CHAPTER XI

NONCLASSICAL AND CONDITIONAL SYMMETRIES

Peter J. Olver, Evgenii M. Vorob'ev

Lie's classical theory of symmetries of differential equations is an inspiring source for various generalizations aiming to find the ways for obtaining explicit solutions, conservation laws, linearizing substitutions, etc. This Chapter describes one of the possible extensions of the Lie theory of invariant solutions, first considered by Bluman and Cole [1969] and named the "nonclassical method". This method, and its equivalence to direct reduction methods of Clarkson and Kruskal [1989] and Galaktionov [1990], has become the focus of much research and many applications to physically important partial differential equations. It is clear that other related topics, such as partially invariant solutions, differentially partially invariant solutions, group foliation, and so on, will give rise to efficient and elegant methods of treating differential equations.

The material of the Chapter is split into two parts: theoretical background and the most important results. The first part is written mainly on the basis of the papers of Olver and Rosenau [1987], Olver [1994], and Vorob'ev [1986], [1989], [1991], [1992].

THEORY AND EXAMPLES

In order to discuss our subject, we will employ the standard geometric approach to the theory of symmetries of differential equations. A k–th order system of differential equations is naturally treated as a submanifold $E \subset J^k$ of the k-th order jet space on the space of independent and dependent variables. As described in the earlier chapters of this book, the classical Lie symmetries of differential equations may be characterized by the following features. Through the process of prolongation, which requires the group transformations preserve the intrinsic contact structure on the jet space, they define local groups of contact transformations on the k-th order jet spaces J^k. Such a transformation group will be a symmetry group of the system of differential equations $E \subset J^k$ if the transformations of the symmetry group *leave E invariant.* This implies that the group transformations map solutions of E onto solutions of E. The classical Lie symmetries are sometimes called *external* symmetries.

To date, several extensions of the classical Lie approach have been proposed. Each of them relaxes one or more of the basic properties obeyed by classical symmetry groups. If we relax the restriction that the infinitesimal generators determine geometrical transformations on a finite order jet space, then we are naturally led to the class of *generalized* or *Lie–Bäcklund* symmetries, first used by E. Noether in her famous theorem relating symmetries and conservation laws, Noether [1918]. If we only require that the symmetries preserve the restriction of the contact structure on the system of differential equations E, then we find the *internal* symmetries first discussed at length in the works of É. Cartan [1914], [1915]. Actually, for a wide class of differential equations all internal symmetries are generated by external symmetries — see Anderson, Kamran and Olver, [1993]. Lastly, if the symmetries only leave invariant a certain submanifold of E, we find the class of *conditional* symmetries, treated by Bluman and Cole [1969], Olver and Rosenau [1987] and, subsequently, many others. The term "conditional" is explained by the fact that the submanifold of E is determined by attaching additional differential equations (called differential constraints) to the original system E. The theory of differential constraints has its origins in the work of Yanenko [1964] on gas dynamics; see the book by Sidorov, Shapeev, and Yanenko [1984] for a survey of this method. The most popular way is to append to E a system of first order differential equations defined by the invariant surface conditions associated with a group that is not necessarily a symmetry group of the system, and to require that the resulting overdetermined system admits the prescribed group as a symmetry group. Other types of differential constraints give rise to partially invariant solutions, or separation of variables — see Olver and Rosenau [1986]. They are also can be set up so that the appended system admits *a priori* fixed group of transformations — see Fushchich, Serov, and Chopik [1988], Fushchich and Serov [1988]. Certainly, these examples do not exhaust all possible interesting classes of differential constraints, and the full applicability of the method remains unexplored.

11.1. THE NONCLASSICAL METHOD

We begin by presenting a version of the nonclassical symmetries first discussed in Bluman and Cole [1969], in their treatment of generalized self-similar solutions of the linear heat equation.

11.1.1. Theoretical background.

Consider a k-th order system E of differential equations

$$\Delta_\nu(x, u, u^{(k)}) = 0, \qquad \nu = 1, \dots, l, \tag{11.1}$$

in n independent variables $x = (x_1, \dots, x_n)$, and q dependent variables $u = (u^1, \dots, u^q)$, with $u^{(k)}$ denoting the derivatives of the u's with respect to the x's up to order k. Suppose that $\mathbf{v}$ is a vector field on the space $R^n \times R^q$ of independent and dependent variables:

$$\mathbf{v} = \sum_{i=1}^{n} \xi^i(x, u)\frac{\partial}{\partial x_i} + \sum_{\alpha=1}^{q} \varphi^\alpha(x, u)\frac{\partial}{\partial u^\alpha}. \tag{11.2}$$

(In what follows, the derivatives $\partial/\partial x_i$, $\partial/\partial u^\alpha$ and so on will be for short denoted by ∂_{x_i}, ∂_{u^α} and so on.) The graph of a solution

$$u^\alpha = f^\alpha(x_1, \dots, x_n), \qquad \alpha = 1, .., q, \tag{11.3}$$

to the system defines an n-dimensional submanifold $\Gamma_f \subset R^n \times R^q$ of the space of independent and dependent variables. The solution will be invariant under the one-parameter subgroup generated by $\mathbf{v}$ if and only if Γ_f is an invariant submanifold of this group. By applying the well known criterion of invariance of a submanifold under a vector field we get that (11.3) is invariant under $\mathbf{v}$ if and only if f satisfies the first order system E_Q of partial differential equations:

$$Q^{\alpha}(x,u,u^{(1)}) = \varphi^{\alpha}(x,u) - \sum_{i=1}^{n} \xi^{i}(x,u)\frac{\partial u^{\alpha}}{\partial x_i} = 0, \qquad \alpha = 1,\dots,q, \tag{11.4}$$

known as the *invariant surface conditions*. The q-tuple $Q = (Q^1,\dots,Q^q)$ is known as the *characteristic* of the vector field (11.3). Since all the solutions of (11.4) are invariant under $\mathbf{v}$, the first prolongation $\mathbf{v}^{(1)}$ of $\mathbf{v}$ is tangent to E_Q. Therefore, we conclude that invariant solutions of the system (11.1) are in fact solutions of the joint overdetermined system (11.1), (11.4). In what follows, the k-th prolongation of the invariant surface conditions (11.4) will be denoted by $E_Q^{(k)}$, which is a kth order system of partial differential equations obtained by appending to (11.4) its partial derivatives with respect to the independent variables of orders $j \leqslant k-1$.

For the system (11.1), (11.4) to be compatible, the kth prolongation $\mathbf{v}^{(k)}$ of the vector field $\mathbf{v}$ must be tangent to the intersection $E \cap E_Q^{(k)}$:

$$\mathbf{v}^{(k)}(\Delta_\nu)|_{E\cap E_Q^{(k)}} = 0, \qquad \nu = 1,\dots,l. \tag{11.5}$$

If the equations (11.5) are satisfied, then the vector field (11.2) is called a *nonclassical infinitesimal symmetry* of the system (11.1). The relations (11.5) are generalizations of the relations

$$\mathbf{v}^{(k)}(\Delta_\nu)|_{E} = 0, \qquad \nu = 1,\dots,l, \tag{11.6}$$

for the vector fields of the infinitesimal classical symmetries. Inserting l variables $u^{(n)}$ found from (11.1) into (11.6), taking the Taylor series of the functions $\mathbf{v}^{(k)}(\Delta_\nu)|_E$ with respect to the remaining variables and setting the coefficients of these series equal to zero generate an overdetermined system of linear differential equations of order not larger than k for the coefficients $\xi^i(x,u)$, $\varphi^\alpha(x,u)$ of the vector field (11.2) of the infinitesimal classical symmetries. A similar procedure is applicable to the case of the nonclassical infinitesimal symmetries with an evident difference that in general one has fewer determining equations than in the classical case. Therefore, we expect that nonclassical symmetries are much more numerous than classical ones, since any classical symmetry is clearly a nonclassical one.

The important feature of determining equations for nonclassical symmetries is that they are nonlinear. This implies that the space of nonclassical symmetries does not, in general, form a vector space. Moreover, the Lie bracket of two nonclassical symmetry vector fields is not, as a rule, a nonclassical symmetry. If $\lambda(x,u)$ is an arbitrary function, then the prolongation formulae for vector fields imply that

$$\left(\lambda\mathbf{v}\right)^{(k)}\Big|_{E_Q^{(k)}} = \lambda\mathbf{v}^{(k)}\Big|_{E_Q^{(k)}}. \tag{11.7}$$

Formula (11.7) means that if the vector field $\mathbf{v}$ is a nonclassical symmetry, then $\lambda\mathbf{v}$ is also a nonclassical symmetry yielding the same equations (11.4). This property allows us to normalize any one nonvanishing coefficient of the vector field (11.2) by setting it equal to one when finding nonclassical symmetries.

11.1.2. Nonclassical symmetries of Burgers' equation.

As an example of finding the nonclassical symmetries, consider the system E of first order equations

$$u_t + uv - v_x = 0, \qquad u_x - v = 0 \tag{11.8}$$

obtained from the well-known Burgers' equation $u_t + uu_x - u_{xx} = 0$. If we assume that the coefficient of ∂_t of the vector field (11.2) does not identically equal zero, then for the vector field

$$\mathbf{v} = \partial_t + \xi(t,x,u,v)\partial_x + \varphi(t,x,u,v)\partial_u + \psi(t,x,u,v)\partial_v, \tag{11.9}$$

the invariant surface conditions are

$$u_t + \xi u_x = \varphi, \qquad v_t + \xi v_x = \psi. \tag{11.10}$$

The equations (11.5) take the form:

$$\begin{gathered} D_t\varphi - u_x D_t\xi + v\varphi + u\psi - D_x\psi - v_x D_x\xi = 0, \\ D_x\varphi - u_x D_x\xi - \psi = 0, \end{gathered} \tag{11.11}$$

where $D_t = \partial_t + u_t\partial_u + v_t\partial_v$, $D_x = \partial_x + u_x\partial_u + v_x\partial_v$. The variables u_t, u_x, v_t, v_x found from (11.8), (11.10):

$$u_t = \varphi - \xi v, \qquad u_x = v, \qquad v_x = \varphi - \xi v + uv, \qquad v_t = \psi - \xi(\varphi - \xi v + uv)$$

must be substituted into (11.11). After substituting, the latter becomes an underdetermined system of two differential equations for three unknown functions $\xi(t,x,u,v)$, $\varphi(t,x,u,v)$, $\psi(t,x,u,v)$. It is evident that this is a generic situation for first order systems in two independent variables. It means that there is a rich variety of nonclassical symmetries and the problem is how one can explicitly obtain them.

