter arise from the invariance of relativity under space-time diffeomorphisms, and so are considered “physically trivial” and not indicative of any underlying integrability property of the Einstein equations.

To start with, suppose that we have a system of differential equations

$$\Delta(x, u^{(n)}) = 0$$

(1)

involving the independent variables $x = (x^1, \ldots, x^p)$ and dependent variables $(u^1, \ldots, u^q)$. In general, to avoid distracting technicalities, we assume (1) is both involutive, [14], and
locally solvable, [12]. We will work exclusively with infinitesimal symmetries in characteristic form. Thus, a (generalized) evolutionary vector field

\[ v_Q = \sum_{\alpha=1}^{q} Q_\alpha \frac{\partial}{\partial u_\alpha} \]  \hspace{1cm} (2)

forms an infinitesimal symmetry of (1) if and only if it satisfies the infinitesimal determining equations

\[ \text{pr} v_Q[\Delta] = 0 \quad \text{whenever} \quad \Delta = 0. \]  \hspace{1cm} (3)

In other words, the infinitesimal invariance criterion holds on all solutions to the system (1), taking into account the equations and all their derivatives. Further,

\[ \text{pr} v_Q = \sum_{\alpha=1}^{q} \sum_{#J \geq 0} D_J Q_\alpha \frac{\partial}{\partial u_\alpha^J}, \]  \hspace{1cm} (4)

denotes the standard jet space prolongation of the evolutionary vector field (2) whose coefficients are obtained by applying the iterated total derivative operators \( D_J = D_{j_1} \cdots D_{j_k} \) — where \( J = (j_1, \ldots, j_k) \) are symmetric multi-indices of order \( ^\dagger 0 \leq k = \#J \) — to the individual components of the characteristic. Similarly, if \( h(x) \) is a smooth function of the independent variables, we denote its partial derivatives (which coincide with its total derivatives) by \( h_J(x) = \partial_J h(x) = D_J h \), where again \( J = (j_1, \ldots, j_k) \) is a symmetric multi-index.

Let us now state the first main result.

**Theorem 1.** Given a system of differential equations, suppose

\[ Q[x, u^{(n)}; \ldots h_J(x) \ldots] \]

is the characteristic of an infinitesimal symmetry \( v_Q \) that depends on finitely many derivatives of an arbitrary function \( h(x) \) of the independent variables. If \( F(x, u^{(n)}) \) is any differential function, then the characteristic

\[ \tilde{Q}(x, u^{(n)}) = Q[x, u^{(n)}; \ldots D_J F \ldots] \]  \hspace{1cm} (5)

obtained by replacing the derivatives of \( h \) by the corresponding total derivatives of \( F \) also determines an infinitesimal symmetry \( v_{\tilde{Q}} \) of the system. Thus, any such system of differential equations automatically admits an infinite family of higher order symmetries depending upon an arbitrary function \( F \) of the independent variables, the dependent variables, and their derivatives of arbitrarily high order.

**Proof:** First, since \( h(x) \) is an arbitrary function of all the independent variables, its partial derivatives \( h_J(x) \) can assume any values\(^\dagger\). The \( h_J \) can therefore be treated as

---

\(^\dagger\) When the order \( \#J = 0 \), by convention \( u_\alpha^J = u_\alpha \) and \( D_J \) is the identity operator.

\(^\dagger\) All expressions only involve finitely many of the \( h_J \), and so no convergence issues arise.
algebraically independent quantities appearing in the algebraic relations prescribed by the
symmetry determining equations (3). Since they are independent, they can be replaced
by any other quantities, \( h_J \mapsto c_J \), independent or not, without affecting these algebraic
relations.