We will restrict ourselves to finding nonclassical symmetries (11.9) for which $\xi = \xi(t,x,u)$, $\varphi = \varphi(t,x,u)$, $\psi = \psi(t,x,u,v)$, which corresponds to the nonclassical symmetries of Burgers' equation itself. In that case the system (11.11) is written as

$$\begin{gathered} \varphi_t + \varphi_u(\varphi - \xi v) - v\big(\xi_t + \xi_u(\varphi - \xi v)\big) + \varphi v + uv - \\ \psi_x - \psi_u v - \psi_v(\varphi - \xi v) + (\xi_x + v\xi_u)(\varphi - \xi v + uv) = 0, \\ \varphi_x + v\varphi_u - v(\xi_x + v\xi_u) - \psi = 0. \end{gathered} \tag{11.12}$$

The second equation of (11.12) implies that the coefficient ψ is at most a quadratic function of v. After inserting ψ and its derivatives, as determined by the second equation in (11.12), into the first equation we obtain the equation:

$$\begin{gathered} \varphi_t + \varphi_u(\varphi - \xi v) - v\xi_t + \varphi v + u\big(\varphi_x + v\varphi_u - v(\xi_x + v\xi_u)\big) - \\ \varphi_{xx} - v\varphi_{ux} + v(\xi_{xx} + v\xi_{ux}) - v\big(\varphi_{xu} + v\varphi_{uu} - v(\xi_{xu} + v\xi_{uu})\big) - \\ (\varphi - \xi v)(\varphi_u\xi_x - 2v\xi_u) + \xi_x(\varphi - \xi v) = 0. \end{gathered} \tag{11.13}$$

The function in the left-hand side of (11.13) is a third order polynomial in v; its coefficients yield the equations:

$$\begin{gathered} \xi_{uu}=0, \qquad \varphi_{uu}=2\xi_{xu}+2(u-\xi)\xi_u, \\ 2\varphi\xi_u-2\xi\xi_x-\xi_t-2\varphi_{xu}+\xi_{xx}+u\xi_x+\varphi=0, \\ 2\varphi\xi_x+\varphi_t+u\varphi_x-\varphi_{xx}=0. \end{gathered} \tag{11.14}$$

¿From the first two equations of (11.14), we find that

$$\xi=a(t,x)u+b(t,x), \qquad \varphi=\frac{a(1-a)}{3}u^3+(a_x-ab)u^2+\alpha(t,x)u+\beta(t,x), \tag{11.15}$$

where $a(t,x)$, $b(t,x)$, $\alpha(t,x)$, $\beta(t,x)$ are arbitrary functions. After inserting (11.15) into the third equation of (11.14) we get a third order polynomial in u with the coefficient of u^3 equal to $a(1-a)(1+2a)$. From this we deduce that the function $a(t,x)$ is constant and it can take three values: $a=0$, $a=1$, $a=-1/2$. The other three coefficients of this polynomial imply the equations:

$$\begin{gathered} ab(1+2a)=0, \qquad (2a+1)(\alpha+b_x)=0, \\ b_t+2bb_x-b_{xx}+2\alpha_x-(1+2a)\beta=0, \end{gathered} \tag{11.16}$$

and the fourth equation of (11.14) yields the equations:

$$\begin{gathered} a(1+2a)b_x=0, \qquad a(b_t+2bb_x-b_{xx})-\alpha_x=0, \\ \alpha_t+\beta_x-\alpha_{xx}+2\alpha b_x=0, \qquad \beta_t-\beta_{xx}+2\beta b_x=0. \end{gathered} \tag{11.17}$$

In the case $a=0$ the functions $b(t,x)$, $\alpha(t,x)$, $\beta(t,x)$ satisfy the system:

$$\begin{gathered} \alpha_x=0, \qquad \alpha+b_x=0, \qquad b_t+2bb_x-\beta=0, \\ \alpha_t+\beta_x+2\alpha b_x=0, \qquad \beta_t-\beta_{xx}+2\beta b_x=0, \end{gathered}$$

which can easily be solved and produces the vector field

$$\mathbf{v}=(At^2+2Bt+C)\partial_t+(Atx+Bx+Dt)\partial_x+(Ax-Atu-Bu+D)\partial_u$$

with A, B, C, D parameters. This vector field belongs in fact to the five-dimensional Lie algebra of the classical symmetries of Burgers' equation. In the case $a=1$ the equations (11.16), (11.17) yield $b=0$, $\alpha=0$, $\beta=0$ so there exists only the nonclassical vector field $\mathbf{v}=\partial_t+u\partial_x$, with invariant solutions $u=(x-x_0)/(t-t_0)$.

Let $a=-1/2$, then the functions $b(t,x)$, $\alpha(t,x)$, $\beta(t,x)$ are found as solutions of the system:

$$b_t+2bb_x-b_{xx}+2\alpha_x=0, \qquad \alpha_t+\beta_x-\alpha_{xx}+2\alpha b_x=0, \qquad \beta_t-\beta_{xx}+2b_x\beta=0,$$

which admits solutions $\alpha=0$, $\beta=0$, and $b(t,x)$ satisfying Burgers' equation $b_t+2bb_x-b_{xx}=0$. This allows us to successively generate solutions of Burgers' equation from the solutions of the same equation. The process reminds one of the generation of solutions to soliton equations by successive Bäcklund transformations. Setting $b(t,x) = 0$ we can get the family of vector fields of the nonclassical symmetries:

$$\mathbf{v} = 4\partial_t - 2u\partial_x + (-u^3 + (\gamma t + \delta)u + \epsilon - \gamma x)\partial_u \tag{11.19}$$

with parameters γ, δ, ϵ.

Now assume that the coefficient of ∂_t in (11.2) equals zero and try to find the infinitesimal nonclassical symmetries of the form

$$\mathbf{v} = \partial_x + \phi(t,x,u)\partial_u + \psi(t,x,u,v)\partial_v,$$

for which the invariant surface conditions are the following ones:

$$u_x = \phi, \qquad v_x = \psi. \tag{11.20}$$

Relations (11.5) lead to the system of equations for the functions ϕ, ψ:

$$\phi_t + u\phi\phi_u - \psi\phi_u + \phi^2 + \psi u - \psi_x - \phi\psi_u - \psi_v\psi = 0, \qquad \phi_x + \phi_u\phi - \psi = 0. \tag{11.21}$$

These are not differential equations, since the arguments t, x, u, and v in (11.21) are tied by the relation $\phi(t,x,u) - v = 0$, which is a consequence of system (11.8), (11.20). There are no established methods to solve such systems. Severely restricting the class of solutions, one can regard (11.21) as a system of differential equations. Exact solutions of (11.21) yielding invariant solutions that are not invariant under classical symmetries have not yet been obtained.

11.1.3. Nonclassical symmetries and direct reduction methods.

Clarkson and Kruskal [1989] proposed a direct method for determining ansätze which reduce the partial differential equation to a single ordinary differential equation. This method was generalized by Galaktionov, [1990], who showed how to effect reductions to two (or more) coupled ordinary differential equations, and was applied to the study of blow-up of solutions to parabolic equations. In Arrigo, Broadbridge and Hill [1993], and Olver, [1994], it was proved that these reduction methods are equivalent to particular cases of the nonclassical symmetry method.

For simplicity, consider a partial differential equation (11.1) in two independent variables x, t. The differential equation admits a *direct reduction* if there exist functions $z = \zeta(x,t)$, $u = U(x,t,w)$, such that the Clarkson–Kruskal ansatz

$$u(x,t) = U(x,t,w(z)) = U\big(x,t,w(\zeta(x,t))\big) \tag{11.22}$$

reduces (11.1) to a single ordinary differential equation for $w = w(z)$. Let $\mathbf{w} = \tau(x,t)\partial_t + \xi(x,t)\partial_x$ be any vector field such that $\mathbf{w}(\zeta) = 0$, i.e. $\zeta(x,t)$ is the unique (up to functions thereof) invariant of the one-parameter group generated by $\mathbf{w}$. Applying $\mathbf{w}$ to the ansatz (11.22), we find

$$\tau u_t + \xi u_x = \tau U_t + \xi U_x \equiv V(x,t,w). \tag{11.23}$$

On the other hand, assuming $U_w \neq 0$, we can solve (11.22) for $w = W(x,t,u)$ using the Implicit Function Theorem. (We avoid singular points, and note that if $U_w \equiv 0$, the ansatz would not explicitly depend on w.) Substituting this into the right hand side of (11.23), we find that if u has the form (11.22), then it satisfies a first order quasi-linear partial differential equation of the form

$$\mathbf{w}(u) = \tau(x,t)u_t + \xi(x,t)u_x = \varphi(x,t,u). \tag{11.24}$$

Conversely, if u satisfies an equation of the form (11.24), then it can be shown that u satisfies a direct reduction type ansatz (11.22). Therefore, there is a one-to-one correspondence between ansätze of the direct reduction form (11.22) with $U_w \neq 0$ and quasi-linear first order differential constraints (11.24). Solutions $u = f(x,t)$ to (11.24) are just the functions which are invariant under the one-parameter group generated by the vector field

$$\mathbf{v} = \tau(x,t)\partial_t + \xi(x,t)\partial_x + \varphi(x,t,u)\partial_u. \tag{11.25}$$

Note in particular that $\mathbf{w}$ generates a group of "fiber-preserving transformations", meaning that the transformations in x and t do not depend on the coordinate u.

In the direct method, one requires that the ansatz (11.22) reduces the partial differential equation (11.1) to an ordinary differential equation. In the nonclassical method of Bluman and Cole, one requires that the differential constraint (11.24) which requires the solution to be invariant under the group generated by $\mathbf{w}$ be compatible with the original partial differential equation (11.1), in the sense that the overdetermined system of partial differential equations defined by (11.1), (11.24), has no integrability conditions. The following result demonstrates the equivalence of the nonclassical method (with projectable symmetry generator) and the direct method.

THEOREM. *The ansatz (11.22) will reduce the partial differential equation (11.1) to a single ordinary differential equation for $w(z)$ if and only if the overdetermined system of partial differential equations defined by (11.1), (11.24), is compatible.*

The proof and generalizations to Galaktionov's "nonlinear separation" are discussed in Olver [1994]. See also Arrigo, Broadbridge and Hill [1993], and Zidowitz [1993].

As an example, the ansatz $u = w(z) - t^2$, where $z = x - \frac{1}{2}t^2$ reduces the Boussinesq equation

$$u_{tt} + \tfrac{1}{2}(u^2)_{xx} + u_{xxxx} = 0$$

to a fourth order ordinary differential equation

$$w'''' + ww'' + (w')^2 - w' + 2.$$

This reduction follows from the constraint $tu_x + u_t + 2t = 0$ arising from the nonclassical symmetry $\mathbf{v} = t\partial_x + \partial_t - 2t\partial_u$.

11.2 MULTIDIMENSIONAL MODULES OF NONCLASSICAL SYMMETRIES

As was mentioned above, the Lie bracket $[\mathbf{v}, \mathbf{w}]$ of two infinitesimal nonclassical symmetries $\mathbf{v}$ and $\mathbf{w}$ is not in general a nonclassical infinitesimal symmetry. The easiest way to be convinced of this fact is to consider the Lie bracket of two distinct vector fields (11.19). For nonclassical symmetries the analog of multidimensional Lie algebras of the classical symmetries is multidimensional module or differential system. Let us consider the simplest case of the latter. Consider a two-dimensional differential system $\mathbf{g}$ (or distribution) on the space of independent and dependent variables, spanned by two independent vector fields $\mathbf{v}$ and $\mathbf{w}$. Thus $\mathbf{g}$ is defined as the set of all vector fields $\mathbf{u}$ which can be represented in the form $\mathbf{u} = f(x,u)\mathbf{v} + g(x,u)\mathbf{w}$. Suppose that $\mathbf{g}$ is involutive, i.e. closed under the Lie bracket: $[\mathbf{v}, \mathbf{w}] \subset \mathbf{g}$. In this case, Frobenius' Theorem implies that we can find a new basis of $\mathbf{g}$ vector fields of which have vanishing Lie bracket. Denote by $E_{\mathbf{g}}$ the union of the invariant surface conditions (11.4) for $\mathbf{v}$ and $\mathbf{w}$; the solutions of $E_{\mathbf{g}}$ are the $\mathbf{g}$ invariant functions. As in the case of one-dimensional nonclassical modules, $E_{\mathbf{g}}$ is invariant under $\mathbf{g}$. For $\mathbf{g}$ to be a two-dimensional module of nonclassical symmetries it is necessary that the basis vector fields $\mathbf{v}$ and $\mathbf{w}$ satisfy the equations (11.5) with $E_Q^{(k)}$ replaced by $E_{\mathbf{g}}^{(k)}$.

11.2.1. Two-dimensional modules of nonclassical symmetries of nonlinear acoustics equations.

The following system of equations:

$$uu_t + u_x + v_y = 0, \qquad u_y - v_t = 0 \tag{11.26}$$

describes a sound beam propagating in a nonlinear medium. Consider the two-dimensional involutive module $\mathbf{g}$ of vector fields with the basis

$$\mathbf{v} = \partial_t + \xi(u,v)\partial_y, \qquad \mathbf{w} = \partial_x + \zeta(u,v)\partial_y.$$

The vector fields $\mathbf{v}$ and $\mathbf{w}$ commute, therefore the coefficients $\xi(u,v)$, $\zeta(u,v)$ are found from equations (11.5), and the system $E_{\mathbf{g}}$ is

$$\begin{aligned} u_t + \xi u_y &= 0, \qquad u_x + \zeta u_y = 0,\\ v_t + \xi v_y &= 0, \qquad v_x + \zeta v_y = 0. \end{aligned} \tag{11.27}$$

The system (11.26), (11.27) can be treated as a system of six linear homogeneous algebraic equations for six first derivatives of the functions ξ, ζ. For this sytem to admit nontrivial solutions, the equation

$$\zeta(u,v) = -\big(1 + u\xi^2(u,v)\big)/\xi(u,v)$$

must hold. In this case all derivatives can be expressed through the derivative u_y :

$$\begin{gathered} u_t = -\xi u_y, \qquad u_x = \frac{1+u\xi^2}{\xi} u_y, \qquad v_t = u_y \\ v_x = -\frac{1+u\xi^2}{\xi^2} u_y, \qquad v_y = -\frac{u_y}{\xi}. \end{gathered} \tag{11.28}$$

Calculating $\mathbf{v}^{(1)}(uu_t + u_x + v_y)$, $\mathbf{v}^{(1)}(u_y - v_t)$, $\mathbf{w}^{(1)}(uu_t + u_x + v_y)$, $\mathbf{w}^{(1)}(u_y - v_t)$ and inserting (11.28), we find that they are identically zero. So the system (11.26) admits the two-dimensional module of nonclassical symmetries with the basis vector fields

$$\mathbf{v} = \partial_t + \xi(u,v)\partial_y, \qquad \mathbf{w} = \partial_x - \frac{1 + u\xi(u,v)^2}{\xi(u,v)}\partial_y,$$

where $\xi(u,v)$ is an arbitrary function.

11.2.2. Bäcklund transformations and two-dimensional modules of nonclassical symmetries for the sine-Gordon equation.

It is well known that if $u = f(x,y)$ is any particular solution to the sine-Gordon equation

$$2u_{xy} = \sin 2u, \tag{11.29}$$

then the system of equations

$$u_x = -f_x + \sin(u - f), \qquad u_y = f_y + \sin(u + f) \tag{11.30}$$

determines the Bäcklund transformation for the sine-Gordon equation (11.29). The equation (11.29) can be treated as a compatibility condition for the system (11.30). The function $u(x,y)$ found from the system (11.30) is a solution of the equation (11.29). From the point of view of the nonclassical symmetries this can be interpreted as follows.

Consider the vector fields

$$\mathbf{v} = \partial_x + (-f_x + \sin(u - f))\partial_u, \qquad \mathbf{w} = \partial_y + (f_y + \sin(u + f))\partial_u.$$

Provided the function $f(x,y)$ is a solution of (11.29) these vector fields have vanishing Lie bracket, so they give rise to a two-dimensional involutive module $\mathbf{g}$ with $E_{\mathbf{g}}$ given by (11.30). In the case considered, the relation $E \cap E_{\mathbf{g}}^{(2)} = E_{\mathbf{g}}^{(2)}$ holds, hence $\mathbf{g}$ is a two-dimensional module of the nonclassical symmetries for the sine-Gordon equation (11.29) with all invariant functions under $\mathbf{g}$ automatically satisfying (11.29).

11.3. PARTIAL SYMMETRIES

A further step towards generalization of the classical symmetries is consideration of tangent transformations instead of point ones.

11.3.1. Contact transformations and modules of partial symmetries.

According to Bäcklund's Theorem, in the case of one unknown function $u(x)$, transformations of J^k that preserve the contact structure (contact transformations) are prolongations of either point transformations or contact transformations on J^1. The infinitesimal contact transformations on the space J^1 are in one-to-one correspondence with their characteristic functions $Q(x, u, u^{(1)})$, which generates the contact vector field

$$\mathbf{v}_Q = -Q_{u_{x_i}}\partial_{x_i} + (Q - u_{x_i}Q_{u_{x_i}})\partial_u + (Q_{x_i} + u_{x_i}Q_u)\partial_{u_{x_i}}.$$

This vector field is the first prolongation of a point transformation if and only if Q is an affine function of the derivative coordinates $u_{x_1}, \dots, u_{x_n}$. A function $u = f(x)$ is invariant with respect to $\mathbf{v}_Q$ if and only if it satisfies the invariant surface condition

$$Q(x, u, u^{(1)}) = 0. \tag{11.31}$$

Let E_Q denote the submanifold of J^1 determined by equation (11.31), and $E_Q^{(k)}$ its prolongation to J^k. The contact vector field $\mathbf{v}_Q$ is called a *partial symmetry* of the differential equation

$$\Delta(x, u, u^{(k)}) = 0 \tag{11.32}$$

if $\mathbf{v}_Q^{(k)}$ is tangent to the intersection $E \cap E_Q^{(k)}$.

Partial symmetries admit the natural structure of a one-dimensional module. Clearly, if Q generates a partial symmetry, so does $\tilde{Q} = gQ$, where g is an arbitrary function on J^1, since

$$\mathbf{v}_{gQ}^{(k)}\big|_{E_Q^{(k)}} = g\mathbf{v}_Q^{(k)}\big|_{E_Q^{(k)}}. \tag{11.33}$$

Relation (11.33) implies that, as long as it depends explictily on the derivative coordinates, the characteristic function of the infinitesimal partial symmetry can be chosen in the form $Q = -u_{x_j} + \phi(x, u, \tilde{u}^{(1)})$ for some index j with $\tilde{u}^{(1)}$ denoting the set of first derivatives of u with the derivative u_{x_j} omitted.

Now consider the r-tuple $Q = (Q_1, \dots, Q_r)$ of functions on J^1 satisfying the relation

$$\operatorname{rank} ||\partial Q_i / \partial u_{x_j}|| = r \leqslant n \tag{11.34}$$

and r contact vector fields $\mathbf{v}_{Q_1}, \dots, \mathbf{v}_{Q_r}$. Let E_Q now denote the system of differential equations

$$Q_1(x, u, u^{(1)}) = 0, \quad \dots, \quad Q_r(x, u, u^{(1)}) = 0 \tag{11.35}$$

satisfied by functions $u = f(x)$ invariant under all of the vector fields $\mathbf{v}_{Q_1}, \dots, \mathbf{v}_{Q_r}$. The system (11.35) is compatible iff the relations

$$(Q_i, Q_j)|_{E_Q} = 0, \qquad 1 \leqslant i \leqslant j \leqslant r, \tag{11.36}$$

hold true. Here (Q_i, Q_j) is the "Lagrange" bracket of the functions Q_i and Q_j, defined as the characteristic function of the Lie bracket $[\mathbf{v}_{Q_i}, \mathbf{v}_{Q_j}]$, i.e.

$$[\mathbf{v}_{Q_i}, \mathbf{v}_{Q_j}] = \mathbf{v}_{(Q_i, Q_j)}. \tag{11.37}$$

Since $\mathbf{v}_{Q_i}(Q_j) = (Q_i, Q_j) + Q_j \partial Q_i / \partial u$, the submanifold E_Q is invariant under $\mathbf{v}_{Q_1}, \dots, \mathbf{v}_{Q_r}$ provided (11.36) are satisfied.

Any smooth function R defined on J^1 which vanishes on E_Q can be represented in the form $R = a_i Q_i$ for functions $a_i(x, u, u^{(1)})$. Therefore, the restriction of the vector field

$$\mathbf{v}_R = a_i \mathbf{v}_{Q_i} + Q_i \mathbf{v}_{a_i} - R\partial_u \tag{11.38}$$

to E_Q equals $a_i \mathbf{v}_{Q_i}|_{E_Q}$. Relation (11.38) makes valid the following assertion. Denote by $I(Q)$ the ideal of the functions on J^1 vanishing on E_Q and by $A(Q)$ the family of contact vector fields generated by the functions in $I(Q)$. Then the restriction of $A(Q)$ to E_Q is an r-dimensional module $\mathbf{g}$ of vector fields over the ring of smooth functions on E_Q. The basis of $\mathbf{g}$ consists of the restrictions of the vector fields $\mathbf{v}_{Q_1}, \dots, \mathbf{v}_{Q_r}$ to E_Q. Moreover, relations (11.36)–(11.38) imply that $\mathbf{g}$ is an involutive module: $[\mathbf{g}, \mathbf{g}] \subset \mathbf{g}$.

It is easily deduced from (11.34), (11.35) that the same module $\mathbf{g}$ is generated by the functions $\tilde{Q}_1 = -u_{x_{i_1}} + \psi_1(x, u, \tilde{u}^{(1)})$, ..., $\tilde{Q}_r = -u_{x_{i_r}} + \psi_r(x, u, \tilde{u}^{(1)})$, where $i_1, \dots, i_r$ are the indices of r linearly independent columns of the matrix (11.34). The functions $\tilde{Q}_1, \dots, \tilde{Q}_r$ are obtained by solving (11.35) with respect to the derivative coordinates $u_{x_{i_1}}, \dots, u_{x_{i_r}}$. They generate the contact vector fields $\mathbf{v}_{\tilde{Q}_1}, \dots, \mathbf{v}_{\tilde{Q}_r}$, which commute when restricted to E_Q.

In the case the relations

$$\mathbf{v}_{Q_i}(\Delta)|_{E \cap E_Q} = 0, \qquad i = 1, \dots, r, \tag{11.39}$$

are satisfied, the module $\mathbf{g}$ is called *an r-dimensional module of infinitesimal partial symmetries of* (11.32).

11.3.2. Two-dimensional modules of partial symmetries for a family of nonlinear heat equations.

Consider the family of nonlinear heat equations

$$u_t = (f(u)u_x)_x + g(u) \tag{11.40}$$

in one space variable x with $f(u)$, $g(u)$ smooth functions. We try to find two-dimensional modules of partial symmetries of (11.40). The characteristic functions Q_1, Q_2 taken as

$$Q_1 = -u_t + a(t, x, u), \qquad Q_2 = -u_x + b(t, x, u) \tag{11.41}$$

imply the compatibility condition (11.36) in the form

$$a_x + a_u b - b_t - a b_u = 0. \tag{11.42}$$

The system E_Q in the case considered, precisely,

$$u_t = a(t, x, u), \qquad u_x = b(t, x, u), \tag{11.43}$$

admits one-parameter family of solutions. One of the six equations determining the system $E \cap E_Q^{(2)}$, which is geometrically a two-dimensional surface in the eight-dimensional space J^2, looks like

$$a = (b_x + b b_u) f + b^2 f' + g. \tag{11.44}$$

If this equation is not a differential consequence of (11.39), we have to solve functional equations for the functions $a(t, x, u)$, $b(t, x, u)$ obtained by restricting (11.39) to the two-dimensional surface given by (11.44) in the space R^3 having coordinates t, x, u. To avoid this difficult problem, we treat (11.44) as a differential equation satisfied by $a(t,x,u)$, $b(t,x,u)$. In this approach relations (11.39) are automatically satisfied, and in order that $\mathbf{v}_{Q_1}$, $\mathbf{v}_{Q_2}$ generate the two-dimensional module $\mathbf{g}$ of partial symmetries of (11.40) the functions $a(t,x,u)$ and $b(t,x,u)$ must satisfy differential equations (11.42), (11.44).

If we substitute the function $a(t,x,u)$ given by (11.44) into (11.42), we obtain the equation

$$b_t = (b_{xx} + 2bb_{xu} + b^2 b_{uu})f + (3bb_x + 2b^2 b_u)f' + b^3 f'' + bg' - gb_u \qquad (11.45)$$

for the function $b(t,x,u)$. Let us try to find this function in the form: $b(t,x,u) = \theta(t)h(u)$, i.e. independent of x and admitting the separation of variables t, u. After substituting $b(t,u) = \theta(t)h(u)$ in (11.45) we obtain the relation

$$\dot{\theta}(t) = \theta^3(t)h(u)(f(u)h(u))'' + \theta(t)h(u)(g(u)/h(u))'.$$

This equation has nontrivial solutions provided

$$h(u)(f(u)h(u))'' = \lambda, \qquad h(u)(g(u)/h(u))' = \mu, \qquad (11.46)$$

with λ, μ constant. Equations (11.46) contain three unknown functions, and they can be treated from various points of view. For example, if we regard the function $h(u)$ as given, then (11.46) are equations for $f(u)$ and $g(u)$.

The function $b(t,u) = \theta(t)h(u)$ yields the invariant solutions

$$u(t,x) = F(\theta(t)x + \phi(t)) \qquad (11.47)$$

to (11.40) as it can be seen from the second equation in (11.43). The function $\theta(t)$ satisfies the ordinary differential equation $\dot{\theta} = \lambda\theta^3 + \mu\theta$ integrated explicitly, and the function $\phi(t)$ is a solution of the equation obtained after substituting (11.47) either into (11.40) or into the first equation in (11.41). If we take $h(u) = u^{-1}$, $f(u) = u^2+u$, $g(u) = u/2$, we obtain the family $u(t,x) = \sqrt{2((x+c)\exp t + \exp 2t)}$ of invariant solutions to the equation $u_t = ((u^2+u)u_x)_x + u/2$.