Furthermore, according to the prolongation formula (4), the coefficients of \( pv \) \( Q \) are
obtained by total differentiation and, as noted above, the partial derivatives of \( h \) coincide
with its total derivatives. Thus, if we write out their explicit formulas
\[
D_J Q_\alpha = F_{\alpha,I} \left[ x, u^{(n)}; \ldots \partial_J h(x) \ldots \right]
\]
as functions of the jet coordinates and the partial (total) derivatives of \( h \), then, replacing
\( h \) by \( F \) in \( Q \) as in (5) leads to the same algebraic expressions for its total derivatives,
\[
D_J \hat{Q}_\alpha = F_{\alpha,I} \left[ x, u^{(n)}; \ldots D_J F \ldots \right],
\]
in terms of the jet coordinates and the total derivatives of \( F \). By the preceding remarks, we
can thus replace each \( h_J(x) \) in the determining equations (3) for \( v_\alpha \) by the corresponding
total derivative \( D_J F \) without affecting their validity. We conclude that \( \hat{v}_\alpha \) also satisfies
the symmetry determining equations for the system of differential equations. \( Q.E.D. \)

Next, suppose we have a variational problem
\[
I[u] = \int L(x, u^{(n)}) \, dx \tag{6}
\]
with Euler–Lagrange equations
\[
\Delta_\alpha = E_\alpha(L) = 0, \quad \alpha = 1, \ldots, q, \tag{7}
\]
where \( E_\alpha \) denotes the variational derivative or Euler operator corresponding to the depen-
dent variable \( u^\alpha \). According to Noether’s Second Theorem, \( I[u] \) admits an infinitesimal
symmetry \( v_\alpha \) whose characteristic depends linearly\(^\dagger\) on an arbitrary function \( h(x) \) —
meaning that
\[
Q_\alpha = D_\alpha h, \quad \alpha = 1, \ldots, q, \tag{8}
\]
where \( D_1, \ldots, D_q \) are linear differential operators, which may depend on \( (x, u^{(n)}) \) — if and
only if its Euler–Lagrange equations satisfy the corresponding Noether dependency
\[
\sum_{\alpha=1}^q D_\alpha^* E_\alpha(L) = 0, \tag{9}
\]
where \( D_\alpha^* \) denotes the formal adjoint of the differential operator \( D_\alpha \), [12]. The existence
of a nontrivial Noether dependency implies that the Euler–Lagrange equations of such a
variational problem form an underdetermined system of differential equations.

\(^\dagger\) If \( Q \) depends nonlinearly on \( h \), then, using the argument in the proof of Noether’s Second
Theorem given in [11, 12], its linearization with respect to \( h \) also forms a symmetry, and so there
is no loss in generality assuming linearity of \( Q \) in \( h \).
Remark: Except in completely trivial cases, the existence of a Noether dependency and thus such a symmetry generator requires that the number of dependent variables \( q \geq 2 \).

According to \([12; \text{Theorem 4.14}]\), any variational symmetry is also a symmetry of the Euler–Lagrange equations. Thus, one can adapt the argument used to justify Theorem 1 to prove that, when we replace \( h \) by an arbitrary differential function \( F(x, u^{(n)}) \) as in (5), the resulting vector field \( \hat{v}_Q \) remains a variational symmetry and a symmetry of the Euler–Lagrange equations. Since the argument leading to the Noether dependency (9) can be reversed, every linear dependency of the Euler–Lagrange equations produces a corresponding infinite-dimensional symmetry group of the variational problem in the form (8). Thus, we deduce the second main theorem, which can be viewed as a counterpart of the aforementioned results in \([1, 8]\).

**Theorem 2.** If \( E(L) = 0 \) is any underdetermined system of Euler–Lagrange equations, then it admits generalized symmetries of arbitrarily high order depending upon one or more arbitrary differential functions.

This result resolves a mystery concerning Noether’s Second Theorem, which relies on infinitesimal symmetries that involve one or more arbitrary functions of the \( p \) independent variables. However, one can perform a “hodograph-type” change of variables in which the roles of independent and dependent variables are interchanged; see \([12; \text{Theorem 4.8}]\) for how the Euler operators behave. Consequently, such a change of variables does not affect the existence of Noether dependencies for the transformed variational problem. On the other hand, the transformed symmetries no longer involve functions solely of the new independent variables, and so Noether’s Second Theorem implies the existence of an ostensibly different symmetry generator depending on arbitrary functions of the new independent variables. What Theorems 1 and 2 imply is that, if the variational problem or system of differential equations admits an infinite-dimensional symmetry group depending on arbitrary functions of any \( p \) of the independent and dependent variables \( (x, u) = (x^1, \ldots, x^p, u^1, \ldots, u^q) \), then it automatically admits the enlarged symmetry group depending, in the same manner, on all of the independent and dependent variables and, in fact, on any finite collection of jet variables also. In other words, Noether’s Second Theorem does not, in fact, rely on any artificial distinction between independent and dependent variables!