11.3.3. Partial symmetries and multidimensional integrable differential equations.

Consider the second order partial differential equation (11.32) in two independent variables. In this case partial symmetries allow us to associate new differential equations of the second order in four independent variables with equation (11.32). The solutions of the Cauchy problem for these associated equations can be expressed through solutions of the appropriately posed Cauchy problems for the original equation (11.32). In particular, each second order linear equation in two independent variables gives rise to multidimensional nonlinear equations which are linearizable in the sense described below. Let us consider the case of an evolution equation, and suppose that the characteristic function of the partial symmetry is taken in the form: $Q = -u_t + \phi(t,x,u,u_x)$. Then the relation

$$\mathbf{v}_Q^{(2)}(\Delta)|_{E\cap E_Q^{(2)}} = 0 \qquad (11.48)$$

satisfied by the partial symmetry $\mathbf{v}_Q$ is actually a differential equation of the second order for the function $\phi(t,x,u,u_x)$. Indeed, by the prolongation formula, the coefficients of the contact vector field $\mathbf{v}_Q^{(2)}$ depend on the derivatives of the function ϕ at most of the second order. Moreover, the variables u_t, u_{tx}, and u_{xx} can be found as functions of the variables t, x, u, u_x from the equations determining $E_Q^{(2)}$ and from (11.32). (We leave aside the case when the equation $u_t = \phi(t,x,u,u_x)$ is an intermediate integral of (11.32).) For example, if (11.32) is the linear heat equation $u_t = u_{xx}$, then equation (11.48) takes the form:

$$\phi_t = \phi^2\phi_{pp} + 2p\phi\phi_{up} + p^2\phi_{uu} + 2\phi\phi_{xp} + 2p\phi_{xu} + \phi_{xx}, \tag{11.49}$$

where $p = u_x$. Suppose that the Cauchy problem :

$$\phi|_{t=0} = \phi_0(x,u,p), \tag{11.50}$$

for (11.49) and the Cauchy problem :

$$u|_{t=0} = a(x) \tag{11.51}$$

for (11.32) are both well-posed. Let us determine which Cauchy data $a(x)$ generate solutions of (11.32) invariant under the vector field $\mathbf{v}_Q$ determined by the given solution ϕ of equation (11.49). Since invariant solutions of equation (11.32) are the common solutions to (11.32) and the equation $u_t - \phi(t,x,u,u_x) = 0$, we can consider this joint system at $t = 0$ and obtain the equation

$$a''(x) = \phi_0(x, a(x), a'(x))$$

for the initial function $a(x)$. The general solution of the latter equation depends on two constants c_1, c_2, so there appears the two-parameter family $u(t,x,c_1,c_2)$ of invariant solutions of the Cauchy problem (11.32), (11.51). Suppose that the system of equations

$$u(t,x,c_1,c_2) = v, \qquad u_x(t,x,c_1,c_2) = p$$

uniquely determines c_1 and c_2 as implicit functions of t,x,v,p, then

$$\phi(t,x,v,p) = u_x(t,x,c_1(t,x,v,p), c_2(t,x,v,p))$$

is a solution of the Cauchy problem (11.49), (11.50).

11.3.4. Induced classical symmetries.

Suppose $\mathbf{w}$ is a vector field of infinitesimal classical symmetries of system (11.1). Denote by $\exp(\epsilon\mathbf{w})$ the one-parameter family of transformations of the space $R^n \times R^q$ of independent and dependent variables associated with $\mathbf{w}$ and by $\exp(\epsilon\mathbf{w})_*$ the differential $d(\exp(\epsilon\mathbf{w}))$ of $\exp(\epsilon\mathbf{w})$ treated as a mapping of vector fields on $R^n \times R^q$. Then if $\mathbf{g}$ is a module of the Bluman-Cole infinitesimal symmetries of (11.1), the modules $\mathbf{g}_\epsilon$ =$\exp(\epsilon\mathbf{w})_*(\mathbf{g})$ forms a one-parameter family of modules of nonclassical symmetries for the same system.

To see this, assume that $\mathbf{v}_1, \dots, \mathbf{v}_r$ is a basis of $\mathbf{g}$. Generalizing the construction of Section 11.2.1 we can state that these vector fields must satisfy the relations

$$[\mathbf{v}_i, \mathbf{v}_j] \in \mathbf{g}, \qquad \mathbf{v}_j^{(k)}(E \cap E_{\mathbf{g}}^{(k)})|_{E\cap E_{\mathbf{g}}^{(k)}} = 0, \qquad 1 \leqslant i < j \leqslant r, \qquad j = 1, \dots, r. \tag{11.52}$$

Let $\mathbf{g}_\epsilon = \exp(\epsilon\mathbf{w})_*\mathbf{g}$, which is spanned by $\mathbf{v}_{\epsilon i} = \exp(\epsilon\mathbf{w})_*(\mathbf{v}_i)$. By the general properties of the mapping $\exp(\epsilon\mathbf{w})_*$ the module $\mathbf{g}_\epsilon$ is also an r-dimensional module of vector fields closed under the Lie bracket: $[\mathbf{v}_{\epsilon i}, \mathbf{v}_{\epsilon j}] \in \mathbf{g}$. The only solutions of the system $E_{\mathbf{g}}$ are vector-valued functions $u^{(\alpha)} = f_\alpha(x)$, $\alpha = 1, \dots q$, invariant under $\mathbf{g}$. Indeed, if Γ_f is invariant under $\mathbf{g}$, then $\exp(\epsilon\mathbf{w})(\Gamma_f)$ is a graph of a solution invariant under $\mathbf{g}_\epsilon$, hence $E_{\mathbf{g}_\epsilon} = \exp(\epsilon\mathbf{w})(E_{\mathbf{g}})$. From these considerations it follows that $\mathbf{g}_\epsilon$ is tangent to the intersection $E \cap E_{\mathbf{g}_\epsilon}^{(k)}$.

Modules of partial symmetries of equation (11.32) are treated quite similarly. The basic property of the mapping $\exp(\epsilon\mathbf{w})_*$ allows us to define the concept of vector fields of nonclassical symmetries invariant under $\mathbf{w}$. Such invariant vector field $\mathbf{v}$ satisfies the relation $[\mathbf{w}, \mathbf{v}] = \lambda\mathbf{v}$. Moreover, the flow $\exp(\epsilon\mathbf{w})$ determines the symmetry transformations of the determining equations for the coefficients of nonclassical infinitesimal symmetries.

11.3.5. Partial symmetries and differential substitutions.

The differential equations for the functions ϕ in $Q = -u_t + \phi(t, x, u, u_x)$ and ψ in $Q = -u_x + \psi(t, x, u, u_t)$ —more precisely, the equation (11.48) — inherit the classical Lie symmetries of the original equation (11.32). If the Lie algebra of infinitesimal symmetries of the equation (11.32) is at least two-dimensional, then the quotient equation for solutions of the equation (11.48) invariant under two-dimensional subalgebras is a differential equation in two independent variables just as equation (11.32). There exists a differential substitution of the group nature connecting these two equations.

We describe the origin of this differential substitution taking as an example the linear heat equation

$$u_t = u_{xx}. \tag{11.53}$$

The equation (11.53) admits the infinite dimensional Lie algebra $\mathbf{g}$ of the classical Lie symmetries with the generators:

$$\begin{gathered}\mathbf{v}_1 = \partial_t, \qquad \mathbf{v}_2 = 2t\partial_t + x\partial_x, \qquad \mathbf{v}_3 = 4t^2\partial_t + 4tx\partial_x - (x^2 + 2t)u\partial_u,\\ \mathbf{v}_4 = \partial_x, \qquad \mathbf{v}_5 = -2t\partial_x + xu\partial_u, \qquad \mathbf{v}_6 = u\partial_u, \qquad \mathbf{v}_\alpha = \alpha(t,x)\partial_u,\end{gathered} \tag{11.54}$$

where $\alpha(t, x)$ is an arbitrary solution of the equation (11.53). Consider the two-dimensional subalgebra $\mathbf{g}_2 \subset \mathbf{g}$ with the generators $\mathbf{v} = \mathbf{v}_6$ and $\mathbf{w} = \partial_u$, and the solutions ϕ of the equation (11.49) invariant under $\mathbf{g}_2$. In the space R^5 of the variables t, x, u, p, ϕ, where the first four are the arguments of the function ϕ, the vector fields $\mathbf{v}$ and $\mathbf{w}$ look as follows:

$$\mathbf{v} = u\partial_u + p\partial_p + \phi\partial_\phi, \qquad \mathbf{w} = \partial_u. \tag{11.55}$$

The solutions ϕ of (11.49) invariant under $\mathbf{g}_2$ are common solutions of (11.46) and the system of invariant surface conditions for the vector fields $\mathbf{v}$, $\mathbf{w}$ given by (11.55):

$$\begin{gathered} \phi_u = 0, \qquad \phi_{tu} = \phi_{xu} = \phi_{uu} = \phi_{up} = 0, \\ p\phi_p = \phi, \qquad p\phi_{tp} = \phi_t, \qquad p\phi_{xp} = \phi_x, \qquad \phi_{pp} = 0. \end{gathered} \tag{11.56}$$

We restrict the system (11.49), (11.56) to the submanifold N in the space R^5 determined by the equations $u = 0$, $p = 1$ and coordinatized by the variables t, x. After expressing the outer derivatives ϕ_{pp}, ϕ_{up}, ϕ_{uu}, ϕ_{xp}, ϕ_{xu} through the inner derivatives ϕ_t, ϕ_x, with the aid of (11.56) we obtain the quotient equation:

$$v_t = 2vv_x + v_{xx}. \tag{11.58}$$

This is Burgers' equation for the restriction $v(t, x)$ of the function $\phi(t, x, u, u_x)$ to the submanifold N. If $v(t, x)$ is a solution of the quotient equation (11.57), then the invariant solution of the equation (11.49) has the form $\phi(t, x, u, p) = pv(t, x)$. In their turn, the solutions of the heat equation invariant under the partial symmetry $\mathbf{v}_Q$ of the heat equation with the characteristic function $Q = -u_t + u_x v(t, x)$ are common solutions of the equation (11.53) and the equation

$$u_t = u_x v(t, x)$$

for invariant functions. The latter relation is a variant of the Hopf-Cole substitution linearizing Burgers' equation (11.57). See also the work of Guthrie [1993] for generalizations and additional applications of this method.

11.3.6. Partial symmetries and functionally invariant solutions. Consider a linear differential equation of the second order:

$$\sum_{i,j=1}^{n} a_{ij}(x) u_{x_i x_j} + \sum_{i=1}^{n} b_i(x) u_{x_i} = 0. \tag{11.58}$$

A solution $u(t, x)$ is called *functionally invariant* if $v(t, x) = F(u(t, x))$ is also a solution of the equation (11.58) for an arbitrary function $F(u)$. Functionally invariant solutions are common solutions of the equation (11.58) and the equation for its characteristics :

$$\sum_{i,j=1}^{n} a_{ij}(x) u_{x_i} u_{x_j} = 0. \tag{11.59}$$

For the wave equation

$$u_{tt} = u_{xx} + u_{yy}$$

equation (11.59) takes the form $u_t^2 = u_x^2 + u_y^2$. By direct calculations it is demonstrated that the vector field $\mathbf{v}_Q$ with the characteristic function $Q = u_t - \sqrt{u_x^2 + u_y^2}$ is a nonclassical partial symmetry of the wave equation. We see that functionally invariant solutions are solutions invariant under the partial symmetries.

11.3.7. Nonclassical symmetries and partially invariant solutions.

Let $\mathbf{g}$ be an r-dimensional involutive differential system on $R^n \times R^q$. A function $f: R^n \to R^q$ is called *partially invariant* under $\mathbf{g}$ if the orbit $G \cdot \Gamma_f$ of its graph has dimension strictly less than $n + \min(r, q)$. The quantity $\delta = \dim(G \cdot \Gamma_f) - n$ is called the *deficiency* of a partially invariant function f. Evidently, $0 \le \delta \le \min(q-1, r-1)$. The partially invariant functions satisfy the following system PE_δ of first order differential equations:

$$\operatorname{rank} \left\| \varphi_j^\alpha - \sum_{i=1}^n u_{x_i}^\alpha \xi_{ji} \right\| \le \delta. \tag{11.60}$$

The system PE_δ is invariant under $\mathbf{g}$. Provided $\mathbf{g}$ is a Lie algebra of infinitesimal classical symmetries of the system E, partially invariant solutions of the system E are the common solutions of E and PE_δ. See Ondich [1994] for applications of this method.

11.3.8. Nonclassical symmetries of the second type for the equations of nonlinear acoustics.

The preceding construction admits a natural generalization. A differential system $\mathbf{g}$ is called a *nonclassical symmetry of the second type* of the system E if the intersection $E \cap PE_\delta$ is invariant under $\mathbf{g}$.

The system (11.26) contains two unknown functions, so the only possible value of the deficiency index δ of partially invariant solutions of (11.26) is one. Consider the family of two-dimensional abelian Lie algebras $\mathbf{g}$ of vector fields with the generators:

$$\mathbf{v}_1 = \partial_v, \qquad \mathbf{v}_2 = \partial_t + \xi(t,x,y,u)\partial_x + \eta(t,x,y,u)\partial_y + \varphi(t,x,y,u)\partial_u + \psi(t,x,y,u)\partial_v.$$

We prescribe the coefficients ξ, η, φ, ψ so that the vector fields $\mathbf{v}_1$, $\mathbf{v}_2$ are tangent to the intersection $E \cap PE_1$. The system PE_1 is determined by the equation (11.60) which now looks like

$$\det \begin{Vmatrix} 0 & \varphi - u_t - \xi u_x - \eta u_y \\ 1 & \psi - v_t - \xi v_x - \eta v_y \end{Vmatrix} = 0$$

or

$$\varphi - u_t - \xi u_x - \eta u_y = 0. \tag{11.61}$$

The vector field $\mathbf{v}_1$ is a classical infinitesimal symmetry of the system (11.26) and the coefficients of the vector field $\mathbf{v}_2$ are found from the relations:

$$\mathbf{v}_2^{(1)}(uu_t + u_x + v_y)|_{E \cap PE_1} = 0, \qquad \mathbf{v}_2^{(1)}(u_y - v_t)|_{E \cap PE_1} = 0. \tag{11.62}$$

Finding the variables v_t, v_y, u_t from the equations (11.26), (11.61) and substituting the results into (11.62) yield the determining equations for the coefficients. One of the particular solutions of these equations:

$$\xi = 0, \qquad \eta = 0, \qquad \varphi = y, \qquad \psi = t - y^3/3$$

yields the following partially invariant solutions of (11.26):

$$u(t,x,y) = ty - \frac{y^4}{12} + h(x)y + e(x), \qquad v(t,x,y) = \frac{t^2}{2} - t\left(\frac{y^3}{3} - h(x)\right) +$$
$$\frac{y^6}{60} - h(x)\frac{y^3}{3} - [e(x) + h'(x)]\frac{y^2}{2} - e'(x)y + d(x)$$

with $h(x)$, $e(x)$, $d(x)$ arbitrary functions.

11.4. CONDITIONAL SYMMETRIES

Consider the system E of differential equations

$$\Delta_\nu(x, u, u^{(k)}) = 0, \qquad \nu = 1, \dots, l \tag{11.1}$$

and append it with the system of differential constraints E_Θ:

$$\Theta_\mu(x, u, u^{(k)}) = 0, \qquad \mu = 1, \dots, m. \tag{11.63}$$

Assume that the appended system $E \cap E_\Theta$ (11.1), (11.63) is compatible. A vector field $\mathbf{v}$ is called a *conditional* infinitesimal symmetry of the system E if the system $E \cap E_\Theta$ is invariant under $\mathbf{v}$, i.e., $\mathbf{v}$ is a Lie classical infinitesimal symmetry of $E \cap E_\Theta$. Possible approaches to constructing the appended system (11.63) are explained below, based on ideas contained in Fushchich, Serov, and Chopik [1988] and Fushchich and Serov [1988].

11.4.1. Conditional symmetries of a nonlinear heat equation.

As a first example consider the nonlinear heat equation

$$\Delta \equiv u_t - u u_{xx} - u_x^2 = 0. \tag{11.64}$$

The Lie algebra of the classical symmetries of the equation (11.64) has the generators

$$\mathbf{v}_1 = \partial_t, \qquad \mathbf{v}_2 = \partial_x, \qquad \mathbf{v}_3 = t\partial_t - u\partial_u, \qquad \mathbf{v}_4 = x\partial_x + 2u\partial_u. \tag{11.65}$$

Comparison with that of the linear heat equation reveals the fact that (11.65) does not contain an infinitesimal Galilean-like transformation

$$\mathbf{v} = t\partial_x + xh(u)\partial_u,$$

which is admissible by the linear heat equation $u_t = u_{xx}$ when $h(u) = -u/2$. Indeed, if we apply the prolonged vector field $\mathbf{v}^{(2)}$ to the left-hand side of (11.64), we obtain the function

$$\begin{aligned} \Theta &\equiv -\mathbf{v}^{(2)}(u_t - u u_{xx} - u_x^2) \\ &= u_x + 2u_x h + x u_{xx} h - x u_t h_u + 2u u_x h_u + 2x u_x^2 h_u + x u u_{xx} h_u + u x u_x^2 h_{uu} \end{aligned}$$

that does not vanish being restricted to E:

$$\Lambda \equiv u\Theta|_E = x u_t h + u u_x + 2u u_x h - x u_x^2 h + 2u^2 u_x h_u + x u u_x^2 h_u + x u^2 u_x^2 h_{uu} \not\equiv 0$$

So append the equation

$$\Lambda = 0 \tag{11.66}$$

to (11.64). Let us check the compatibility of (11.64) and (11.66) later on and try to find the function $h(u)$ so that the vector field $\mathbf{v}^{(2)}$ is tangent to the system (11.64), (11.66). The function $\Psi = \mathbf{v}^{(1)}(\Lambda)$ vanishes on the submanifold $\{\Delta = 0, \Lambda = 0\}$ if and only if

$$h(u) = -\frac{1}{2} + \frac{c}{u},$$

with c arbitrary constant. For checking the compatibility we apply the contact vector field $\mathbf{v}_\Lambda^{(2)}$ to Δ and find that the function $\mathbf{v}_\Lambda^{(2)}(\Delta)$ vanishes when restricted to the submanifold $\{\Delta = 0, \Lambda = 0\}$. Therefore the vector field $\mathbf{v}_\Lambda$ is a partial symmetry of equation (11.64) and the system

$$u_t = uu_{xx} + u_x^2, \qquad (-1 + 2c/u)u_t + u_x^2 = 0, \tag{11.67}$$

is Galilean-invariant.

It is interesting to note that if $c \neq 0$, then (11.67) admits only three-dimensional subalgebra $\mathbf{g}_3 = L(\mathbf{v}_1, \mathbf{v}_2, 2\mathbf{v}_3 + \mathbf{v}_4)$ of the initially four-dimensional algebra (11.65) of the classical Lie symmetries of (11.65). Taking the latter fact into account, one can try to preserve the classical Lie symmetry group as a subgroup of the classical symmetry group of the appended system. For a generic function $f(u)$, the family of nonlinear heat equations

$$u_t = (f(u)u_x)_x \tag{11.68}$$

admits the three-dimensional Lie algebra $\mathbf{g}_3$ generated by the vector fields

$$\mathbf{v}_1 = \partial_t, \qquad \mathbf{v}_2 = \partial_x, \qquad \mathbf{v}_3 = 2t\partial_t + x\partial_x. \tag{11.69}$$

The functions u and u_t/u_x^2 are first order differential invariants for $\mathbf{g}_3$. Thus, if we append the equation

$$u_t = g(u)u_x^2, \tag{11.70}$$

for $g(u)$ arbitrary (for the moment), the combined system (11.68), (11.70) admits $\mathbf{g}_3$.

The compatibility condition for the system (11.68), (11.70) is the following ordinary differential equation connecting the functions $f(u)$ and $g(u)$:

$$f^2g'' + f(4g - 3f')g' + (11(f')^2 - ff'')g - 5g'f^2 + 2g^3 = 0.$$

For $f(u)$ given, the function $g(u) = f(u)/u$ is a particular solution of the latter equation, and we are led to the compatible system

$$u_t = (f(u)u_x)_x, \qquad u_t = \frac{f(u)u_x^2}{u}$$

admitting (11.69) as a subalgebra of the Lie algebra of its infinitesimal symmetries.

11.5. WEAK SYMMETRIES

In Olver and Rosenau [1987], a further generalization of the non-classical method was proposed. Since the combined system (11.1), (11.4) is an overdetermined system of partial differential equations, one should, in treating it, take into account any integrability conditions given by equating mixed partials. (The Cartan–Kuranishi

Theorem assures us that, under mild regularity conditions, the integrability conditions can all be found in a finite number of steps; differential Gröbner basis methods, as in Pankrat'ev [1989], Topunov [1989], provide a practical means to compute them.) Therefore, one should compute the symmetry group not of just the system (11.1), (11.4) but also any associated integrability conditions. Thus, we define a *weak symmetry group* of the system (11.1) to be any symmetry group of the overdetermined system (11.1), (11.4) and all its integrability conditions.

11.5.1 An example of a weak symmetry group.

For the Boussinesq equation

$$u_{tt} + uu_{xx} + u_x^2 + u_{xxxx} = 0,$$

consider the scaling group generated by the vector field $\mathbf{v} = x\partial_x + t\partial_t$. This is not a symmetry of the Boussinesq equation, nor is it a symmetry of the combined system

$$u_{tt} + uu_{xx} + u_x^2 + u_{xxxx} = 0, \qquad Q = xu_x + tu_t = 0. \tag{11.71}$$

Nevertheless, if we append the integrability conditions to (11.71), we do find that $\mathbf{v}$ satisfies the weak symmetry conditions. To compute the invariant solutions, we begin by introducing the invariants, $y = x/t$, and $w = u$. Differentiating the formula $u = w(y) = w(x/t)$ and substituting the result into the Boussinesq equation, we come to the following equation

$$t^{-4}w'''' + t^{-2}[(y^2 + w)w'' + (w')^2 + 2yw'] = 0. \tag{11.72}$$

At this point the crucial difference between the weak symmetries and the nonclassical (or classical) symmetries appears. In the latter case, any non-invariant coordinate, e.g. the t here, will factor out of the resulting equation and thereby leave a single ordinary differential equation for the invariant function $w(y)$. For weak symmetries this is no longer true, since we have yet to incorporate the integrability conditions for (11.67). However, we can separate out the coefficients of the various powers of t in the above equation (11.68), leading to an overdetermined system of ordinary differential equations,

$$w'''' = 0, \qquad (y^2 + w)w'' + (w')^2 + 2yw' = 0,$$

for the unknown function w. In this particular case, the resulting overdetermined system does have solutions, namely $w(y) = -y^2$, or $w(y) =$ constant. The latter are trivial, but the former yield a nontrivial similarity solution: $u(x,t) = -x^2/t^2$.