Let us finish by illustrating the preceding results with two examples, that are based on \([12; \text{Examples 5.70 and 5.71}]\).

Systems of differential equations or variational problems for curves, surfaces, etc., that do not depend on any underlying parametrization thereof are called parameter-independent or parametric, cf. \([12]\). In other words, a system of differential equations involving the independent variables \( x = (x^1, \ldots, x^p) \in X \) is parameter-independent if it admits the symmetry pseudo-group of all local diffeomorphisms \( x \mapsto \Psi(x) \) of the base space \( X \). The infinitesimal generator of this symmetry group consists of all vector fields on the base:

\[
\mathbf{v} = \sum_{i=1}^{p} \xi^i(x) \frac{\partial}{\partial x^i},
\]  

(10)
where the $\xi^i(x)$ are arbitrary functions of the independent variables. Their evolutionary representative takes the form

$$v_Q = \sum_{\alpha=1}^{q} \left( \sum_{i=1}^{p} \xi^i(x)u_i^\alpha \right) \frac{\partial}{\partial u^\alpha}, \quad \text{where} \quad u_i^\alpha = \frac{\partial u^\alpha}{\partial x^i},$$

and, by a general principle, is also an infinitesimal symmetry. Theorem 1 thus immediately implies that any parameter-independent system of differential equations admits hierarchies of generalized symmetries depending on $p = \dim X$ arbitrary differential functions.

**Theorem 3.** A system of differential equations $\Delta(x, u^{(n)}) = 0$ is parameter-independent if and only if it admits all generalized infinitesimal symmetry generators of the form

$$v_Q = \sum_{\alpha=1}^{q} \left( \sum_{i=1}^{p} u_i^\alpha F_i(x, u^{(n)}) \frac{\partial}{\partial u^\alpha} \right),$$

where $F_1(x, u^{(n)}), \ldots, F_p(x, u^{(n)})$, are arbitrary differential functions.

In particular, any system of Euler–Lagrange equations (7) arising from a parameter-independent variational problem admits the generalized symmetries (12) along with the $p$ consequential Noether dependencies

$$\sum_{\alpha=1}^{q} u_i^\alpha E_\alpha(L) = 0, \quad i = 1, \ldots, p.$$  

We remark that one can explicitly characterize all nondegenerate parameter-independent systems of differential equations and variational problems in terms of the differential invariants and invariant volume form of the diffeomorphism pseudo-group, cf. [6, 12, 13].

**Example 4.** As in [12; Example 5.70], let $p = 1$ and $q = 2$, so there is a single independent variable $x$ and two dependent variables $u, v$. The solution to a parameter-independent differential equation or variational problem can be regarded as a plane curve $C \subset \mathbb{R}^2$, and is independent of any particular parametrization $x \mapsto (u(x), v(x))$ thereof. In particular, second order system of differential equations is parameter-independent if and only if it is equivalent to one in the form

$$H_1\left(u, v, \frac{v_x}{u_x}, \frac{u_xv_{xx} - u_{xxx}v_x}{u_x^2}\right) = H_2\left(u, v, \frac{v_x}{u_x}, \frac{u_xv_{xx} - u_{xxx}v_x}{u_x^3}\right) = 0.$$  

If we use $u$ to parametrize the curve, so $v = v(u)$, then the system (14) should reduce to a single second order ordinary differential equation of the form

$$H\left(u, v, \frac{dv}{du}, \frac{d^2v}{du^2}\right) = 0.$$  

In other words, unless the system (14) is overdetermined, one of its parameter-independent equations is redundant.
According to Theorem 3, any such system (14) admits the higher order generalized symmetries
\[ v_Q = u_x F(x, u^{(n)}) \frac{\partial}{\partial u} + v_x F(x, u^{(n)}) \frac{\partial}{\partial v}, \]
where \( F(x, u^{(n)}) \) is an arbitrary differential function. (On the other hand, these symmetries do not carry over to the reduced ordinary differential equation (15), which only admits ordinary Lie symmetries.) If the system (14) is the Euler–Lagrange equations for a parametric (parameter-independent) variational problem, which must have the form
\[
I[u] = \int L(x, u, v, u_x, v_x) \, dx = \int G(u, v, \frac{v_x}{u_x}) \, u_x \, dx = \int G(u, v, \frac{dv}{du}) \, du,
\]
for some function \( G \), then it admits the Noether dependency
\[ u_x E_u(L) + v_x E_v(L) = 0. \]