11.5.2. A particular case of the infinitesimal weak symmetries.

An explication of one possible approach to treating the weak symmetries (Dzhamay and Vorob'ev, [1994]) is given below. We will restrict ourselves to the case of differential equations in one unknown function such as equation (11.32):

$$\Delta(x, u, u^{(k)}) = 0. \tag{11.32}$$

Consider the contact vector field $\mathbf{v}_Q$, the function $\Gamma(x, u, u^{(k)}) = \mathbf{v}_Q^{(k)}(\Delta)(x, u, u^{(k)})$, and the system W of differential equations

$$\Delta = 0, \qquad \Gamma = 0, \qquad Q = 0. \tag{11.73}$$

DEFINITION. *A vector field* $\mathbf{v}_Q$ *is an infinitesimal weak symmetry of equation (11.32) if*

(1) $\mathbf{v}_Q$ *is a classical infinitesimal symmetry of system (11.73),*
(2) *system (11.73) is compatible.*

Property (1) can be reformulated by saying that $\mathbf{v}_Q$ is a partial infinitesimal symmetry of the system $\Delta = 0$, $\Gamma = 0$. Note that if we required that $\mathbf{v}_Q$ were a classical infinitesimal symmetry of the latter system, we would have got a conditional infinitesimal symmetry considered in Section 11.4. Property (2) means that system (11.73) implies no extra conditions that may arise by cross differentiation of the equations of system (11.73) and their differential consequences. Since $\Gamma = \mathbf{v}_Q^{(k)}(\Delta)(x, u, u^{(k)})$ and $\mathbf{v}_Q(Q) = Q_u\, Q$, the criterion of tangency of $\mathbf{v}^{(k)}$ to W takes the form:

$$\mathbf{v}_Q^{(k)}(\Gamma)|_W = 0. \tag{11.74}$$

Relation (11.74) and the compatibility conditions for W imply the determining equations for the characteristic function Q.

It is clear that relations (11.73) can be generalized by adding to W the functions $\mathbf{v}_Q^{(k)}(\Gamma)$ and so on.

11.5.3. Weak symmetries of the nonlinear heat equation.

In this Section, the exact solutions of the equation

$$u_t = u_{xx} + u_x^2 + u^2 \tag{11.75}$$

obtained by Galaktionov, [1990], are interpreted as invariant under the weak symmetries of (11.75). Consider the vector field $\mathbf{v}_Q$ with the characteristic function $Q = -p_x + a(t,x)$, where the function $a(t,x)$ needs to be defined. Since the equations of the intersection $E_\Delta \cap E_Q^{(2)}$ are equivalent to the equations

$$\tilde{\Delta} \equiv -p_t + a_x + a^2 + u^2, \qquad p_x = a, \qquad p_{tx} = a_t, \qquad p_{xx} = a_x,$$

the following formula is valid: $\Gamma \equiv \mathbf{v}_Q^{(2)}(\tilde{\Delta}) = a_t + a_{xx} + 2aa_x + 2ua$. Therefore, if the function a is fixed a unique invariant solution $u(t,x)$ is obtained from the equation $\Gamma = 0$:

$$u(t,x) = \frac{a_t - a_{xx} - 2aa_x}{2a}. \tag{11.76}$$

So we can conclude that the system W takes the form:

$$p_t = a_x + a^2 + u^2, \qquad p_x = a(t,x), \qquad u(t,x) = \frac{a_t - a_{xx} - 2aa_x}{2a}. \tag{11.77}$$

The compatibility conditions for system (11.77) are evident:

$$\begin{aligned} &\frac{\partial}{\partial x}\left(\frac{a_t - a_{xx} - 2aa_x}{2a}\right) = a, \\ &\frac{\partial}{\partial t}\left(\frac{a_t - a_{xx} - 2aa_x}{2a}\right) = a_x + a^2 + \left(\frac{a_t - a_{xx} - 2aa_x}{2a}\right)^2. \end{aligned} \tag{11.78}$$

Equations (11.78) admit separation of variables, precisely, the first equation is satisfied if $a(t,x) = \phi(t)\sin x$ with $\phi(t)$ arbitrary function. Hence, the second equation implies the relation

$$\frac{d}{dt}\left(\frac{\dot\phi+\phi}{2\phi}\right) = \left(\frac{\dot\phi+\phi}{2\phi}\right)^2 + \phi^2 \tag{11.79}$$

for the function ϕ.

After the function $\phi(t)$ is found from (11.79), we obtain the infinitesimal weak symmetry $\mathbf{v}_Q$ of equation (11.75) with the characteristic function $Q = -p_x + \phi(t)\sin x$ and the invariant under $\mathbf{v}_Q$ solution

$$u(t,x) = \frac{\dot\phi+\phi}{2\phi} - \phi\sin x.$$

Galaktionov obtained this solution by directly applying his method of generalized separation of variables in the form $u(t,x) = \theta(t) - \phi(t)\sin x$ to equation (11.75).

11.5.4. Discussion.

Weak symmetry groups, while at the outset quite promising, have some critical drawbacks. It can be shown that every group is a weak symmetry group of a given system of partial differential equations, and, moreover, every solution to the system can be derived from some weak symmetry group — see Olver and Rosenau [1987]. Therefore, the generalization is too severe. Nevertheless, it gives some hints as to how to proceed in any practical analysis of such solution methods. What is required is an appropriate theory of overdetermined systems of partial differential equations which will allow one to write down reasonable classes of groups for which the combined system (11.1), (11.4) is compatible, in the sense that is has solutions, or, more restrictively, has solutions that can be algorithmically computed. For example, restricting to scaling groups, or other elementary classes of groups, might be a useful starting point.

A SURVEY OF RESULTS

The following preliminary comments will be helpful for the reader. First, to date only particular solutions of the determining equations for the coefficients of infinitesimal nonclassical symmetries have been obtained and are therefore given below. The reason is partly explained in Section 11.1.2. Second, we do not point out which vector fields are obtained in what papers, so the results are a set theoretic union of those in separate papers. Third, not all of the known results are given but only important for applications or most completely investigated. And last, exact solutions invariant under the nonclassical symmetries can give rise to multiparameter families of solutions with the aid of the classical symmetry transformations.

11.7. BOUSSINESQ EQUATION

$$u_{tt} + uu_{xx} + (u_x)^2 + u_{xxxx} = 0$$

Lie point symmetries.

(Clarkson and Kruskal [1989], Nishitani and Tajiri [1982], Rosenau and Schwarzmeier [1986])

$$X_1 = x\partial_x + 2t\partial_t - 2u\partial_u, \qquad X_2 = \partial_x, \qquad X_3 = \partial_t.$$

Nonclassical conditional symmetries.

(Levi and Winternitz [1989], Fushchich and Serov [1989])

$\mathbf{v}_1 = \partial_t + t\partial_x - 2t\partial_u,$

$\mathbf{v}_2 = t^3\partial_t + (\beta_1 t^7 - xt^2)\partial_x + (2t^2 u + 6x^2 - 2\beta_1 t^5 x - 4\beta_1^2 t^{10})\partial_u,$

(by dilations and coordinate reflections one can transform β_1 into $\beta_1 = 1$ or $\beta_1 = 0$),

$\mathbf{v}_3 = 2t\partial_t + (x + 2t^2)\partial_x - 2(u + 2x + 4t^2)\partial_u,$

$\mathbf{v}_4 = 2p(t)\partial_t + (x + \beta_2 w)\dot{p}\partial_x - [2\dot{p}u + 6\dot{p}px^2 + \beta_2(1 + 12p\dot{p}w)x + \beta_2^2 w + 6\beta_2^2 p\dot{p}w^2]$

$$\partial_u, \qquad \dot{p}^2 = 4p^3 - c, \qquad c = \text{const}, \qquad w(t) = \int_0^t p(s)/\dot{p}(s)^2\, ds,$$

$\mathbf{v}_5 = t^2\partial_x + (t^5 - 2x)\partial_u,$

$\mathbf{v}_6 = \partial_x + (\lambda(t) - \frac{1}{3}\mu(t)x)\partial_u, \qquad \ddot{\mu} = \mu^2, \qquad \ddot{\lambda} = \mu\lambda,$

$\mathbf{v}_7 = x\partial_x + 2u\partial_u,$

$\mathbf{v}_8 = x^3\partial_x + 2(x^2 u + 24)\partial_u.$ ∎

Nonclassical weak symmetries.

(Olver and Rosenau [1987])

$$\mathbf{w} = x\partial_x + t\partial_t$$

Exact solutions invariant under nonclassical symmetries.

$\mathbf{v}_1:$ $\quad u = \phi(z) - t^2, \qquad z = x - t^2/2, \qquad \dddot{\phi} + \phi\dot{\phi} - \phi = 2z + c_1,$

$\mathbf{v}_2:$ $\quad u = \phi(z)t^2 - z^2/t^2, \qquad z = xt - \beta_1 t^6/6, \qquad \ddot{\phi} + \phi^2/2 = c_1 z + c_0$

$(\beta_1 = 0), \qquad \dddot{\phi} + \phi\dot{\phi} - 5\phi = 50z + c_0 \qquad (\beta_1 = 1),$

$\mathbf{v}_3:$ $\quad u = \phi(z)/t - (x/2t + t)^2, \qquad z = x/\sqrt{t} - 2t^{3/2}/3, \qquad \ddddot{\phi} + \phi\ddot{\phi} + (\dot{\phi})^2 +$

$3z\dot{\phi}/4 + 3\phi/2 = 9z^2/8$

$\mathbf{v}_4:$ $\quad u = \phi(z)p^{-1} - (\dot{p}/2p + \beta_2 w\dot{p}/2p), \qquad z = \dfrac{\beta_2 c^{-1} p^{-1/2} \int_0^t p(s)\, ds}{3} + xp^{-1/2},$

$\dddot{\phi} + \phi\ddot{\phi} + \dot{\phi}^2 - 3c\dot{\phi}/4 - 3c\phi/2 = 9c^2 z^2/8,$

$\mathbf{v}_5:$ $\quad u = \phi(t) - x^2/t^2 + t^3 x, \qquad \ddot{\phi} - 2\phi/t^2 + t^6 = 0,$

$\mathbf{v}_6:$ $\quad u = \phi(t) - x^2\mu/6 + \lambda x, \qquad \ddot{\phi} - 2\phi/t^2 + t^6 == 0,$

$\mathbf{v}_7:$ $\quad u = x^2\phi(t), \qquad \ddot{\phi} + 6\phi^2 = 0,$

$\mathbf{v}_8:$ $\quad u = x^2\phi(t) - 12x^{-2}, \qquad \ddot{\phi} + 6\phi^2 = 0,$

$\mathbf{w}:$ $\quad u = -x^2/t^2.$ ∎

11.8 BURGERS' EQUATION

$$u_t + uu_x = u_{xx}$$

Lie point symmetries.

$$X_1 = \partial_t, \qquad X_2 = \partial_x, \qquad X_3 = t\partial_x + \partial_u,$$
$$X_4 = 2t\partial_t + x\partial_x - u\partial_u, \qquad X_5 = t^2\partial_t + tx\partial_x + (x - tu)\partial_u$$

Nonclassical conditional symmetries.

(Arrigo, Broadbridge, and Hill [1993], Pucci [1992], Vorob'ev [1986])

The following general expressions are valid for the vector fields of the nonclassical infinitesimal symmetries:

$$\mathbf{v} = \partial_t + \xi(t,x,u)\partial_x + \phi(t,x,u)\partial_u, \qquad \text{or} \qquad \mathbf{w} = \partial_x + \psi(t,x,u)\partial_u,$$

where the functions ξ, ϕ, and ψ satisfy the equations:

$$\xi_{uu} = 0, \qquad \phi_{uu} - 2\xi_{xu} - 2(u-\xi)\xi_u = 0, \qquad \xi_{xx} - (2\xi - u)\xi_x +$$
$$2\phi\xi_u - \xi_t - 2\phi_{xu} + \phi = 0, \qquad \phi_{xx} + 2\phi\xi_x + u\phi_x + \phi_t = 0,$$

or

$$u\psi_x - \psi_{xx} - \psi^2\psi_{uu} - 2\psi\psi_{xu} + \psi_t + \psi^2 = 0.$$

Particular solutions of the first set of equations generate the infinitesimal nonclassical symmetries:

$$\begin{aligned}
\mathbf{v}_1 =& \partial_t + u\partial_x,\\
\mathbf{v}_2 =& \partial_t + \big(-\tfrac{1}{2}u + a_0t^2 + a_1t + a_2\big)\partial_x + \big(-\tfrac{1}{4}u^3 + \tfrac{1}{2}u^2(a_0t^2 + a_1t + a_2) -\\
& (a_0t + \tfrac{1}{2}a_1)xu + (b_0t + b_2)u + \tfrac{1}{2}a_0x^2 - b_0x + a_0t + a_3\big)\partial_u,\\
\mathbf{v}_3 =& \partial_t - \tfrac{1}{2}u\partial_x - \frac{u^3}{4}\partial_u,\\
\mathbf{v}_4 =& \partial_t - \tfrac{1}{2}u\partial_x - \big(\tfrac{1}{4}u^3 - \tfrac{1}{4}a_1^2u\big)\partial_u,\\
\mathbf{v}_5 =& \partial_t + \big(a_2 - \tfrac{1}{2}u\big)\partial_x + \big(-\tfrac{1}{4}u^3 + \tfrac{1}{2}a_2u^2\big)\partial_u,\\
\mathbf{v}_6 =& \partial_t - \big(\tfrac{1}{2}u + x^{-1}\big)\partial_x - \big(\tfrac{1}{4}u^3 + \frac{u^2}{2x}\big)\partial_u,\\
\mathbf{v}_7 =& \partial_t + \big(-\tfrac{1}{2}u + w(t,x)\big)\partial_x + \big(-\tfrac{1}{4}u^3 + \tfrac{1}{2}w(t,x)u^2\big)\partial_u,
\end{aligned}$$

with a, b parameters, and $w(t,x)$ a solution of Burgers' equation $w_t + 2ww_x - w_{xx} = 0$.

Solutions invariant under the nonclassical symmetries.

$$\mathbf{v}_1: \quad u = x/t,$$

$$\mathbf{v}_3: \quad u = -\frac{4x}{x^2+2t},$$

$$\mathbf{v}_4: \quad u = -\frac{a_1(c_1\exp(a_1x)-1)}{c_1\exp(a_1x)+1\pm c_2\exp(a_1x/2-a_1^2t/4)},$$

$$\mathbf{v}_5: \quad u = -2\frac{1+a_2\exp(a_2x+a_2^2t)}{x+\exp(a_2x+a_2^2t)},$$

$$\mathbf{v}_6: \quad u = -\frac{12t+6x^2}{6xt+x^3}$$

11.9. GENERALIZED KORTEWEG–DEVRIES EQUATION

$$u_t + f(u)u_x^k + u_{xxx} = 0, \qquad k > 0$$

Lie point symmetries.

k	$f(u)$	
arbitrary	arbitrary	$X_1 = \partial_t, \quad X_2 = \partial_x$
	u^n	$X_1, \quad X_2, \quad X_3 = 3t(k+n-1)\partial_t + x(k+n-1)\partial_x + (k-3)u\partial_u$
	$\exp u$	$X_1, \quad X_2, \quad X_4 = 3t\partial_t + x\partial_x + (k-3)\partial_u$
	1	$X_1, \quad X_2, \quad X_5 = 3(k-1)t\partial_t + (k-1)x\partial_x + (k-3)u\partial_u, \quad X_6 = \partial_u$
3	arbitrary	$X_1, \quad X_2, \quad X_7 = 3t\partial_t + x\partial_x$
	u^{-2}	$X_1, \quad X_2, \quad X_7, \quad X_8 = u\partial_u$
	1	$X_1, \quad X_2, \quad X_6, \quad X_7$
1	$u^n + c$	$X_1, \quad X_2, \quad X_9 = 3nt\partial_t + n(2ct+x)\partial_x - 2u\partial_u$
	u	$X_1, \quad X_2, \quad X_{10} = 3t\partial_t + x\partial_x - 2u\partial_u, \quad X_{11} = t\partial_x + \partial_u$
	$\exp u + c$	$X_1, \quad X_2, \quad X_{12} = 3t\partial_t + (2ct+x)\partial_x - 2\partial_u$
	1	$X_1, \quad X_2, \quad X_8, \quad X_{13} = 3t\partial_t + (2t+x)\partial_x,$
		$X_g = g(t,x)\partial_u, \quad g(t,x)$ solves the generalized KdV equation

Nonclassical conditional symmetries.
(Fushchich, Serov, Ahmerov [1991])

$$\mathbf{v} = t^{1/k}\partial_x + F(u)\partial_u$$

$$f(u) = c_1u^{1-k/2} + c_2u^{(1-k)/2}, \qquad F(u) = (\frac{kc_1}{2})^{-1/k}\sqrt{u}$$

$$f(u) = (c_1\log u + c_2)(1-u^2)^{1-k}, \qquad F(u) = (kc_1)^{-1/k}u$$

$$f(u) = (c_1\arcsin u + c_2)(1-u^2)^{(1-k)/2}, \qquad F(u) = (kc_1)^{-1/k}\sqrt{1-u^2}$$

$$f(u) = (c_1\sinh^{-1}u + c_2)(1+u^2)^{(1-k)/2}, \qquad F(u) = (kc_1)^{-1/k}\sqrt{1+u^2}$$

$$f(u) = c_1u, \qquad F(u) = (kc_1)^{-1/k}$$

Exact solutions invariant under the nonclassical symmetries.

$$
\begin{aligned}
u(t,x) &= \left(\frac{x}{2}\Big(\frac{kc_1t}{2}\Big)^{-1/k} + ct^{-1/k} - \frac{c_2}{c_1}\right)^2,\\
u(t,x) &= \exp\left(\frac{k(kc_1)^{-3/k}}{k-2}t^{1-3/k} - \frac{c_2}{c_1} + ct^{-1/k} + (kc_1t)^{-1/k}x\right), \qquad k \neq 2,\\
u(t,x) &= \exp\left((2c_1)^{-3/2}t^{-1/2}\log t - \frac{c_2}{c_1} + ct^{-1/2} + (2c_1t)^{-1/2}x\right), \qquad k = 2,\\
u(t,x) &= \sin\left(-\frac{k(kc_1)^{-3/k}}{k-2} - \frac{c_2}{c_1} + ct^{-1/k} + (kc_1t)^{-1/k}x\right), \qquad k \neq 2,\\
u(t,x) &= \sin\left(-(2c_1)^{-3/2}\frac{\log t}{\sqrt{t}} - \frac{c_2}{c_1} + ct^{-1/2} + (2c_1t)^{-1/2}x\right), \qquad k = 2,\\
u(t,x) &= \sinh\left(\frac{k(kc_1)^{-3/k}}{k-2}t^{1-3/k} - \frac{c_2}{c_1} + ct^{-1/k} + (kc_1t)^{-1/k}x\right), \qquad k \neq 2,\\
u(t,x) &= \sinh\left((2c_1)^{-3/2}\frac{\log t}{\sqrt{t}} - \frac{c_2}{c_1} + ct^{-1/2} + (2c_1t)^{-1/2}x\right), \qquad k = 2\\
u(t,x) &= ct^{-1/k} + x\big(kc_1t\big)^{-1/k}
\end{aligned}
$$

11.10. KADOMTSEV-PETVIASHVILI EQUATION

(Kadomtsev and Petviashvili [1970])

$$(u_t + uu_x + u_{xxx})_x + ku_{yy} = 0, \qquad k = \pm 1$$

Lie point symmetries.
(Tajiri, Nishitani, and Kawamoto [1982], David, Kamran, Levi and Winternitz [1986])

$$X_\alpha = 6\alpha(t)\partial_t + (2x\dot\alpha(t) - ky^2\ddot\alpha(t))\partial_x + 4y\dot\alpha(t)\partial_y + (-4u\dot\alpha(t) + 2x\ddot\alpha(t) - ky^2\dddot\alpha(t))\partial_u,$$
$$X_\beta = \beta(t)\partial_x + \dot\beta(t)\partial_u, \qquad X_\gamma = -y\dot\gamma(t)\partial_x + 2k\gamma(t)\partial_y - y\ddot\gamma(t)\partial_u,$$

where $\alpha(t)$, $\beta(t)$, and $\gamma(t)$ are arbitrary functions.

Nonclassical conditional symmetries.
(Clarkson and Winternitz [1991])

$$\mathbf{v} = \partial_x + [R(y,t)x + S(y,t)]\partial_u, \qquad R \neq 0 \text{ and } S_y \neq 0,$$

where the functions $R(y,t)$ and $S(y,t)$ are solutions of the system

$$kR_{yy} + 3R^2 = 0, \qquad kS_{yy} + 3RS + R_t = 0.$$

Exact solutions invariant under the nonclassical symmetries.

$$u(x,y,t) = w(y,t) + x^2\gamma(y,t) + x\psi(y,t),$$

where

$$\gamma(y,t) = -kW(y+\phi_0(t);0,g(t)),$$
$$\psi(y,t) = W(y+\phi_0(t);0,g(t))\Big(A(t) + B(t)\int^y \frac{dz}{W^2(z+\phi_0;0,g)}\Big)$$
$$-\frac{\dot g}{12g} + y\big(\dot\phi_0 + \dot g\frac{y+2\phi_0}{2g}\big)W(y+\phi_0(t);0,g(t))$$

with $\phi_0(t)$, $g(t)$, $A(t)$, $B(t)$ arbitrary functions, $W(z;0,h)$ the Weierstrass elliptic function.

11.11. KOLMOGOROV-PETROVSKII-PISKUNOV, OR FITZHUGH-NAGUMO EQUATION

$$u_t = u_{xx} + u(1-u)(u-a), \qquad -1 \leqslant a \leqslant 1$$

Lie point symmetries.

$$X_1 = \partial_t, \qquad X_2 = \partial_x$$

Nonclassical conditional symmetries.
(Nucci and Clarkson [1992], Vorob'ev [1986])

$$\mathbf{v}_1 = \partial_t \pm \frac{1}{\sqrt{2}}(3u-a-1)\partial_x + \frac{3}{2}u(1-u)(u-a)\partial_u,$$

$$\mathbf{v}_2 = \partial_t + \alpha(x)\partial_x - \alpha'(x)u\partial_u, \qquad a=-1, \qquad \alpha(x) = \frac{3}{\sqrt{2}}\frac{1+\exp\sqrt{2}x}{1-\exp\sqrt{2}x},$$

$$\mathbf{v}_3 = \partial_t + \alpha(x)\partial_x - \alpha'(x)(u-1/2)\partial_u, \qquad a=1/2, \qquad \alpha(x) = \frac{3}{\sqrt{2}}\frac{1+\exp x/\sqrt{2}}{1-\exp x/\sqrt{2}}$$

$$\mathbf{v}_4 = \partial_t + \alpha(x)\partial_x - \alpha'(x)(u-1)\partial_u, \qquad a=2, \qquad \alpha(x) = \frac{3}{\sqrt{2}}\frac{1+\exp\sqrt{2}x}{1-\exp\sqrt{2}x}$$

Exact solutions invariant under the nonclassical symmetries.

$\mathbf{v}_1$:

$$u = \frac{ac_1\exp((\sqrt{2}x+a^2t)/2) + c_2\exp((\sqrt{2}ax+t)/2)}{c_1\exp((\sqrt{2}x+a^2t)/2) + c_2\exp((\sqrt{2}ax+t)/2) + c_3\exp((\sqrt{2}(a+1)x+at)/2)},$$

$\mathbf{v}_2, \mathbf{v}_4$:

$$u = (c_1\exp((\sqrt{2}x+3t)/2) - c_2\exp((-\sqrt{2}x+3t/2)))w(z),$$
$$z = c_1\exp((\sqrt{2}x+3t)/2) + c_2\exp((-\sqrt{2}x+3t/2)) + c_3$$

$\mathbf{v}_3$:

$$u = (c_1\exp((2\sqrt{2}x+3t)/8) - c_2\exp((-2\sqrt{2}x+3t)/8))w(z),$$
$$z = c_1\exp((2\sqrt{2}x+3t)/8) + c_2\exp((-2\sqrt{2}x+3t)/8) + c_3,$$

where $w(z)$ is the Weierstass function satisfying the equation $w''(z) = 2w(z)^3$. ∎

11.12. NONLINEAR WAVE EQUATION

$$u_{tt} = uu_{xx}$$

Lie point symmetries.

$$X_1 = \partial_t, \qquad X_2 = \partial_x, \qquad X_3 = t\partial_t + x\partial_x, \qquad X_4 = t\partial_t - 2u\partial_u$$

Nonclassical conditional symmetries.
(Fushchich and Serov [1988])

$$\begin{aligned}
\mathbf{v}_1 =& \partial_x + a_1\partial_u, \qquad \mathbf{v}_2 = \partial_t + (a_2x + a_3)\partial_u,\\
\mathbf{v}_3 =& \partial_t + (a_4t + a_5)\partial_x + 2a_4(a_4t + a_5)\partial_u, \qquad \mathbf{v}_4 = \partial_x + [w(t) + f(t)]\partial_u,\\
\mathbf{v}_5 =& t\partial_t + (u + a_7x + a_8)\partial_u, \qquad \mathbf{v}_6 = t\partial_t + [t^3(a_9x + a_{10}) - 2u]\partial_u,\\
\mathbf{v}_7 =& x\partial_x + (u + b_1t + b_2)\partial_u, \qquad \mathbf{v}_8 = x\partial_x + [u + w(t)x^2/2 - f(t)]\partial_u,\\
\mathbf{v}_9 =& (t^2 - 1)\partial_t + 2x\partial_x + (t + 1)u\partial_u, \qquad \mathbf{v}_{10} = t^3\partial_t + (3t^2 - 15x^2 + b_3x + b_4)\partial_u,\\
\mathbf{v}_{11} =& t^2x\partial_x + (t^2u + 3x^2 + b_2t^5 + b_6)\partial_u, \qquad \mathbf{v}_{12} = w(t)\partial_t + \dot{w}(t)u\partial_u,
\end{aligned}$$

where a, b are arbitrary parameters, $w(t)$ and $f(t)$ satisfy the ODE: $\ddot{w} = w^2$, $\ddot{f} = wf$.