**Example 5.** Following [12; Example 5.71], consider the variational problem
\[
I[u] = \int \int \frac{1}{2}(u_x + v_y)^2 \, dx \, dy, \quad \text{with Lagrangian} \quad L = \frac{1}{2}(u_x + v_y)^2, \tag{16}
\]
involving two independent variables \( x, y \), and two dependent variables \( u, v \). Its Euler–Lagrange equations are
\[
E_u(L) = -u_{xx} - v_{xy} = 0, \quad E_v(L) = -u_{xy} - v_{yy} = 0. \tag{17}
\]
The variational problem (16) admits the infinite-dimensional abelian symmetry group with generator
\[
v = -\frac{\partial h}{\partial y} \frac{\partial}{\partial u} + \frac{\partial h}{\partial x} \frac{\partial}{\partial v}, \tag{18}
\]
where \( h(x, y) \) is an arbitrary function of the independent variables. Noether’s Second Theorem produces the evident linear dependency among the Euler–Lagrange equations:
\[
D_y E_u(L) - D_x E_v(L) = 0. \tag{19}
\]

Theorem 1 implies that, for any differential function \( F \) depending on \( x, y, u, v \) and their derivatives, the evolutionary vector field
\[
\tilde{v} = -D_y F \frac{\partial}{\partial u} + D_x F \frac{\partial}{\partial v}
\]
also forms a variational symmetry, and thus a symmetry of the Euler–Lagrange equations, which is easy to check by direct computation. Thus, the underdetermined system (17) admits an infinite hierarchy of generalized symmetries of arbitrarily high order. On the one hand, as explained in [12; Proposition 5.22], since the system is linear, this fact is not so surprising. On the other hand, the same result holds for more complicated variational
problems admitting the same variational symmetry (18); for example the second order variational problem

\[ I[u] = \int \int \left[ \frac{1}{2} (u_{xx} + v_{xy}) (u_{xy} + v_{yy}) + \frac{1}{6} (u_x + v_y)^3 \right] \, dx \, dy, \]

with underdetermined nonlinear fourth order Euler–Lagrange equations

\[ u_{xxxx} + v_{xxyy} = (u_x + v_y)(u_{xx} + v_y), \quad u_{xxyy} + v_{xyyy} = (u_x + v_y)(u_{xy} + v_y), \]

satisfies the same properties as above.

While Theorem 2 implies the existence of higher order symmetries of any under-determined system of Euler–Lagrange equations, this result does not extend to general underdetermined systems of nonlinear partial differential equations. Indeed, if \( H(x, u^{(n)}) \) is any differential function, then the underdetermined system

\[ \Delta_1 = D_x H = 0, \quad \Delta_2 = D_y H = 0, \] (20)

satisfies the same linear dependency:

\[ D_y \Delta_1 - D_x \Delta_2 = 0. \]

An evolutionary infinitesimal generator \( v = Q \partial_u + R \partial_v \) will be a symmetry of (20) provided

\[ D_x \left[ \text{pr} \, v(H) \right] = D_y \left[ \text{pr} \, v(H) \right] = 0, \]

whenever (20) holds. It is clear that, by making \( H \) sufficiently complicated, one can ensure that there are no symmetries. Thus, such an underdetermined system does not admit an infinite-dimensional symmetry group of the required form, and hence Theorem 1 does not apply.

Acknowledgments: Thanks to Jim Stasheff for correspondence on Noether’s Second Theorem that inspired me to revisit it here. Also thanks to Stephen Anco, Peter Hydon, Yvette Kosmann-Schwarzbach, and Juha Pohjanpelto for references and additional remarks.
References