Exact solutions invariant under the nonclassical symmetries.

$$\begin{aligned}
\mathbf{v}_1 :& \quad u = \phi(t) + a_1x, \qquad \ddot{\phi} = 0,\\
\mathbf{v}_2 :& \quad u = \phi(t) + t(a_2x + a_3), \qquad \ddot{\phi} = 0,\\
\mathbf{v}_3 :& \quad u = \phi(z) + 2a_4x, \qquad z = a_4t^2/2 + a_5t - x, \qquad (\phi - 2a_4z - a_5^2)\ddot{\phi} = a_4\dot{\phi},\\
\mathbf{v}_4 :& \quad u = w(t)x^2/2 + f(t)x + \phi(t), \qquad \ddot{\phi} = w\phi,\\
\mathbf{v}_5 :& \quad u = t\phi(x) - (a_7x + a_8), \qquad \ddot{\phi} = 0,\\
\mathbf{v}_6 :& \quad u = t^{-2}\phi(x) + t^3(a_9x + a_{10})/5, \qquad \ddot{\phi} = 6,\\
\mathbf{v}_7 :& \quad u = x\phi(t) - (b_1t + b_2), \qquad \ddot{\phi} = 0,\\
\mathbf{v}_8 :& \quad u = w(t)x^2/2 + \phi(t)x + f(t), \qquad \ddot{\phi} = w(t)\phi,\\
\mathbf{v}_9 :& \quad u = (t - 1)\phi((t + 1)x/(t - 1)), \qquad \ddot{\phi} = 0,\\
\mathbf{v}_{10} :& \quad u = t^3\phi(x) + 3x^2t^{-2} - (b_3x + b_4)/5t^2, \qquad \ddot{\phi} = 0,\\
\mathbf{v}_{11} :& \quad u = x\phi(t) + 3x^2/t^2 - b_5t^2 - b_6t^{-2}, \qquad t^2\ddot{\phi} = 64,\\
\mathbf{v}_{12} :& \quad u = w(t)\phi(x), \qquad \ddot{\phi} = 1.
\end{aligned}$$

11.13 ZABOLOTSKAYA-KHOKHLOV EQUATION

$$u_{tx} - (uu_x)_x - u_{yy} = 0$$

The infinitesimal Lie point symmetry algebra of this equation contains the subalgebra with the generators:

$$X_1 = \partial_t, \qquad X_2 = \partial_x, \qquad X_3 = \partial_y, \qquad X_4 = y\partial_x + 2t\partial_y,$$
$$X_5 = t\partial_t + x\partial_x + y\partial_y, \qquad X_6 = 4t\partial_t + 2x\partial_x + 3y\partial_y - 2u\partial_u, \qquad X_7 = t\partial_x - \partial_u$$

Nonclassical conditional symmetries.

(Extracted from Fushchich, Chopik and Mironiuk [1990] and adapted to the case of three independent variables)

The equation considered attached by the first order equation:

$$u_t u_x - u u_x^2 - u_y^2 = 0,$$

which is its characteristic equation, forms the compatible system while such a system is incompatible in the case of four independent variables, i.e. for the equation $u_{tx} - (u_x)_x - u_{yy} - u_{zz} = 0$. The system admits the infinite-dimensional Lie algebra of classical infinitesimal symmetries whose vector fields can be presented as

$$X = \sum_1^8 a_i(u) X_i$$

with $a_i(u)$ smooth functions, X_i for $1 \leqslant i \leqslant 7$ given above, and

$$X_8 = y\partial_t + 2(x + 2ut)\partial_y.$$

Acknowledgments.

The authors have a pleasure to thank Mikhail V. Foursov for his help in checking the results of this Survey by using the "Mathematica" program SYMMAN (Dzhamay *et al*, [1994]) .

References

Anderson, I., Kamran, N., Olver, P.J., *Internal, external and generalized symmetries*, Adv. in Math. **100** (1993), 53–100.

Arrigo, D.J., Broadbridge, P., Hill, J.M., *Nonclassical symmetry solutions and the methods of Bluman–Cole and Clarkson–Kruskal*, J. Math. Phys. **34** (1993), 4692–4703.

Bluman, W., Cole, J.D., *The general similarity solutions of the heat equation*, J. Math. Mech. **18** (1969), 1025–1042.

Bluman, G.W., Kumei, S, *Symmetries and Differential Equations*, Springer, N.Y., 1989.

Cartan, É., *Sur l'équvalence absolue de certaines systèmes d'équationes différentielles et sur certaines familles de courbes*, Bull. Soc. Math. France **42** (1914), 12-48; Oeuvres Completes, part II, vol.2, Gauthier-Villars, Paris, 1953, pp. 133-1168..

Cartan, É., *Sur l'intégration de certaines systèmes indéterminés d'équations différentielles*, J. Reine Angew. Math. **145** (1915), 86-91; Oeuvres Completes, part II,vol.2, Gauthier-Villars, Paris, pp. 1169-1174.

Clarkson, P.A., Kruskal, M., *New similarity reductions of the Boussinesq equation*, J. Math.Phys. **30** (1989), 2201-2213.

Clarkson, P.A., Winternitz, P., *Nonclassical symmetry reductions for the Kadomtsev–Petviashvili equation*, Physica D **49** (1991), 257–272.

David, D., Kamran, N., Levi, D., Winternitz, P., *Subalgebras of loop algebras and symmetries of the Kadomtsev–Petviashvili equation*, Phys. Rev. Lett. **55** (1985), 2111–2113.

Dzhamay, A.V., Vorob'ev, E.M., *Infinitesimal weak symmetries of differential equations in two independent variables*, J. Phys. A, (1994) (to appear).

Dzhamay, A.V., Foursov, M.V., Grishin, O.V., Vorob'ev, E.M., Zhikharev, V.N., *A "Mathematica" program SYMMAN for symmetry analysis of systems of partial differential equations*, New Computer Technologies in Control Systems, Proceedings of the International Workshop, Pereslavl-Zalessky, Russia, 1994, pp. 25–27.

Fuschich, V.I., Serov, N.I., Chopik, V.I., *Conditional symmetry and nonlinear heat equations*, Rep. Ukr.Acad. Sci. (1988), no. A9, 17–21. (Russian)

Fuschich, V.I., Serov, N.I., *Conditional symmetries and explicit solutions of the nonlinear acoustic equation*, Rep. Ukr.Acad. Sci. (1988), no. A10, 27–31. (Russian)

Fuschich, V.I., Serov, N.I., *Conditional symmetry and explicit solutions of the Boussinesq equation*, Symmetries and solutions of differential equations of mathematical physics, Institute of Mathematics, Kiev, 1989. (Russian)

Fushchich, V.I., Serov, N.I., Ahmerov, T.K., *On the conditional symmetry of the generalized KdV equation*, Rep. Ukr.Acad. Sci. (1991), no. A12. (Russian)

Fushchich, V.I., Chopik, V.I., Mironiuk, P.I., *Conditional symmetry and explicit solutions of the three-dimensional acoustic equation*, Rep. Ukr.Acad. Sci. (1991), no. A9, 25–28. (Russian)

Galaktionov, V.A., *On new exact blow–up solutions for nonlinear heat conduction equations with source and applications*, Diff. Int. Eqs. **3** (1990), 863–874.

Guthrie, G., *Miura Transformations and Symmetry Groups of Differential Equations*, Ph.D. Thesis, University of Canterbury, Christchurch, New Zealand, 1993.

Kadomtsev, V.V., Petviashvili, V.I., *On the stability of solitary waves in weakly dispersing media*, Sov. Phys. Dokl. **15** (1970), 539–541.

Levi, D., Winternitz, P., *Nonclassical symmetry reduction: example of the Boussinesq equation*, J. Phys. A **22** (1989), 2915–2924.

Noether, E, *Invariante Variationsprobleme*, Nachr. König. Gesell. Wissen. Göttingen Math.-Phys. Kl. (1918), 235-257..

Nishitani, T., Tajiri, M., *On similarity solutions of the Boussinesq equation*, Phys. Lett. A **89** (1982), 379–380.

Nucci, M.C., Clarkson, P.A., *The nonclassical method is more general than the direct method for symmetry reductions. An example of the Fitzhugh–Nagumo equation*, Physics Letters **164** (1992), 49–56.

Olver, P.J., *Applications of Lie Groups to Differential Equations*, Second Edition, Springer–Verlag, New York, 1993..

Olver, P.J., *Direct reduction and differential constraints*, Proc. Roy. Soc. London A **444** (1994), 509–523.

Olver, P.J., Rosenau, P., *The construction of special solutions to partial differential equations*, Phys.Lett. A **114** (1986), 107–112.

Olver, P.J., Rosenau, P., *Group-invariant solutions of differential equations*, J. Math. Mech. **47** (1987), 263–278.

Ondich, J., *The reducibility of partially invariant solutions of partial differential equations*, European J. Appl. Math. (1994) (to appear).

Pankrat'ev, E.V., *Computations in differential and difference modules*, Acta Appl. Math. **16** (1989), 167–189.

Pucci, E., *Similarity reductions of partial differential equations*, J. Phys. A **25** (1992), 2631–2640.

Rosenau, P., Schwarzmeier, J.L., *On similarity solutions of Boussinesq-type equations*, Phys.Lett. A **115** (1986), 75–77.

Sidorov, A.F., Shapeev, V.P., Yanenko, N.N., *Method of Differential Constraints and Its Applications to Gas Dynamics*, Nauka, Novosibirsk, 1984.

Tajiri, M., Nishitani, T., Kawamoto, S., *On solutions of the Kadomtsev–Petviashvili equation*, J. Phys. Soc. Japan **51** (1982), 2350–2356.

Topunov, V.L., *Reducing systems of linear differential equations to a passive form*, Acta Appl. Math. **16** (1989), 181–206.

Vorob'ev, E.M., *Partial symmetries of systems of differential equations*, Dokl.Akad.Nauk SSSR **287** (1986), 408–411 (Russian); English transl. in Soviet Math. Dokl. **33** (1986), 408–411.

Vorob'ev, E.M., *Partial symmetries and multidimensional integrable differential equations*, Diff.Eqs. **25** (1989), 322. (Russian)

Vorob'ev, E.M., *Reduction and quotient equations for differential equations with symmetries*, Acta Appl. Math. **23** (1991), 1–24.

Vorob'ev, E.M., *Symmetries of compatibility conditions for systems of differential equations*, Acta Appl. Math. **26** (1992), 61–86.

Yanenko, N.N., *Theory of consistency and methods of integrating systems of nonlinear differential equations*, Proceedings of the Fourth All-Union Mathematics Congress, Leningrad, 1964, pp. 247–250. (Russian)

Zidowitz, S., *Conditional symmetries and the direct reduction of partial differential equations*, Modern Group Analysis: Advanced Analytical and Computational Methods in Mathematical Physiscs (N.H.Ibragimov, M. Torrisi, and A. Valenti, eds.), Kluwer Acad. Publ., The Netherlands, 1993, pp. 387–393.

This research was supported in part by NSF Grants DMS 91–16672 and DMS 92–04192 and by the Grant of Russian State Committee of Higher Education.